Graphs and Combinatorics (2016) 32:2295–2313
DOI 10.1007/s00373-016-1729-4

ORIGINAL PAPER

Chorded Cycles

Megan Cream[1] · **Ralph J. Faudree**[2] ·
Ronald J. Gould[3] · **Kazuhide Hirohata**[4]

Received: 3 October 2015 / Revised: 5 July 2016 / Published online: 12 August 2016

Abstract A *chord* is an edge between two vertices of a cycle that is not an edge on the cycle. If a cycle has at least one chord, then the cycle is called a *chorded cycle*, and if a cycle has at least two chords, then the cycle is called a *doubly chorded cycle*. The minimum degree and the minimum degree-sum conditions are given for a graph to contain vertex-disjoint chorded (doubly chorded) cycles containing specified elements of the graph, i.e., specified vertices, specified edges as cycle-edges, specified paths, or specified edges as chords. Furthermore, the minimum degree condition is given for a graph to be partitioned into chorded cycles containing specified edges as cycle-edges.

Keywords Chorded cycles · Doubly chorded cycles · Degree-sum · Minimum degree

---

M. Cream, R. Gould, and K. Hirohata would like to dedicate this paper to the memory of our colleague and friend, Ralph J. Faudree.

✉ Megan Cream
mcream@spelman.edu

Ronald J. Gould
rg@mathcs.emory.edu

Kazuhide Hirohata
hirohata@ece.ibaraki-ct.ac.jp

1 Spelman College, 350 Spelman Lane SW, Atlanta, GA 30314, USA

2 University of Memphis, Memphis, USA

3 Emory University, 400 Dowman Drive, Atlanta, GA 30322, USA

4 National Institute of Technology, Ibaraki College, Ibaraki, Japan

1 Introduction

We only consider finite simple graphs. In this paper, "disjoint" means "vertex-disjoint." The study of cycles and systems of disjoint cycles in graphs is well established. Recently, there have been numerous papers considering cycles with additional properties, i.e., cycles containing a specific set of vertices, cycles containing a specific set of edges, or cycles containing a set of disjoint paths (see the survey [7]). Let G be a graph, and let C be a cycle in G. A *chord* is an edge between two vertices of C that is not an edge of C. If C has at least one chord, then it is called a *chorded cycle*, and if C has at least two chords, then it is called a *doubly chorded cycle*. Another natural additional property for cycles is that of containing at least one chord or at least some integer t chords. The study of chorded cycles has been increasing (see for example [1,5,8,9]). In this paper, we extend several well-known results on disjoint cycles containing specified elements such as edges or vertices to similar results on disjoint chorded (doubly chorded) cycles containing these elements.

For $u \in V(G)$, the set of neighbors of u in G is denoted $N_G(u)$, and $\deg_G(u) = |N_G(u)|$ is the degree of u in G. Let H be a subgraph of G, and let S be a subset of $V(G)$. For $u \in V(G) - V(H)$, we denote the neighborhood of u in H as $N_H(u) = N_G(u) \cap V(H)$ and $\deg_H(u) = |N_H(u)|$. For $u \in V(G) - S$, $N_S(u) = N_G(u) \cap S$. Furthermore, $N_G(S) = \cup_{w \in S} N_G(w)$ and $N_H(S) = N_G(S) \cap V(H)$. Let A, B be subgraphs of G. Then $N_G(A) = N_G(V(A))$ and $N_B(A) = N_G(A) \cap V(B)$. Let $G - H$ denote the subgraph of G induced by $V(G) - V(H)$, and let $G - S$ denote the subgraph of G induced by $V(G) - S$. If $S = \{u\}$, then we write $G - u$ for $G - S$. For graphs G_1, G_2, and G_3, $G_1 \cup G_2$ denotes the union of G_1 and G_2, $G_1 + G_2$ denotes the join of G_1 and G_2, and $G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2$. For distinct $x, y \in V(G)$, $G + xy$ denotes the graph obtained from G by adding the edge xy ($G + xy = G$, if $xy \in E(G)$). Similarly, if $xy \in E(G)$, then $G - xy$ denotes the graph obtained from G by removing the edge xy. For $e \in E(G)$, $V(e)$ denotes the set of both end-vertices of e. Let C be a cycle with a given orientation, and let $x, y \in V(C)$. Then, according to the orientation, x^+ denotes the first successor of x on C, and $x \overrightarrow{C} y$ denotes the subpath on C from x to y. For $\{u_1, u_2, \ldots, u_k\} \subseteq V(G)$, let $\deg_H(u_1, u_2, \ldots, u_k) = \sum_{i=1}^{k} \deg_H(u_i)$. If $H = G$, then we denote $\deg_G(u_1, u_2, \ldots, u_k) = \deg_H(u_1, u_2, \ldots, u_k)$. The minimum degree of G is denoted by $\delta(G)$. For a non-complete graph G, let

$$\sigma_2(G) = \min \{\deg_G(u) + \deg_G(v) \mid u, v \in V(G), u \neq v, uv \notin E(G)\},$$

and $\sigma_2(G) = \infty$ when G is a complete graph. A cycle of length l is called an l-cycle. The *complement* of G is denoted $\overline{G}$. For terminology and notation not defined here, see [6].

First, we note the following result on disjoint cycles containing specified vertices.

Theorem 1 [4] *Let G be a graph of order $n \geq 6k - 3$ for an integer $k \geq 1$. If*

$$\delta(G) \geq \frac{n}{2},$$

then for any k distinct vertices $v_1, \dots, v_k$ in G, there exist k disjoint cycles $C_1, \dots, C_k$ such that $v_i \in V(C_i)$ and $3 \leq |V(C_i)| \leq 5$ for all $1 \leq i \leq k$.

We also note a similar result on disjoint cycles containing specified edges.

Theorem 2 [3] *Let G be a graph of order n, and for $2 \leq k \leq n/4$, let $e_1, \dots, e_k$ be any k independent edges in G. Suppose that*

$$\delta(G) \geq \frac{n}{2} + k - 1.$$

Then there exist k disjoint cycles $C_1, \dots, C_k$ such that $e_i \in E(C_i)$ and $3 \leq |V(C_i)| \leq 4$ for all $1 \leq i \leq k$. Furthermore, G contains k disjoint cycles $C'_1, \dots, C'_k$ such that $e_i \in E(C'_i)$ for all $1 \leq i \leq k$ and $V(G) = \bigcup_{i=1}^{k} V(C'_i)$.

Note that when $k = 1$, a packing result in Theorem 2 holds under the same minimum degree condition. (See [3] for more details).

In this paper, we extend Theorems 1 and 2 to disjoint chorded (doubly chorded) cycles for a graph of sufficiently large order with respect to k. In addition, we show when k disjoint chorded cycles containing specified edges can be extended to span the entire vertex set of a graph. We also consider the question of when k independent edges can be chords for k disjoint cycles, one per cycle.

2 Results

First, we extend Theorem 1 to k disjoint chorded cycles for a graph of sufficiently large order with respect to k. A vertex of a graph G with n vertices is *r-pancyclic* if it is contained in an l-cycle for every $r \leq l \leq n$, and G is *vertex r-pancyclic* if every vertex in G is r-pancyclic. The following theorem will be useful in our proof.

Theorem 3 [11] *Let G be a graph of order $n \geq 4$ with $\sigma_2(G) \geq n$. Then G is vertex 4-pancyclic unless n is even and $G = K_{n/2,n/2}$.*

Theorem 4 *Let G be a graph of order $n \geq 16k - 5$ for an integer $k \geq 1$. If*

$$\delta(G) \geq \frac{n}{2},$$

then for any k distinct vertices $v_1, \dots, v_k$ in G, there exist k disjoint chorded cycles $C_1, \dots, C_k$ such that $v_i \in V(C_i)$ and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k$.

Proof Suppose that $k = 1$, then $n \geq 11$. If n is even and $G = K_{n/2,n/2}$, then we can find a chorded 6-cycle containing v_1. Hence, G is vertex 4-pancyclic by Theorem 3. Now consider a 5-cycle $C = w_1, w_2, w_3, w_4, w_5, w_1$, where $w_1 = v_1$. If C is a chorded cycle, then the theorem holds for $k = 1$. Thus, we may assume that C is not a chorded cycle, and then $w_1w_3, w_2w_4 \notin E(G)$. We claim that $|N_G(w_1) \cap N_G(w_3)| \geq 2$. Suppose that $|N_G(w_1) \cap N_G(w_3)| \leq 1$. Then $\deg_G(w_1) + \deg_G(w_3) \leq (n-2) + 1 = n - 1 < n$, a contradiction to the degree condition. Thus the claim holds. Similarly,

$|N_G(w_2) \cap N_G(w_4)| \geq 2$. Let x (resp. y) be a common neighbor of w_1 and w_3 (resp. w_2 and w_4) in $V(G) - V(C)$. If $x = y$, then x, w_3, w_4, w_5, w_1, x is a 5-cycle containing $v_1 = w_1$ and xw_4 as a chord. If $x \neq y$, then $x, w_3, w_4, y, w_2, w_1, x$ is a 6-cycle containing $v_1 = w_1$ and w_2w_3 as a chord. This completes the case where $k = 1$. Hence we may assume that $k \geq 2$.

Suppose that the theorem does not hold. Let G be an edge-maximal counterexample. Since a complete graph of order at least $4k$ contains the desired k disjoint chorded cycles, we may assume that G is not complete. Let x and y be nonadjacent vertices in G, and define $G' = G + xy$. Then G' is not a counterexample by the edge-maximality of G. Hence G' contains the desired k disjoint chorded cycles $C_1, \ldots, C_k$. Without loss of generality, we may assume that $xy \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ disjoint chorded cycles $C_1, \ldots, C_{k-1}$ such that $v_i \in V(C_i)$, $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k-1$, and $v_k \notin \cup_{i=1}^{k-1} V(C_i)$. We choose $C_1, \ldots, C_{k-1}$ such that

$$\sum_{i=1}^{k-1} |V(C_i)| \text{ is a minimum.} \tag{1}$$

Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$ and let $H = G - \mathscr{C}$. Then since $k \geq 2$ and $|V(\mathscr{C})| \leq 6(k-1)$,

$$\begin{aligned} |V(H)| &= |V(G)| - |V(\mathscr{C})| \\ &\geq n - 6(k-1) \\ &\geq (16k-5) - 6k + 6 \\ &= 10k + 1 \\ &\geq 21. \end{aligned}$$

Claim 1. *For any* $x \in V(H)$, $\deg_{C_i}(x) \leq 4$ *for all* $1 \leq i \leq k-1$.

Proof Suppose that $\deg_{C_i}(x) \geq 5$ for some $x \in V(H)$ and some $1 \leq i \leq k-1$. We consider the following cases. In each case, we find a chorded cycle C_i' containing the specified vertex either v_i or v_k such that $|V(C_i')| < |V(C_i)|$, contradicting (1).

Case 1 $|V(C_i)| = 5$.
Let $C_i = w_1, w_2, w_3, w_4, w_5, w_1$. Without loss of generality, we may assume that $v_i = w_1$. In this case, x must be adjacent to all of the vertices in C_i. If $x \neq v_k$, then x, w_5, w_1, w_2, x is a 4-cycle containing v_i and xw_1 as a chord. If $x = v_k$, then x, w_2, w_3, w_4, x is a 4-cycle containing v_k and xw_3 as a chord.
Case 2 $|V(C_i)| = 6$.
Let $C_i = w_1, w_2, w_3, w_4, w_5, w_6, w_1$. Without loss of generality, we may assume that x is adjacent to all of the vertices in $\{w_1, w_2, w_3, w_4, w_5\}$. First suppose that $x \neq v_k$. If $v_i \in \{w_1, w_2, w_3, w_4\}$, then x, w_1, w_2, w_3, w_4, x is a 5-cycle containing v_i and xw_2 as a chord. If $v_i \in \{w_5, w_6\}$, then x, w_5, w_6, w_1, w_2, x is a 5-cycle containing v_i and xw_1 as a chord. Now suppose that $x = v_k$. If $v_i \in \{w_1, w_2, w_6\}$, then x, w_3, w_4, w_5, x is a 4-cycle containing v_k and not containing v_i, with xw_4 as a chord. If $v_i = w_3$, then x, w_4, w_5, w_6, w_1, x is a 5-cycle containing v_k and not containing v_i, with xw_5

as a chord. If $v_i \in \{w_4, w_5\}$, then x, w_1, w_2, w_3, x is a 4-cycle containing v_k and not containing v_i, with xw_2 as a chord.

This completes the proof of the claim. □

Now, $\delta(G) \geq n/2 \geq (16k-5)/2 = 8k - 5/2$, and $\deg_{\mathscr{C}}(v_k) \leq 4(k-1)$ by Claim 1. Then $\deg_H(v_k) \geq 8k - 5/2 - 4(k-1) = 4k + 3/2 \geq 19/2$, since $k \geq 2$. Let $U = \{u_1, u_2, u_3\} \subset N_H(v_k)$. For any $u \in V(H)$, $\deg_{\mathscr{C}}(u) \leq 4(k-1)$ by Claim 1. Thus,

$$\deg_H(u) \geq \delta(G) - 4(k-1). \tag{2}$$

Case 1 There exist two distinct vertices $u, u' \in U$ such that $N_{H-v_k}(u) \cap N_{H-v_k}(u') = \emptyset$.

Without loss of generality, we may assume that $u = u_1$ and $u' = u_2$. The vertices u_1 and u_1 may be adjacent in G, and there exists at most one edge among u_1, u_2, and u_3, otherwise there exists a chorded 4-cycle containing v_k in H. Also, $|N_{H-(U\cup\{v_k\})}(u_i) \cap N_H(v_k)| \leq 1$ for all $i \in \{1, 2\}$. By (2),

$$\begin{aligned}
|V(H)| &\geq \deg_H(v_k) + 1 + (\deg_H(u_1) - 3) + (\deg_H(u_2) - 3) \\
&= \deg_H(v_k) + \deg_H(u_1) + \deg_H(u_2) - 5 \\
&\geq 3(\delta(G) - 4(k-1)) - 5 \\
&= 3\delta(G) - 12k + 7.
\end{aligned}$$

Then

$$\begin{aligned}
n &= |V(H)| + |V(\mathscr{C})| \\
&\geq 3\delta(G) - 12k + 7 + 4(k-1) \\
&= 3\delta(G) - 8k + 3 \\
&\geq 3\left(\frac{n}{2}\right) - 8k + 3 \\
\Longrightarrow n &\leq 16k - 6, \text{ a contradiction.}
\end{aligned}$$

Case 2 There exist at least two pairs (u, u') and (u', u'') for pairwise distinct u, u', $u'' \in U$ such that $x \in N_{H-v_k}(u) \cap N_{H-v_k}(u')$, $y \in N_{H-v_k}(u') \cap N_{H-v_k}(u'')$, and $x \neq y$.

Without loss of generality, we may assume that $u = u_1$, $u' = u_2$, and $u'' = u_3$, that is, $x \in N_{H-v_k}(u_1) \cap N_{H-v_k}(u_2)$, $y \in N_{h-v_k}(u_2) \cap N_{H-v_k}(u_3)$, and $x \neq y$. Then $v_k, u_1, x, u_2, y, u_3, v_k$ is a 6-cycle containing v_k and $v_k u_2$ as a chord.

Case 3 The vertices u_1, u_2, and u_3 all have exactly one common neighbor in $H - v_k$. There exist no edges among u_1, u_2, and u_3, otherwise there is a chorded 4-cycle containing v_k in H. Also, $|N_{H-(U\cup\{v_k\})}(u_i) \cap N_H(v_k)| \leq 1$ for all $i \in \{1, 2, 3\}$, similar to Case 1. By (2),

$$\begin{aligned}
|V(H)| &\geq \deg_H(v_k) + 1 + (\deg_H(u_1) - 2) + (\deg_H(u_2) - 3) + (\deg_H(u_3) - 3) \\
&= \deg_H(v_k) + \deg_H(u_1) + \deg_H(u_2) + \deg_H(u_3) - 7 \\
&\geq 4(\delta(G) - 4(k-1)) - 7 \\
&= 4\delta(G) - 16k + 9.
\end{aligned}$$

Then

$$\begin{aligned} n &= |V(H)| + |V(\mathscr{C})| \\ &\geq 4\delta(G) - 16k + 9 + 4(k-1) = 4\delta(G) - 12k + 5 \\ &\geq 4\left(\frac{n}{2}\right) - 12k + 5 = 2n - 12k + 5. \\ \Longrightarrow n &\leq 12k - 5, \text{ a contradiction.} \end{aligned}$$

This completes the proof of the theorem. □

Next, we extend the packing result in Theorem 2 to disjoint doubly chorded cycles for a graph of sufficiently large order with respect to k. In order to prove the extension, we will use the following lemma.

Lemma 1 *Let G be a graph of order $n \geq 18k - 3$ for an integer $k \geq 1$ with*

$$\delta(G) \geq \frac{n}{2} + k - 1.$$

Suppose that $D \subseteq V(G)$ with $3 \leq |D| \leq 6k - 1$. Then for any $X \subseteq D$ with $|X| = 3$, some pair of vertices in X have a common neighbor in $H = G - D$.

Proof Let $X = \{u_1, u_2, u_3\}$. Suppose that no two vertices in X have a common neighbor in H. Then

$$\begin{aligned} 3\left(\frac{n}{2} + k - 1 - (|D| - 1)\right) &\leq 3(\delta(G) - (|D| - 1|)) \\ &\leq \deg_H(u_1, u_2, u_3) \\ &\leq n - |D|. \\ \Longrightarrow n &\leq 4|D| - 6k \leq 4(6k-1) - 6k \\ &= 18k - 4, \text{ a contradiction.} \end{aligned}$$

□

Theorem 5 *Let G be a graph of order $n \geq 18k - 3$ for an integer $k \geq 1$. If*

$$\delta(G) \geq \frac{n}{2} + k - 1,$$

then for any k independent edges $e_1, \ldots, e_k$ in G, there exist k disjoint doubly chorded cycles $D_1, \ldots, D_k$ such that e_i is a cycle-edge of D_i and $4 \leq |V(D_i)| \leq 6$ for all $1 \leq i \leq k$.

Remark The minimum degree condition is sharp in the following sense. Let $H = K_{n/2-k+1} + K_{2k-2} + K_{n/2-k+1}$, ($n$ is even). Consider the graph G obtained from H by adding an edge e between two $K_{n/2-k+1}$'s. Take $k-1$ independent edges in K_{2k-2} and e as the specified k independent edges. Then $\delta(G) = (n/2-k+1-1)+(2k-2) = n/2 + k - 2$. However, there is no doubly chorded cycle containing e as a cycle-edge without the vertices in K_{2k-2}.

Proof By Theorem 2, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $3 \leq |V(C_i)| \leq 4$ and $e_i \in E(C_i)$ for all $1 \leq i \leq k$. Let $\mathscr{C} = \cup_{i=1}^{k} C_i$ and let $H = G - \mathscr{C}$. Since $|V(\mathscr{C})| \leq 4k$, we have $|V(H)| \geq (18k-3) - 4k = 14k - 3$. By applying Lemma 1 to each C_i, we extend each C_i to a doubly chorded cycle. We consider the following cases.

Case 1 C_i is a 3-cycle.
Let $C_i = u_1, u_2, u_3, u_1$ and let $e_i = u_1u_2$. By Lemma 1, some pair of vertices in $\{u_1, u_2, u_3\}$ must have a common neighbor in H, say x_1. We consider the following two pairs: (u_1, u_2) and (u_2, u_3). Note that (u_1, u_3) is equivalent to (u_2, u_3) by symmetry.
Subcase 1.1 (u_1, u_2).
Consider the set $X = \{x_1, u_2, u_3\}$. By Lemma 1, some pair of vertices in X must have a common neighbor in $H - x_1$, say x_2. We consider the following two pairs: (x_1, u_2) and (x_1, u_3). Note that (u_2, u_3) is equivalent to (x_1, u_2) by symmetry.
First suppose that (x_1, u_2). Then, since there are no doubly chorded cycles containing e_i as a cycle-edge, consider the set $X' = \{u_1, x_1, x_2\}$. By Lemma 1, some pair of vertices in X' must have a common neighbor in $H' = H - \{x_1, x_2\}$, say x_3. If $x_3 \in N_{H'}(u_1) \cap N_{H'}(x_1)$, then $u_1, x_3, x_1, x_2, u_2, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_1x_1 and u_2x_1 as chords. If $x_3 \in N_{H'}(x_1) \cap N_{H'}(x_2)$, then $u_1, x_1, x_3, x_2, u_2, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_2x_1 and x_1x_2 as chords. If $x_3 \in N_{H'}(u_1) \cap N_{H'}(x_2)$, then $u_1, x_3, x_2, x_1, u_2, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_1x_1 and u_2x_2 as chords.
Next suppose that (x_1, u_3). Then $u_1, u_2, x_1, x_2, u_3, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_1x_1 and u_2u_3 as chords.
Subcase 1.2 (u_2, u_3).
Consider the set $X = \{u_1, u_3, x_1\}$. By Lemma 1, some pair of vertices in X must have a common neighbor in $H' = H - x_1$, say x_2. If $x_2 \in N_{H'}(u_1) \cap N_{H'}(u_3)$, then $u_1, u_2, x_1, u_3, x_2, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_1u_3 and u_2u_3 as chords. If $x_2 \in N_{H'}(u_1) \cap N_{H'}(x_1)$, then $u_1, u_2, u_3, x_1, x_2, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_1u_3 and u_2x_1 as chords. If $x_2 \in N_{H'}(u_3) \cap N_{H'}(x_1)$, then $u_1, u_2, x_1, x_2, u_3, u_1$ is a 5-cycle containing e_i as a cycle-edge and u_2u_3 and u_3x_1 as chords.

Case 2 C_i is a 4-cycle.
Let $C_i = u_1, u_2, u_3, u_4, u_1$ and without loss of generality, let $e_i = u_1u_2$. If C_i has a chord, then this is Case 1. Hence we may assume that C_i has no chords. Consider the set $X = \{u_2, u_3, u_4\}$. By Lemma 1, some pair of vertices in X must have a common neighbor in H, say x_1. Let $H' = H - x_1$. We consider the following three pairs: (u_2, u_3), (u_3, u_4), and (u_2, u_4).
Subcase 2.1 (u_2, u_3).
Since there are no doubly chorded cycles, consider the set $X' = \{x_1, u_3, u_4\}$. By Lemma 1, some pair of vertices in X' must have a common neighbor in H', say x_2. If $x_2 \in N_{H'}(x_1) \cap N_{H'}(u_3)$, then $u_1, u_2, x_1, x_2, u_3, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_2u_3 and u_3x_1 as chords. If $x_2 \in N_{H'}(u_3) \cap N_{H'}(u_4)$, then $u_1, u_2, x_1, u_3, x_2, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_2u_3 and u_3u_4 as chords. If $x_2 \in N_{H'}(x_1) \cap N_{H'}(u_4)$, then $u_1, u_2, u_3, x_1, x_2, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_2x_1 and u_3u_4 as chords.

Subcase 2.2 (u_3, u_4).
Since there are no doubly chorded cycles, consider the set $X' = \{u_2, u_3, x_1\}$. By Lemma 1, some pair of vertices in X' must have a common neighbor in H', say x_2. If $x_2 \in N_{H'}(u_2) \cap N_{H'}(u_3)$, then $u_1, u_2, x_2, u_3, x_1, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_2u_3 and u_3u_4 as chords. If $x_2 \in N_{H'}(u_3) \cap N_{H'}(x_1)$, then $u_1, u_2, u_3, x_2, x_1, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_3u_4 and u_3x_1 as chords. If $x_2 \in N_{H'}(u_2) \cap N_{H'}(x_1)$, then $u_1, u_2, x_2, x_1, u_3, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_2u_3 and u_4x_1 as chords.
Subcase 2.3 (u_2, u_4).
Since there are no doubly chorded cycles, consider the set $X' = \{u_2, u_3, x_1\}$. By Lemma 1, some pair of vertices in X' must have a common neighbor in H', say x_2. If $x_2 \in N_{H'}(u_2) \cap N_{H'}(u_3)$ or $x_2 \in N_{H'}(u_2) \cap N_{H'}(x_1)$, then we have a doubly chorded 6-cycle containing e_i as a cycle-edge by the arguments used in Subcase 2.1. If $x_2 \in N_{H'}(u_3) \cap N_{H'}(x_1)$, then $u_1, u_2, x_1, x_2, u_3, u_4, u_1$ is a 6-cycle containing e_i as a cycle-edge and u_2u_3 and u_4x_1 as chords.
By applying Lemma 1 repeatedly to each C_i, we have the desired system of disjoint doubly chorded cycles. This completes the proof of the theorem. □
Next, we extend Theorem 5 to disjoint doubly chorded cycles containing specified paths. Given k independent paths $P_1, \ldots, P_k$, where P_i has order $r_i \geq 2$ for all $1 \leq i \leq k$, let $r = \sum_{i=1}^{k} r_i$. Then the number of interior vertices in the path system is $r - 2k$.

Theorem 6 *Let G be a graph of order $n \geq 16k + r - 3$ for an integer $k \geq 1$ with*

$$\delta(G) \geq \frac{n}{2} + r - k - 1,$$

and let $P_1, \ldots, P_k$ be any k independent paths in G. Then there exist k disjoint doubly chorded cycles $D_1, \ldots, D_k$ such that D_i contains P_i as a path along the cycle and $r_i + 2 \leq |V(D_i)| \leq r_i + 4$ for all $1 \leq i \leq k$.

Proof Consider the graph G' obtained by replacing each path P_i in G by an edge e_i. Then

$$|V(G')| = n - (r - 2k) \geq (16k + r - 3) - r + 2k = 18k - 3,$$

and

$$\delta(G') \geq \delta(G) - (r - 2k) \geq \left(\frac{n}{2} + r - k - 1\right) - r + 2k = \frac{n}{2} + k - 1.$$

Then by Theorem 5, there exist k disjoint doubly chorded cycles $D_1, \ldots, D_k$ in G' such that e_i is a cycle-edge in D_i and $4 \leq |V(D_i)| \leq 6$ for all $1 \leq i \leq k$. Now replace the edge e_i by the path P_i for all $1 \leq i \leq k$, that is, insert the interior vertices of each P_i back into G' to form the original graph G. For all $1 \leq i \leq k$, let D_i' be the doubly chorded cycle obtained from D_i by the addition of the interior vertices of P_i. Then D_i' contains P_i as a path on the cycle, and $r_i + 2 = 4 + (r_i - 2) \leq |V(D_i')| \leq 6 + (r_i - 2) = r_i + 4$ for all $1 \leq i \leq k$. □

Next, we prove the following theorem which shows the existence of disjoint chorded cycles containing specified edges as cycle-edges and spanning the entire vertex set of a graph. In order to prove the theorem, we need the following lemma.

Lemma 2 [3] *Let* $P = u_1, u_2, \ldots, u_q$ *be a path, and* A, B *be subsets of* $V(P)$ *such that* $|A| + |B| > q$. *Then there exists a number* h *with* $1 \leq h < q$ *such that either* $u_h \in A$ *and* $u_{h+1} \in B$ *or* $u_h \in B$ *and* $u_{h+1} \in A$, *unless* q *is odd and* $A = B = \{u_1, u_3, \ldots, u_q\}$.

Theorem 7 *Let* G *a graph of order* $n \geq 4k$ *for an integer* $k \geq 1$ *with*

$$\delta(G) \geq \frac{n+k}{2},$$

and let $e_1, \ldots, e_k$ *be any* k *independent edges in* G. *Suppose that* G *contains* k *disjoint chorded cycles* $C_1, \ldots, C_k$ *such that* e_i *is a cycle-edge of* C_i *for all* $1 \leq i \leq k$. *Then there exist* k *disjoint chorded cycles* $C'_1, \ldots, C'_k$ *such that* e_i *is a cycle-edge of* C'_i *for all* $1 \leq i \leq k$ *and* $V(G) = \cup_{i=1}^{k} V(C'_i)$.

Remark Since a chorded cycle must have at least four vertices, $n \geq 4k$ is necessary. The minimum degree condition is also sharp in the following sense. Let $G = K_{p+k-1} + \overline{K}_p$, with $n \geq 5k+1$, $p = (n-k+1)/2 \geq 2k+1$, and $n \not\equiv k \pmod 2$. Take k independent edges in K_{p+k-1}. Then $\delta(G) = (n+k-1)/2$, and G contains k disjoint chorded cycles containing the specified edges as cycle-edges. However, the chorded cycle system cannot be extended to span $V(G)$. To see this, contract the k specified edges to k vertices. Now if a spanning chorded cycle system existed, then we could replace each contracted vertex by its original specified edge to give the desired spanning chorded cycle system. However, the contracted K_{p+k-1} graph has $(n+k-1)/2 - k = (n-k-1)/2$ vertices, while the $\overline{K}_p$ graph has $(n-k+1)/2$ vertices. Hence, the disjoint chorded cycles cannot span $V(G)$.

Proof We choose k disjoint chorded cycles $C_1, \ldots, C_k$ such that e_i is a cycle-edge of C_i for all $1 \leq i \leq k$, and

$$\sum_{i=1}^{k} |V(C_i)| \quad \text{is as large as possible.} \tag{3}$$

For all $1 \leq i \leq k$, let $P_i = C_i - e_i$ and let $\mathscr{P} = \cup_{i=1}^{k} P_i$. Note that $|V(P_i)| = |V(C_i)| \geq 4$ since C_i is a chorded cycle for all $1 \leq i \leq k$. Let $W = \cup_{i=1}^{k} V(P_i)$. We may assume that $V(G) \neq W$, otherwise the theorem holds.

Claim 1. *For any* $x \in V(G) - W$, $\deg_{P_i}(x) \leq (|V(P_i)| + 1)/2$ *for all* $1 \leq i \leq k$.

Proof Let each C_i be a chorded cycle with a given orientation. Suppose that $u, u^+ \in N_{C_i}(x)$ with $uu^+ \neq e_i$ for some $1 \leq i \leq k$. Then there exists a longer cycle containing e_i as a cycle-edge and uu^+ as a chord, contradicting (3). Hence $|V(P_i)| \geq 2\deg_{P_i}(x) - 1$, and therefore the claim holds. □

Claim 2. *The number of components in $G - W$ is exactly one.*

Proof Suppose that the claim does not hold. Let H_1 and H_2 be distinct components of $G - W$. By Claim 1, for any $x \in V(G) - W$,

$$\deg_{\mathscr{P}}(x) \leq \frac{|V(\mathscr{P})| + k}{2}. \tag{4}$$

Consider $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$. By the minimum degree condition and (4), we have

$$\begin{aligned}|V(G)| &\geq |V(H_1)| + |V(H_2)| + |V(\mathscr{P})| \\ &\geq (\deg_G(x_1) - \deg_{\mathscr{P}}(x_1) + 1) + (\deg_G(x_2) - \deg_{\mathscr{P}}(x_2) + 1) + |V(\mathscr{P})| \\ &\geq 2\left(\frac{n+k}{2} - \frac{|V(\mathscr{P})| + k}{2} + 1\right) + |V(\mathscr{P})| \\ &= n + k - |V(\mathscr{P})| - k + 2 + |V(\mathscr{P})| \\ &= n + 2 > n, \quad \text{a contradiction.}\end{aligned}$$

□

Now, let H be the component of $G - W$.

Claim 3. $|V(H)| \geq 3$.

Proof Suppose that the claim does not hold, that is, $|V(H)| \leq 2$. First, suppose that $|V(H)| = 1$ and let $V(H) = \{x\}$. By Claim 2, $n = |V(\mathscr{P})| + |V(H)| = |V(P_1)| + \cdots + |V(P_k)| + 1$. Therefore, we have

$$\begin{aligned}\deg_{\mathscr{P}}(x) = \deg_G(x) &\geq \frac{n+k}{2} \\ &= \frac{(|V(P_1)| + \cdots + |V(P_k)| + 1) + k}{2} \\ &= \frac{(|V(P_1)| + 1) + \cdots + (|V(P_k)| + 1) + 1}{2}.\end{aligned}$$

Hence by Claim 1, there must exist two neighbors of x that are adjacent on P_i for some $1 \leq i \leq k$. Then there exists a longer chorded cycle containing e_i as a cycle-edge, contradicting (3).

Next, suppose that $|V(H)| = 2$ and let $V(H) = \{x_1, x_2\}$. By Claim 2, $n = |V(\mathscr{P})| + |V(H)| = |V(P_1)| + \cdots + |V(P_k)| + 2$. For $x \in \{x_1, x_2\}$, we have

$$\begin{aligned}\deg_{\mathscr{P}}(x) = \deg_G(x) - 1 \geq \frac{n+k}{2} - 1 &= \frac{n+k-2}{2} \\ &= \frac{(|V(P_1)| + \cdots + |V(P_k)| + 2) + k - 2}{2} \\ &= \frac{|V(P_1)| + \cdots + |V(P_k)| + k}{2} \\ &= \frac{(|V(P_1)| + 1) + \cdots + (|V(P_k)| + 1)}{2}.\end{aligned}$$

Hence, by Claim 1, $\deg_{P_i}(x) = (|V(P_i)| + 1)/2$ for all $1 \le i \le k$. Consider $i = 1$, and let $P_1 = u_1, u_2, \ldots, u_r$ and $e_1 = u_1 u_r$. Then by Lemma 2, r is odd and $N_{P_1}(x_1) = N_{P_1}(x_2) = \{u_1, u_3, \ldots, u_r\}$. Thus $u_1, x_1, x_2, u_3, u_4, \ldots, u_r, u_1$ is a longer cycle containing e_1 as a cycle-edge and u_1x_2 as a chord, contradicting (3). Therefore, the claim holds. □

Claim 4. *H contains a Hamiltonian cycle.*

Proof For all $x \in V(H)$, we have

$$\begin{aligned}\deg_H(x) = \deg_G(x) - \deg_{\mathscr{P}}(x) &\ge \frac{n+k}{2} - \frac{|V(\mathscr{P})| + k}{2} \\ &= \frac{n - |V(\mathscr{P})|}{2} \\ &= \frac{|V(H)|}{2}\end{aligned}$$

Thus the claim holds by Dirac's theorem [2]. □

Claim 5. *If* $|V(H)| \ge 4$, *then* $\deg_H(x) \ge 3$ *for all* $x \in V(H)$.

Proof From the proof of Claim 4, $\deg_H(x) \ge |V(H)|/2$, for all $x \in V(H)$. If $|V(H)| \ge 5$, then the claim holds. Hence we only need to consider the case where $|V(H)| = 4$. In this case, we may assume that there exist two distinct vertices $x_1, x_2 \in V(H)$ such that $\deg_H(x_1) = \deg_H(x_2) = 2$, otherwise the claim holds. By Claim 2, $n = |V(\mathscr{P})| + |V(H)| = |V(P_1)| + \cdots + |V(P_k)| + 4$. Then for all $x \in \{x_1, x_2\}$,

$$\begin{aligned}\deg_{\mathscr{P}}(x) &= \deg_G(x) - \deg_H(x) \\ &\ge \frac{n+k}{2} - 2 = \frac{n+k-4}{2} \\ &= \frac{(|V(P_1)| + \cdots + |V(P_k)| + 4) + k - 4}{2} \\ &= \frac{|V(P_1)| + \cdots + |V(P_k)| + k}{2} \\ &= \frac{(|V(P_1)| + 1) + \cdots + (|V(P_k)| + 1)}{2}.\end{aligned}$$

By Claim 1, equality holds in the above inequality. Consider $i = 1$ and let $P_1 = u_1, u_2, \ldots, u_r$ and $e_1 = u_1u_r$. Let $x_1P^*x_2$ be a path from x_1 to x_2 in H. Then $u_1, x_1P^*x_2, u_3, u_4, \ldots, u_r, u_1$ is a longer cycle containing e_1 as a cycle-edge and u_1x_2 as a chord, contradicting (3). □

Since $\delta(G) \ge (n+k)/2$, G is $(k+1)$-connected. Then $|N_{\mathscr{P}}(H)| \ge k+1$.

Claim 6. *There exist two independent edges between H and* P_i *for some* $1 \le i \le k$.

Proof We consider the following two cases. Note that $|V(H)| \ge 3$ by Claim 3.
Case 1 $|V(H)| \ge k+1$.

Since G is $(k+1)$-connected, we have $|N_H(\mathscr{P})| \geq k+1$. Then we may assume that $|N_H(\mathscr{P})| = k+1$. If there are $k+1$ independent edges between H and $\mathscr{P}$, then the claim holds. Hence we may assume that there are not $k+1$ such edges. By Hall's Theorem [10], there exists some $S \subseteq N_H(\mathscr{P})$ such that $|N_{\mathscr{P}}(S)| < |S|$. Then $(N_H(\mathscr{P}) - S) \cup N_{\mathscr{P}}(S)$ is a vertex-cut of cardinality at most k, a contradiction of the connectivity of G.
Case 2 $3 \leq |V(H)| \leq k$.
Let $V(H) = \{x_1, x_2, \ldots, x_t\}$ for $3 \leq t \leq k$. Since G is $(k+1)$-connected, $N_H(\mathscr{P}) = V(H)$. Now, we may assume that $|N_{\mathscr{P}}(H)| = k+1$. Then there exists some $x \in V(H)$ such that $|N_{P_i}(x)| \geq 2$ for some $1 \leq i \leq k$, say $x = x_1$ and $i = 1$, otherwise the claim holds. If $N_{P_1}(x_i) \neq \emptyset$ for some $2 \leq i \leq t$, then the claim also holds. Hence $N_{P_1}(x_i) = \emptyset$ for all $2 \leq i \leq t$, and then $N_{\mathscr{P}-P_1}(x_i) \neq \emptyset$ for all $2 \leq i \leq t$. Since $|\cup_{i=2}^{t} N_{\mathscr{P}-P_1}(x_i)| \leq k-1$, $\{x_1\} \cup (\cup_{i=2}^{t} N_{\mathscr{P}-P_1}(x_i))$ is a vertex-cut of cardinality at most k, a contradiction of the connectivity of G. □

By Claim 6, without loss of generality, we may assume that there are two independent edges between H and P_1. Let $P_1 = u_1, u_2, \ldots, u_r$ and $e_1 = u_1u_r$. In addition, let $xu_s, x'u_t \in E(G)$ for distinct $x, x' \in V(H)$ and $s < t$. We choose u_s, u_t so that $t - s$ is minimized. Then $t - s \geq 2$ by (3). Let $P = u_1, u_2, \ldots, u_s$, $Q = u_t, u_{t+1}, \ldots, u_r$, and $W' = W - \{u_i | s < i < t\}$. Now, we have

$$\deg_G(x, u_{s+1}) \geq 2\delta(G) \geq n + k. \tag{5}$$

By the minimality of $t - s$, $N_{G-W'}(x) \subseteq V(H) - \{x\}$ and $N_{G-W'}(u_{s+1}) \subseteq V(G) - W' - V(H) - \{u_{s+1}\}$. Then it follows from (5) that

$$\deg_{W'}(x, u_{s+1}) \geq (n + k) - (n - |W'| - 2) = |W'| + k + 2.$$

Similarly, $\deg_{W'}(x', u_{t-1}) \geq |W'| + k + 2$. Combining these two inequalities, we obtain

$$\deg_{W'}(x, x', u_{s+1}, u_{t-1}) \geq 2(|W'| + k + 2). \tag{6}$$

On the other hand, by Claim 1, it follows that for $R \in \{P, Q, P_2, \ldots, P_k\}$,

$$\deg_R(x, x') \leq 2\left(\frac{|V(R)| + 1}{2}\right) = |V(R)| + 1. \tag{7}$$

Now, let xP^*x' be a path from x to x' in H. If $u_{s+1}u_h, u_{t-1}u_{h+1} \in E(G)$ for some $1 \leq h < s$, then $u_1, \ldots, u_h, u_{s+1}, \ldots, u_{t-1}, u_{h+1}, \ldots, u_s, xP^*x', u_t, \ldots, u_r, u_1$ is a longer cycle containing e_1 as a cycle-edge and u_hu_{h+1} as a chord, contradicting (3). If $u_{s+1}u_{h+1}, u_{t-1}u_h \in E(G)$ for some $1 \leq h < s$, then again there exists a longer chorded cycle containing e_1 as a cycle-edge, contradicting (3). Thus, by Lemma 2, $\deg_P(u_{s+1}, u_{t-1}) \leq |V(P)| + 1$ and similarly, $\deg_Q(u_{s+1}, u_{t-1}) \leq |V(Q)| + 1$.
Next, let $P_i = u'_1, \ldots, u'_{r'}$ with $e_i = u'_1u'_{r'}$ for all $2 \leq i \leq k$. By Claim 4, H contains a Hamiltonian cycle, say C, with a given orientation. If $|V(H)| \geq 4$, then a chord

of C is incident to x, by Claim 5. Let xy (possibly $y = x'$) be such a chord. Then $y \in V(x\overrightarrow{C}x')$ or $y \in V(x'\overrightarrow{C}x)$. Without loss of generality, we may assume that $y \in V(x\overrightarrow{C}x')$. If $|V(H)| = 3$, without loss of generality, we may assume that $x\overrightarrow{C}x'$ is a path of length two in C. Then note that $xx' \in E(C)$.
If $u_{s+1}u'_h, u_{t-1}u'_{h+1} \in E(G)$ for some $1 \le h < r'$, then there exist two disjoint chorded cycles with specified edges as cycle-edges as follows: $u_1, \dots, u_s, x\overrightarrow{C}x'$, $u_t, \dots, u_r, u_1$ is a chorded cycle containing e_1 as a cycle-edge, and $u'_1, \dots, u'_h$, $u_{s+1}, \dots, u_{t-1}, u'_{h+1}, \dots, u'_{r'}, u'_1$ is a cycle containing e_i as a cycle-edge and $u'_hu'_{h+1}$ as a chord, contradicting (3). If $u_{s+1}u'_{h+1}, u_{t-1}u'_h \in E(G)$ for some $1 \le h < r'$, then again there exist two disjoint chorded cycles with specified edges as cycle-edges, contradicting (3).
By the above observations and Lemma 2, for all $2 \le i \le k$, we obtain

$$\deg_{P_i}(u_{s+1}, u_{t-1}) \le |V(P_i)| + 1.$$

Consequently, for $R \in \{P, Q, P_2, \dots, P_k\}$,

$$\deg_R(u_{s+1}, u_{t-1}) \le |V(R)| + 1. \tag{8}$$

It follows from (7) and (8) that

$$\sum_{R\in\{P,Q,P_2,\dots,P_k\}} \deg_R(x, x', u_{s+1}, u_{t-1}) \le 2(|W'| + k + 1). \tag{9}$$

Since

$$\deg_{W'}(x, x', u_{s+1}, u_{t-1}) = \sum_{R\in\{P,Q,P_2,\dots,P_k\}} \deg_R(x, x', u_{s+1}, u_{t-1}),$$

by (6) and (9),

$$\begin{aligned} 2(|W'| + k + 2) &\le 2(|W'| + k + 1) \\ |W'| + k + 2 &\le |W'| + k + 1, \text{ a contradiction.} \end{aligned}$$

This completes the proof of the theorem. □
By Theorem 7, we have the following corollaries.

Corollary 8 *Let G be a graph of order $n \ge 4k$ for an integer $k \ge 1$ with $\delta(G) \ge (n+k)/2$. Suppose that G contains k disjoint chorded cycles $C_1, \dots, C_k$. Then there exists k disjoint chorded cycles $C'_1, \dots, C'_k$ in G such that $V(G) = \cup_{i=1}^k V(C'_i)$.*

Corollary 9 *Let G be a graph of order $n \ge 4k$ for an integer $k \ge 1$ with $\delta(G) \ge (n+k)/2$. Let $v_1, \dots, v_k$ be any k distinct vertices in G. Suppose that G contains k disjoint chorded cycles $C_1, \dots, C_k$ such that $v_i \in V(C_i)$ for all $1 \le i \le k$. Then there exist k disjoint chorded cycles $C'_1, \dots, C'_k$ such that $v_i \in V(C'_i)$ for all $1 \le i \le k$, and $V(G) = \cup_{i=1}^k V(C'_i)$.*

Proof Let $e_i = v_i w_i$ be a cycle-edge of C_i, for all $1 \leq i \leq k$. We apply Theorem 7 to extend the chorded cycle system, keeping e_i as a cycle-edge. Hence the corollary holds. □

By Theorems 5 and 7, we also have the following theorem, similar to Theorem 2.

Theorem 10 *Let G be a graph of order $n \geq 18k - 3$ for an integer $k \geq 2$, and let $e_1, e_2, \ldots, e_k$ be any k independent edges in G. Suppose that $\delta(G) \geq n/2 + k - 1$. Then G contains k disjoint doubly chorded cycles $D_1, D_2, \ldots, D_k$ such that e_i is a cycle-edge of D_i and $4 \leq |V(D_i)| \leq 6$ for all $1 \leq i \leq k$. Furthermore, there exist k disjoint chorded cycles $C_1, C_2, \ldots, C_k$ such that e_i is a cycle-edge of C_i for all $1 \leq i \leq k$, and $V(G) = \cup_{i=1}^{k} V(C_i)$.*

Finally, we prove the following theorem which shows the existence of disjoint chorded cycles containing specified edges as chords.

Theorem 11 *Let G be a graph of order $n \geq 6k + 1$ for an integer $k \geq 2$. If*

$$\sigma_2(G) \geq n + 3k - 2 \text{ and } \delta(G) \geq 6k - 3,$$

then for any k independent edges $e_1, e_2, \ldots, e_k$ in G, there exist k disjoint cycles $C_1, C_2, \ldots, C_k$ such that e_i is a chord of C_i and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k$.

Here, we make the following conjecture.

Conjecture 1 *Let G be a graph of order $n \geq 6k$ for an integer $k \geq 1$. If*

$$\sigma_2(G) \geq n + 3k - 2,$$

then for any k independent edges $e_1, \ldots, e_k$, there exist k disjoint cycles $C_1, \ldots, C_k$ such that e_i is a chord of C_i and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k$.

Remark The degree-sum condition is sharp in the following sense. Let $H = K_{2k} + K_{2k-1} + K_{n-4k+1}$. Consider k independent edges $e_1, e_2, \ldots, e_k$ in K_{2k}, and say $e_i = x_i y_i$ for all $1 \leq i \leq k$. Now, consider a graph G obtained from H where, for all $1 \leq i \leq k$, y_i is adjacent to every vertex in K_{n-4k+1}. Since $x_i z \notin E(G)$ for any $z \in V(K_{n-4k+1})$, we have

$$\begin{aligned}\sigma_2(G) &= \deg_G(x_i) + \deg_G(z)\\ &= (2k - 1 + 2k - 1) + (n - 4k + 2k - 1 + k)\\ &= n + 3k - 3.\end{aligned}$$

For any e_i to be a chord of a cycle requires at least two neighbors in K_{2k-1}. Hence, there are not k disjoint chorded cycles in G containing the specified edges as chords. Before proving Theorem 11, we first prove the following lemma and the following theorem, which is the extension of Theorem 11 to the case where $k = 1$.

Lemma 3 *Let G be a graph of order $n \geq 4$ with $\sigma_2(G) \geq n$. Then any edge $e \in E(G)$ lies either on a 3-cycle or a 4-cycle.*

Proof Let $e = uu' \in E(G)$. If e lies on a 3-cycle, then the lemma holds. Hence, we may assume that e does not exist on any 3-cycle. Since $\sigma_2(G) \geq n$, G is 2-connected. Without loss of generality, we may assume that $ux \in E(G)$ for $x \in V(G) - \{u, u'\}$. Then $u'x \notin E(G)$, otherwise e lies on a 3-cycle. By the degree-sum condition, $\deg_G(u') + \deg_G(x) \geq n$ and therefore $|N_G(u') \cap N_G(x)| \geq 2$. Let $y \in N_G(u') \cap N_G(x)$ for $y \in V(G) - \{u, u', x\}$. Then e lies on the 4-cycle u, x, y, u', u. □

Theorem 12 *Let G be a graph of order $n \geq 6$ with $\sigma_2(G) \geq n + 1$. Then for any $e \in E(G)$, there exists a cycle C such that e is a chord of C and $4 \leq |V(C)| \leq 6$.*

Remark In Theorem 12, 6-cycles are necessary. Consider the graph G of order n which is obtained from K_{n-2} and an edge $e = xy$ such that x is adjacent to exactly two vertices in K_{n-2} and y is adjacent to every vertex in $V(G) - N_G(x)$. Then for $z \in V(G) - (N_G(x) \cup \{x\})$, $\sigma_2(G) = \deg_G(x) + \deg_G(z) = 3 + (n - 2) = n + 1$, and the shortest cycle to have e as a chord in G is a 6-cycle.

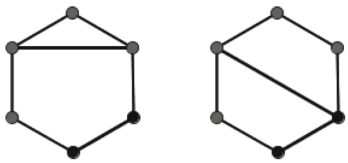
Proof Let e be any edge in the graph G. By Lemma 3, e lies on either a 3-cycle or a 4-cycle. Note that the degree-sum condition implies that G is 3-connected and $\delta(G) \geq 3$. We consider the following two cases.

Case 1 The edge e lies on a 3-cycle, say $C = u_1, u_2, u_3, u_1$.
Let $e = u_1u_2$. Letting $R = G - \{u_1, u_2, u_3\}$, we know u_1, u_2, and u_3 must each have at least one neighbor in R, say u_1', u_2', and u_3' respectively. If $u_1' = u_2'$ then $u_1', u_2, u_3, u_1, u_1'$ is a 4-cycle containing e as a chord. Next suppose that $u_1' \neq u_2'$. If $u_1'u_2' \in E(G)$, then $u_1', u_2', u_2, u_3, u_1, u_1'$ is a 5-cycle containing e as a chord. Now if $u_1'u_2' \notin E(G)$, then u_1' and u_2' must share some neighbor $w \in R$. Then $u_1', w, u_2', u_2, u_3, u_1, u_1'$ is a 6-cycle containing e as a chord.

Case 2 The edge e lies on a 4-cycle, $C = u_1, u_2, u_3, u_4, u_1$.
Let $e = u_1u_2$ and note that C does not have a chord, otherwise e lies on a 3-cycle and we may refer back to Case 1. From the degree condition, the nonadjacent vertices u_2 and u_4 have at least three common neighbors, therefore there exists a vertex $w \neq u_1$ or u_3 that is adjacent to both u_2 and u_4. Similarly, there is a vertex $w' \neq u_2$ or u_4 that is adjacent to both u_1 and u_3. If $w = w'$ then w, u_2, u_3, u_4, u_1, w is a 5-cycle containing e as a chord. Otherwise, $w, u_4, u_1, w', u_3, u_2, w$ is a 6-cycle containing e as a chord.
This completes the proof of the theorem. □

Proof (of Theorem 11) Suppose that the theorem does not hold. Let G be an edge-maximal counterexample. Since a complete graph of order at least $4k$ contains the desired k disjoint chorded cycles, we may assume that G is not complete. Let $uv \notin E(G)$ for distinct $u, v \in V(G)$, and define $G' = G + uv$. Then G' is not a counterexample by the edge-maximality of G, and thus G' contains k disjoint chorded cycles $C_1, \ldots, C_k$ such that e_i is a chord of C_i and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k$. Without loss of generality, we may assume that $uv \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ disjoint cycles $C_1, C_2, \ldots, C_{k-1}$ such that e_i is a chord of C_i and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k - 1$. Note that there exist two distinct types of chorded 6-cycles

Fig. 1 The two types of chorded 6-cycles



which are shown in Fig. 1. Let $C_i = u_1, u_2, \ldots, u_p, u_1$ for $1 \le i \le k-1$ and $4 \le p \le 6$, and let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$. Note that in $G - \mathscr{C}$, there exists a 3, 4, or 5-cycle, say C, containing e_k as an edge with an adjacency to a vertex in $R = G - \mathscr{C} - C$ from at least one end-vertex of e_k. Let $C = w_1, w_2, \ldots, w_q, w_1$ for $3 \le q \le 5$, and let $e_k = w_1 w_2$. □

We choose $\mathscr{C}$ such that

1. $\sum_{i=1}^{k-1} |V(C_i)|$ is a minimum.
2. Subject to (1), $|V(C)|$ is a minimum.
3. Subject to (1) and (2), the number of Type II chorded 6-cycles is a maximum.

Without loss of generality, we may assume that $w_1 x \in E(G)$ for $x \in V(R)$. By condition (2), $|N_C(w_2)| = 2$. Since $\delta(G) \ge 6k-3$ and $|N_{\mathscr{C}}(w_2)| \le 6(k-1)$, we have $|N_{G-\mathscr{C}}(w_2)| \ge 6k - 3 - 6(k-1) = 3$. Hence $N_R(w_2) \ne \emptyset$, and let $y \in N_R(w_2)$. If $y = x$, then $G - \mathscr{C}$ contains a cycle C' such that e_k is a chord of C' and $4 \le |V(C')| \le 6$, a contradiction. Hence we may assume that $y \ne x$. Note that $|V(R)| = |V(G - \mathscr{C} - C)| \ge 6k + 1 - 6(k-1) - 5 = 2$. For the nonadjacent pairs of vertices (w_1, y) and (w_2, x), we first consider the adjacencies from these pairs to the cycles in $\mathscr{C}$.

Claim 1. *Let* $1 \le i \le k-1$. *Then the following statements hold.*

(i) $\deg_{C_i}(w_1, w_2, x, y) \le 14$ *for any chorded* 4-*cycle* C_i,
(ii) $\deg_{C_i}(w_1, w_2, x, y) \le 16$ *for any chorded* 5-*cycle* C_i, *and*
(iii) $\deg_{C_i}(w_1, w_2, x, y) \le 18$ *for any chorded* 6-*cycle* C_i.

Proof For $1 \le i \le k-1$, let $M = N_{C_i}(w_1) \cap N_{C_i}(w_2)$ and $Z = V(C_i) - V(e_i)$. Note that if C_i is a chorded p-cycle for $p \in \{4, 5\}$, then without loss of generality, we may assume that $e_i = u_1 u_3$ and if C_i is a chorded 6-cycle, then we may assume that either $e_i = u_1 u_3$ (Type I) or $e_i = u_1 u_4$ (Type II). We consider the following cases. □

Case 1 C is a 3-cycle.
Subcase 1.1 C_i is a chorded 4-cycle.
Suppose that $M \cap Z \ne \emptyset$. By symmetry, we may assume that $u_4 \in M \cap Z$, that is, $w_1 u_4, w_2 u_4 \in E(G)$. Then w_1, u_4, w_2, w_3, w_1 is a 4-cycle containing e_k as a chord. We claim that $|N_{V(e_i)}(v)| \le 1$ for all $v \in \{x, y\}$. If $|N_{V(e_i)}(v)| = 2$ for some $v \in \{x, y\}$, say $v = x$, then u_1, x, u_3, u_2, u_1 is a 4-cycle containing e_i as a chord, and we have the two desired cycles containing the specified edges as chords. Hence the claim holds. Then $\deg_{C_i}(w_1, w_2) \le 8$ and $\deg_{C_i}(x, y) \le 6$. If $M \cap Z = \emptyset$, then $\deg_{C_i}(w_1, w_2) \le 6$ and $\deg_{C_i}(x, y) \le 8$. Therefore, in both cases, we have $\deg_{C_i}(w_1, w_2, x, y) \le 14$.

Subcase 1.2 C_i is a chorded 5-cycle.
If $M \cap Z \neq \emptyset$, then there exists a 4-cycle containing e_k as a chord, contradicting condition (1). Hence $M \cap Z = \emptyset$ and thus, $\deg_{C_i}(w_1, w_2) \leq 7$. If $|N_{V(e_i)}(v)| = 2$ for some $v \in \{x, y\}$, similarly, (1) is contradicted. Hence $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$, and then $\deg_{C_i}(x, y) \leq 8$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 15 < 16$.
Subcase 1.3 C_i is a chorded 6-cycle.
Since $M \cap Z = \emptyset$, $\deg_{C_i}(w_1, w_2) \leq 8$. Also, $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$, otherwise there is a 4-cycle or 5-cycle (depending on the type of chorded 6-cycle, C_i) containing e_i as a chord, contradicting (1). Hence, $\deg_{C_i}(x, y) \leq 10$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 18$.

Case 2 C is a 4-cycle.
Subcase 2.1 C_i is a chorded 4-cycle.
If $\deg_{C_i}(v) \leq 3$ for all $v \in \{x, y\}$ then claim (i) holds. Hence, without loss of generality, we may assume that $\deg_{C_i}(x) = 4$. Then $M \cap Z = \emptyset$, otherwise we have the two desired cycles containing the specified edges as chords. Hence $\deg_{C_i}(w_1, w_2) \leq 6$. Noting that $\deg_{C_i}(x, y) \leq 8$, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 14$.
Subcase 2.2 C_i is a chorded 5-cycle.
If $M \cap \{u_4, u_5\} \neq \emptyset$, then there exists a 5-cycle containing e_k as a chord and e_i is left on a 3-cycle, contradicting (2). Hence $M \cap \{u_4, u_5\} = \emptyset$, and then $\deg_{C_i}(w_1, w_2) \leq 8$. Since $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$, $\deg_{C_i}(x, y) \leq 8$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 16$.
Subcase 2.3 C_i is a chorded 6-cycle.
If $M \cap Z \neq \emptyset$, then there exists a 5-cycle containing e_k as a chord, contradicting (1). Hence $M \cap Z = \emptyset$, thus $\deg_{C_i}(w_1, w_2) \leq 8$. Since $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$, $\deg_{C_i}(x, y) \leq 10$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 18$.

Case 3 C is a 5-cycle.
Subcase 3.1 C_i is a chorded 4-cycle.
By the same arguments as Subcase 1.1, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 14$.
Subcase 3.2 C_i is a chorded 5-cycle.
Since $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$, we have $\deg_{C_i}(x) \leq 4$ and $\deg_{C_i}(y) \leq 4$. If $\deg_{C_i}(x) \leq 3$ and $\deg_{C_i}(y) \leq 3$, then claim (ii) holds. First we may suppose that $\deg_{C_i}(x) = 4$ and $\deg_{C_i}(y) \leq 3$ by symmetry. Let $N_{C_i}(x) = \{u_2, u_3, u_4, u_5\}$. Then x, u_5, u_1, u_2, u_3, x is a 5-cycle containing e_i as a chord. If $u_4 \in M$, then $w_1, u_4, w_2, w_3, w_4, w_5, w_1$ is a 6-cycle containing e_k as a chord, and we have the two desired cycles containing the specified edges as chords. Hence $u_4 \notin M$, thus $\deg_{C_i}(w_1, w_2) \leq 9$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 16$.
Next, suppose that $\deg_{C_i}(x) = \deg_{C_i}(y) = 4$. Under this assumption we must consider two cases. First assume that $N_{V(e_i)}(x) \cap N_{V(e_i)}(y) = \emptyset$. Then either u_4 or u_5 can be replaced by x (resp. y) depending on the adjacencies from x (resp. y) to C_i, and there exists a 5-cycle containing e_i as a chord. If $M \cap \{u_4, u_5\} \neq \emptyset$, then there exists a 6-cycle containing e_k as a chord, and we have the two desired cycles with the specified edges as chords. Hence $M \cap \{u_4, u_5\} = \emptyset$, and $\deg_{C_i}(w_1, w_2) \leq 8$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 16$. Now assume that $N_{V(e_i)}(x) \cap N_{V(e_i)}(y) \neq \emptyset$. Without loss of generality, we may assume that $u_1 \in N_{V(e_i)}(x) \cap N_{V(e_i)}(y)$. Then

x, u_1, u_2, u_3, u_4, x is a 5-cycle containing e_i as a chord. If $u_5 \in M$ then there exists a 6-cycle containing e_k as a chord, and we have the desired two cycles containing the specified edges as chords. If $vu_5 \notin E(G)$ for all $v \in \{w_1, w_2\}$, then $\deg_{C_i}(w_1, w_2) \leq 8$, and $\deg_{C_i}(w_1, w_2, x, y) \leq 16$. Hence, without loss of generality, we may assume that $w_1u_5 \in E(G)$ and $w_2u_5 \notin E(G)$. If $u_4 \in M$, then $w_1, u_5, y, w_2, u_4, w_1$ is a 5-cycle containing e_k as a chord, and e_i is left on a 3-cycle, contradicting (2). Hence, at most one vertex in $\{w_1, w_2\}$ is adjacent to u_4. Then $\deg_{C_i}(w_1, w_2) \leq 8$, and therefore, $\deg_{C_i}(w_1, w_2, x, y) \leq 16$.
Subcase 3.3 C_i is a chorded 6-cycle.
First suppose that C_i is a Type I chorded 6-cycle. If $M \cap \{u_4, u_5, u_6\} \neq \emptyset$, then there exists a 6-cycle containing e_k as a chord, and e_i is left on a 3-cycle, contradicting (2). Hence $M \cap \{u_4, u_5, u_6\} = \emptyset$, thus $\deg_{C_i}(w_1, w_2) \leq 9$. Also, $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$. If $\deg_{C_i}(x, y) \leq 9$, then claim (iii) holds. So we may assume that $\deg_{C_i}(x, y) \geq 10$. Now, we consider two cases. First, suppose that $N_{V(e_i)}(x) \cap N_{V(e_i)}(y) = \emptyset$. Without loss of generality, we may assume that $xu_1 \in E(G)$ and $yu_3 \in E(G)$. Then $u_1, x, u_2, u_3, y, u_6, u_1$ is a 6-cycle containing e_i as a chord. This cycle is a Type II chorded 6-cycle, which contradicts (3). Next, suppose that $N_{V(e_i)}(x) \cap N_{V(e_i)}(y) \neq \emptyset$. Without loss of generality, we may assume that $u_1 \in N_{V(e_i)}(x) \cap N_{V(e_i)}(y)$. Then $u_1, y, u_2, u_3, u_4, x, u_1$ is a Type II 6-cycle containing e_i as a chord, contradicting (3). Next, suppose that C_i is a Type II chorded 6-cycle. If $M \cap \{u_2, u_3, u_5, u_6\} \neq \emptyset$, then there exists a chorded 6-cycle containing e_k as a chord, and e_i is left on a 4-cycle, contradicting (2). Hence $\deg_{C_i}(w_1, w_2) \leq 8$. Also, $|N_{V(e_i)}(v)| \leq 1$ for all $v \in \{x, y\}$. Hence $\deg_{C_i}(x, y) \leq 10$. Therefore, we have $\deg_{C_i}(w_1, w_2, x, y) \leq 18$.
This completes the proof of Claim 1. □
Recall that C is a q-cycle for $3 \leq q \leq 5$ containing e_k as a cycle-edge.

Claim 2. $\deg_R(w_1, w_2, x, y) \leq 2|V(R)| - 2$.

Proof Note that $N_R(w_1) \cap N_R(w_2) = \emptyset$, otherwise $G - \mathscr{C}$ contains a q'-cycle for some $4 \leq q' \leq 6$ (depending on $|V(C)|$) containing e_k as a chord, a contradiction. Hence $|N_R(w_1) \cup N_R(w_2)| = |N_R(w_1)| + |N_R(w_2)|$. Suppose that there exists an edge between $N_R(w_1)$ and $N_R(w_2)$. If C is a 3-cycle (resp. 4-cycle), then $G - \mathscr{C}$ contains a 5-cycle (resp. 6-cycle) containing e_k as a chord, a contradiction. If C is a 5-cycle, then $G - \mathscr{C}$ contains a 4-cycle with e_k as a cycle-edge, contradicting (2). Hence, there are no edges between $N_R(w_1)$ and $N_R(w_2)$, thus $|N_R(x)| \leq |V(R)| - 1 - |N_R(w_2)|$ and $|N_R(y)| \leq |V(R)| - 1 - |N_R(w_1)|$. Therefore, we have

$$\begin{aligned}\deg_R(w_1, w_2, x, y) &\leq |N_R(w_1)| + |N_R(w_2)| + (|V(R)| - 1 - |N_R(w_2)|)\\ &\quad + (|V(R)| - 1 - |N_R(w_1)|)\\ &= 2|V(R)| - 2\end{aligned}$$

This completes the proof of Claim 2. □

Claim 3. $\deg_C(w_1, w_2, x, y) \leq 2q + 2$.

Proof By (2), $\deg_C(w_1) = 2$ and $\deg_C(w_2) = 2$. Since $N_R(w_1) \cap N_R(w_2) = \emptyset$, $\deg_C(x) \leq q - 1$ and $\deg_C(y) \leq q - 1$. Hence, $\deg_C(w_1, w_2, x, y) \leq 2 + 2 + 2(q - 1) = 2q + 2$. □

Suppose that there are r chorded 4-cycles, s chorded 5-cycles, and t chorded 6-cycles in $\mathscr{C}$. Then by Claims 1, 2, and 3, we have

$$\begin{aligned} 2(n+3k-2) \le 2\sigma_2(G) &\le \deg_G(w_1, w_2, x, y) \\ &\le (14r+16s+18t) + 2(|V(R)|-2) + (2q+2) \\ &= 14r+16s+18t+2(n-4r-5s-6t-q)+2q \\ &= 2n+6(r+s+t) \\ &= 2n+6(k-1) \\ \Longrightarrow n+3k-2 &\le n+3k-3, \text{ a contradiction.} \end{aligned}$$

This completes the proof of the theorem. □

Corollary 13 *Let G be a graph of order $n \ge 9k-5$ for an integer $k \ge 2$. If*

$$\delta(G) \ge \frac{n+3k-2}{2},$$

then for any k independent edges $e_1, e_2, \ldots, e_k$, there exist k disjoint cycles $C_1, C_2, \ldots, C_k$ such that e_i is a chord of C_i and $4 \le |V(C_i)| \le 6$ for all $1 \le i \le k$.

Acknowledgments The authors would like to thank the referees for their helpful comments and suggestions.

References

1. Bialostocki, A., Finkel, D., Gyárfás, A.: Disjoint chorded cycles in graphs. Discrete Math. **308**, 5886–5890 (2008)
2. Dirac, G.A.: Some theorems on abstract graphs. Proc. Lond. Math. Soc. **2**, 69–81 (1952)
3. Egawa, Y., Faudree, R.J., Györi, E., Ishigami, Y., Schelp, R., Wang, H.: Vertex-disjoint cycles containing specified edges. Gr. Comb. **16**, 81–92 (2000)
4. Egawa, Y., Enomoto, H., Faudree, R.J., Li, H., Schiermeyer, I.: Two-factors each component of which contains a specified vertex. J. Gr. Theory **43**, 188–198 (2003)
5. Finkel, D.: On the number of independent chorded cycles in a graph. Discrete Math. **308**, 5265–5268 (2008)
6. Gould, R.J.: Graph Theory. Dover Pub. Inc., Mineola (2012)
7. Gould, R.J.: A look at cycles containing specified elements of a graph. Discrete Math. **309**, 6299–6311 (2009)
8. Gould, R.J., Hirohata, K., Horn, P.: Independent cycles and chorded cycles in graphs. J. Comb. **4**, 105–122 (2013)
9. Gould, R.J., Horn, P., Magnant, C.: Multiply chorded cycles. SIAM J. Discrete Math. **28**, 160–172 (2014)
10. Hall, P.: On representatives of subsets. J. Lond. Math. Soc. **10**, 26–30 (1935)
11. Randerath, B., Schiermeyer, I., Tewes, M., Volkmann, L.: Vertex pancyclic graphs. Discrete Appl. Math **120**, 219–237 (2002)