Minimum Degree and Dominating Paths

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Abstract: A *dominating path* in a graph is a path P such that every vertex outside P has a neighbor on P. A result of Broersma from 1988 implies that if G is an n-vertex k-connected graph and $\delta(G) > \frac{n-k}{k+2} - 1$, then G contains a dominating path. We prove the following results. The lengths of dominating paths include all values from the shortest up to at least $\min\{n-1, 2\delta(G)\}$. For $\delta(G) > an$, where a is a constant greater than 1/3, the minimum length of a dominating path is at most logarithmic in n when n is sufficiently large (the base of the logarithm depends on a). The preceding results are sharp. For constant s and $c' < 1$, an s-vertex dominating path is guaranteed by $\delta(G) \geq n - 1 - c'n^{1-1/s}$ when n is sufficiently large, but $\delta(G) \geq n - c(s \ln n)^{1/s} n^{1-1/s}$ (where $c > 1$) does not even guarantee a dominating set of size s. We also obtain minimum-degree conditions for the existence of a spanning tree obtained from a dominating path by giving the same number of leaf neighbors to each vertex. © 2016 Wiley Periodicals, Inc. J. Graph Theory 84: 202–213, 2017

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1. INTRODUCTION

Many results in extremal graph theory study minimum-degree conditions that force the occurrence of various structures. For example, connected graphs have spanning trees, but large minimum degree yields spanning trees with additional properties. The survey paper on spanning trees by Ozeki and Yamashita [12] has an extensive bibliography on conditions for spanning trees of various types.

We seek a spanning tree whose nonleaf vertices form a short path. Forcing of spanning trees with many leaves or few leaves have both been well studied; we sketch the history to set the context.

A *dominating set* in a graph G is a set $S \subseteq V(G)$ such that every vertex outside S has a neighbor in S. A *connected dominating set* is a dominating set that induces a connected subgraph; it is just the set of nonleaf vertices in a spanning tree. Let $\ell(n,k)$ denote the least t such that every n-vertex graph with minimum degree at least k has a spanning tree with at least t leaves. It is known that $\ell(n,k) = \frac{k-2}{k+1}n + c_k$ for $k \leq 6$, with $(c_2, c_3, c_4, c_5) = (2, 2, \frac{8}{5}, 2)$ [10, 11]. When k is large and fixed, Alon and Wormald [1, 2] showed probabilistically the existence of k-regular graphs with no dominating set of size less than $\frac{1+\ln(k+1)}{k+1}n$. Hence the bound $\ell(n,k) \geq (1 - \frac{(1+o(1))\ln k}{k})n$ [4] is asymptotically sharp.

On the other hand, spanning paths are spanning trees with the fewest leaves. Dirac [6] proved that an n-vertex graph with minimum degree at least $(n-1)/2$ has a spanning path, containing a dominating path with $n-2$ vertices.

We combine these streams of research by seeking minimum-degree conditions for dominating paths and by seeking short dominating paths for given minimum degree. The first problem has a prior solution. Broersma [3] proved a difficult result about cycles passing within a fixed distance of every vertex. He stated without proof an analogue for paths, from which he stated the following corollary, where we have changed notation to fit our context.

Corollary 1.1 (Broersma [3]). *Fix $k, l \in \mathbb{N}$, and let G be a k-connected n-vertex graph. If the degree-sum is at least $n - 2k - 1 - (l - 1)k(k + 2)$ for any set S of $k + 2$ vertices such that the distance between any two vertices of S is more than $2l$, then G contains a path P such that every vertex has distance at most l from P.*

Setting $l = 1$ yields a minimum-degree threshold for dominating paths.

Corollary 1.2. *If G is a k-connected n-vertex graph, and $\delta(G) > \frac{n-k}{k+2} - 1$, then G contains a dominating path.*

Furthermore, the threshold is sharp infinitely often.

Example 1.3. For $n \equiv k \bmod (k + 2)$ with $n \geq 3k + 4$, let $t = \frac{n-k}{k+2}$, and begin with $n - k$ vertices in disjoint t-cliques $Q_1, \ldots, Q_{k+2}$. Add k central vertices adjacent to all but one vertex q_i in each Q_i. The resulting graph is k-connected, and each q_i has degree $t - 1$. Every path misses at least one Q_j completely, leaving its vertex q_j undominated. This construction is essentially that of Broersma [3].

For $l > 1$, Broersma's corollary implies that $\delta(G) > \frac{n-k}{k+2} - 1 - (l - 1)k$ yields a path within distance l of every vertex. This threshold is also sharp, by a more general construction analogous to the construction Broersma presented for sharpness of his result on cycles.

Let H be the tree formed by $k + 2$ copies of the path P_l with a common endpoint; let $v_1, \ldots, v_{k+2}$ be the leaves. Form H' by expanding each vertex of H into a k-clique. To reach a total of n vertices, add disjoint cliques $Q_1, \ldots, Q_{k+2}$ of size $t - (l - 1)k$. Make the k vertices of H' that arose by expanding v_i adjacent to all but one vertex q_i in Q_i. The vertex q_i has degree $t - 1 - (l - 1)k$, the graph is k-connected, every path P misses some "branch" of H, and the vertex q_j for that branch has distance $l + 1$ from P.

We begin in Section 2 with a short, self-contained proof of Corollary 1.2 for $k = 1$ (Theorem 2.1). We also give a short proof of a slightly weaker result for $k = 2$: $\delta(G) \geq (n + 1)/4$ is sufficient when G is 2-connected (Theorem 2.2). We also show that in a connected n-vertex graph having a dominating path, the lengths of dominating paths include all values from the shortest up to at least $\min\{n - 1, 2\delta(G)\}$ (Lemma 2.3), and this is sharp.

Next, we consider guaranteeing a short dominating path. When $\delta(G) > a \cdot n + \log_{a/(1-a)} n$ with $a > 1/2$ and n is sufficiently large, some dominating path has at most $\log_{a/(1-a)} n$ vertices (Lemma 3.1); this value is $\log_2 n$ when $a = 2/3$. We use this to prove our main result (Theorem 3.2): when $\delta(G) > an$ with $a > 1/3$ and n is sufficiently large, some dominating path has at most $c_a \log_{1/(1-a)} n$ vertices.

This order of growth is sharp: for fixed $p, \epsilon \in (0, 1)$, in the random graph with edge probability p any set of size $(1 + \epsilon) \log_{1/(1-p)} n$ is a dominating set with probability tending to 1 (as $n \to \infty$), while sets of size $(1 - \epsilon) \log_{1/(1-p)} n$ dominate with probability tending to 0 (Dreyer [8]). This result was strengthened to two-point concentration for the domination number by Wieland and Godbole [13], even when p tends (slowly) to 0.

To guarantee a dominating set or dominating path with a constant number of vertices, the minimum degree must be very high. For a fixed integer s and a fixed constant c greater than 1, we show that when n is sufficiently large there are graphs with minimum degree at least $n - c(s \ln n)^{1/s} n^{1-1/s}$ having no dominating set of size at most s and hence no such dominating path (Theorem 4.2). However, when $\delta(G) \geq n - 1 - c' n^{1-1/s}$ an

s-vertex dominating path is guaranteed (Theorem 4.1; here $c' < 1$, and again n must be sufficiently large).

We also study the distribution of vertices off the dominating path. Flandrin et al. [9] asked for a degree condition for the existence of a path whose first vertex is adjacent to all vertices off the path. Such a path yields a spanning broom, where a *broom* is a tree formed by identifying an endpoint of a path with a center of a star. Chen et al. [5] proved that if G is an n-vertex graph with $n \geq 42$ and $\delta(G) \geq (n-2)/3$, then G has a spanning broom.

A *caterpillar* is a tree whose nonleaf vertices form a path called the *spine*. A dominating path is the spine of a spanning caterpillar. A caterpillar is *balanced* if all spine vertices have the same number of leaf neighbors. It is *nearly balanced* if the numbers of leaf neighbors of spine vertices differ by at most 1. In Section 5, we obtain minimum-degree conditions for balanced or nearly balanced spanning caterpillars. For example, if n is sufficiently large and is divisible by $p+1$, then $\delta(G) \geq (1 - \frac{p}{(p+1)^2})n$ implies that an n-vertex graph G contains a balanced spanning caterpillar with $\frac{n}{p+1}$ spine vertices. The special case $p = 1$ is interesting; when n is large and even, $\delta(G) \geq 3n/4$ guarantees a spanning tree consisting of a path P with $n/2$ vertices and a matching joining $V(P)$ to the remaining vertices.

Our results leave several open questions.

Question 1. *For $a > 1/3$, what is the smallest constant c_a such that for large n, every n-vertex graph G with $\delta(G) \geq a \cdot n$ has a dominating path with at most $c_a \log_{1/(1-a)} n$ vertices?*

Question 2. *For $a > 1/(k+2)$ and n sufficiently large, what is the least s such that every k-connected n-vertex graph G with $\delta(G) \geq a \cdot n$ has a dominating path with at most s vertices?* (Since G is connected, the value is at most logarithmic in n. If $k \in O(\log n)$, then the results [8, 13] on domination in random graphs imply that s cannot be sublogarithmic, but the question becomes more interesting when k grows faster than $\log n$.)

Question 3. *For fixed $s \in \mathbb{N}$ and n sufficiently large, what is the least t such that every n-vertex graph with minimum degree at least t has a dominating path with at most s vertices?* (The value is at least $n - O((s \ln n)^{1/s} n^{1-1/s})$ and at most $n - \Omega(n^{1-1/s})$.)

Question 4. *For $n, p \in \mathbb{N}$, what is the least t such that every n-vertex connected graph G with $\delta(G) \geq t$ has a nearly balanced spanning caterpillar in which each spine vertex has p or $p+1$ leaf neighbors?* (The value is trivially at most $n-1$.)

Question 5. *For even n, what is the least t such that every n-vertex graph G with $\delta(G) \geq t$ has a balanced spanning caterpillar whose spine has $n/2$ vertices?* (The value is at most $3n/4$ and at least $n/2$.) *Does the same degree condition yield a nearly balanced spanning caterpillar with s spine vertices whenever $n/2 \leq s < 2n/3$?*

2. EXISTENCE OF DOMINATING PATHS

Let $V(G)$ and $E(G)$ denote the vertex and edge set of a graph G (we consider only simple graphs). We write $N_G(v)$ for the neighborhood of a vertex v in G and $d_G(v)$ for its degree. For $S, T \subseteq V(G)$, we extend this notation by letting $N_G(T) = \bigcup_{v \in T} N_G(v) - T$

and letting $N_S(T) = N_G(T) \cap S$ and $d_S(T) = |N_S(T)|$. Also $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum of the vertex degrees.

Theorem 2.1. *For $n \geq 2$, every connected n-vertex graph G with $\delta(G) > \frac{n-1}{3} - 1$ has a dominating path, and the inequality is sharp.*

Proof. The sharpness construction given in Example 1.3 was stated for $n \equiv 1 \bmod 3$. To generalize, let Q_i in the construction for $k = 1$ be a clique with $\lfloor (n+2-i)/3 \rfloor$ vertices, for $i \in \{1, 2, 3\}$. The three cliques together then have $n-1$ vertices, and $\delta(G) = \left\lfloor \frac{n-1}{3} \right\rfloor - 1$.

Now suppose that G is an n-vertex connected graph with $\delta(G) \geq n/3 - 1$ that contains no dominating path; note that $n \leq 3t + 3$, where $t = \delta(G)$. Suppose first that G is 2-connected. Dirac [6] proved that G must then have a cycle with at least $\min\{n, 2\delta(G)\}$ vertices. A path with at least $n - t$ vertices is a dominating path, so we may assume $t < n/2$.

When G is 2-connected, we have $t \geq 2$, and G has a cycle C of length at least $2t$. If $V(C)$ is not the vertex set of a dominating path, then some vertex u and its neighbors are not on C. Since G is connected, there is a shortest path P_u to u from $V(C)$. Adding to P_u a path along C at one end and another neighbor of u at the other end (available since $t \geq 2$) yields a path P with at least $2t + 3$ vertices. If P is not a dominating path, then $V(P)$ omits some other vertex and its neighborhood, which requires $n \geq 3k + 4$, a contradiction.

Hence G must have a cut-vertex v. Each component of $G - v$ has at least t vertices, so $G - v$ has at most three components. Since $G - v$ has at most $3t + 2$ vertices, having three components with at least t vertices requires one with exactly t vertices. Such a component of $G - v$ must be a complete graph with all vertices adjacent to v. The other two components have order at most $k + 2$, so a vertex w in such a component H is nonadjacent to at most one other vertex of H if w is also nonadjacent to v. Hence in this case G has a dominating path consisting of v and two vertices each from two largest components of $G - v$.

In the remaining case, $G - v$ has two components. If either has a cut-vertex w, then $G - v - w$ consists of three nearly complete components yielding a dominating path as in the preceding paragraph. If each component of $G - v$ is 2-connected, then each has a cycle that is spanning or has at least $2t - 2$ vertices, since deleting v leaves minimum degree at least $t - 1$. Since each component has at least t vertices, each has at most $2t + 2$ vertices. We obtain a path through v that omits very few vertices and is dominating. □

For 2-connected graphs, another method gives a short proof of almost the optimal threshold. Dirac's Theorem implies also that if $\delta(H) > |V(H)|/2$, then H is *Hamiltonian-connected*, meaning that any two vertices are the endpoints of a spanning path. For a path P from u to v and $R \subseteq V(P)$, let R^+ denote the set of immediate successors of vertices of R along P, and let R^- denote the set of immediate predecessors. Note that $|R^+| = |R^-| = |R|$ when R contains no endpoint of P.

Theorem 2.2. *Every 2-connected n-vertex graph G with $\delta(G) \geq \frac{n+1}{4}$ has a dominating path, and the conclusion fails when $\delta(G) = \frac{n-6}{4}$.*

Proof. The last part of the claim follows from Example 1.3 with $k = 2$.

Let P be a longest path in such a graph G, with vertex set S, first vertex u, and last vertex v. Let $s = |S|$ and $t = \delta(G) \geq 2$. If $s \geq 3t - 1$, then P is a dominating path, since a vertex with no neighbor on P requires $n \geq s + 1 + t \geq 4t > n$, a contradiction. Thus, we may assume $s < 3t - 1$. If P is not a dominating path, then let H be a component of $G - S$ containing a vertex z with no neighbor on P, and let $T = V(H)$.

Let $D = N_S(T)$, $A = N_S(u)$, and $B = N_S(v)$; note that $|A|, |B| \geq t$. Every vertex of T having a neighbor in S is the endpoint of a 3-vertex path in H, since it has a path to z and $d_H(z) \geq 2$. Since P is a longest path, the sets A^-, B^{++}, D and D^+ are thus pairwise disjoint. Hence $s \geq |A| + |B| + 2|D| \geq 2t + 2|D|$.

Since $|T| \leq n - s$ and $\delta(H) \geq t - |D|$, we have $2\delta(H) - |T| \geq 4t - n$. With $t > n/4$, we have $2\delta(H) > |V(H)|$, so H is Hamiltonian-connected. Since G is 2-connected and $|T| \geq 2$, there exist vertices x' and y' in T having respective distinct neighbors x and y in P. Let Q be a spanning x', y'-path in H. Note that $P - \{x, y\}$ consists of three (possibly empty) paths. Combining the two longest with x, y, and Q yields a path with at least $\frac{2}{3}(s-2) + 2 + |T|$ vertices. Since $|V(P)| = s$, we obtain $|T| \leq \frac{s-2}{3} < t - 1$, contradicting $|T| \geq t + 1$. □

In the introduction, we noted that a graph G with a dominating path has dominating paths of all orders from the smallest through $\min\{n, 2\delta(G) + 1\}$. This follows immediately from a standard lemma in the theory of long paths and cycles. A longer path containing the vertices of a dominating path will also dominate. We use this result in Theorem 5.2.

Lemma 2.3. *If P is a path in a connected n-vertex graph G and $|V(P)| < \min\{n, 2\delta(G) + 1\}$, then G has a path with $|V(P)| + 1$ vertices that contains $V(P)$.*

Proof. Let u and v be the first and last vertices of P. If u or v has a neighbor outside P, then P extends by one vertex. Hence we may assume $A, B \subseteq V(P)$, where $A = N(u)$ and $B = N(v)$. If some vertex of A follows a vertex of B, then $V(P)$ is contained in a cycle, and the fact that G is connected yields a path with $|V(P)| + 1$ vertices containing $V(P)$. In the remaining case, A^-, B, and $\{v\}$ are pairwise disjoint subsets of $|V(P)|$, which cannot happen since P has at most $2\delta(G)$ vertices. □

The graph $K_{k,n-k}$ with $k < n/2$ shows that the guarantee in Lemma 2.3 cannot be extended beyond $2\delta(G) + 1$; we have $\delta(K_{k,n-k}) = k$, and longest paths in $K_{k,n-k}$ have $2k + 1$ vertices.

3. SHORT DOMINATING PATHS WHEN $\delta(G) > an$

In light of both Lemma 2.3 and the theme of finding small connected dominating sets, we now seek small dominating paths, which by Lemma 2.3 yields dominating paths with all orders from the smallest up to $\min\{n, 2\delta(G) + 1\}$. We consider $\delta(G) > n/3$ but first develop a tool that applies when the minimum degree is more than half the number of vertices. We will apply it to obtain short dominating paths when the minimum degree is smaller.

Lemma 3.1. *Let G be a connected n-vertex graph with $\delta(G) \geq a \cdot n + \log_{a/(1-a)} n$, where $a > 1/2$. For n sufficiently large, G has a dominating path with at most $\log_{a/(1-a)} n$ vertices, starting from any vertex x_1.*

Proof. We grow a dominating path $\langle x_1, \ldots, x_r \rangle$ from x_1, where $r \leq \log_{a/(1-a)} n$. Having grown $\langle x_1, \ldots, x_j \rangle$, let $T_j = N_G(x_j) - \{x_1, \ldots, x_{j-1}\}$, and let S_j be the set of vertices in G not dominated by $\{x_1, \ldots, x_j\}$. Select x_{j+1} as a vertex of T_j having the most neighbors in S_j.

We claim that x_{j+1} has at least $\frac{2a-1}{a}|S_j|$ neighbors in S_j. Otherwise, in the complement of G each vertex of T_j has at least $\frac{1-a}{a}|S_j|$ neighbors in S_j. Now, some vertex of S_j has degree at least $\frac{1-a}{a}|T_j|$ in the complement. Since $|T_j| \geq a \cdot n + \log_{a/(1-a)} n - (j-1)$, this contradicts $\delta(G) \geq a \cdot n$ when $j \leq \log_{a/(1-a)} n$.

We thus have $|S_{j+1}| \leq \frac{1-a}{a}|S_j|$. Inductively, $|S_{j+1}| \leq (\frac{1-a}{a})^j |S_1|$. Thus S_{r+1} becomes empty for some r with $r \leq \log_{a/(1-a)} n$, at which point we end with an r-vertex dominating path. □

Lemma 3.1 helps us handle one of the cases in our main result, giving an improved upper bound on the minimum length of a dominating path even when the minimum degree is just a bit larger than needed to guarantee having a dominating path.

Theorem 3.2. Let G be a connected n-vertex graph with $\delta(G) \geq a \cdot n$, where $1/3 < a < 1$. There is a constant c_a such that if n is sufficiently large, then G contains a dominating path with at most $c_a \log_{1/(1-a)} n$ vertices.

Proof. We consider two cases, depending on the connectivity $\kappa(G)$. Let $k = \kappa(G)$.

Case 1. $k \leq \log_{1/(1-a)} n$. Let Q be a smallest separating set. If $G - Q$ has at least three components, then one has at most $(n-k)/3$ vertices, with maximum degree less than $(n+2k)/3$, which contradicts $a > \frac{1}{3}$ when n is sufficiently large. Hence $G - Q$ has only two components, H_1 and H_2. Since $a > \frac{1}{3}$, we have $\frac{n}{3} < |V(H_i)| < \frac{2n}{3}$ when n is sufficiently large.

Let $Q = \{v_1, \ldots, v_k\}$. Since Q is a smallest cutset, there is a matching from Q into $V(H_1)$; let it be $\{v_1w_1, \ldots, v_kw_k\}$. Since $a > 1/3$, we have $\delta(H_1) > (\frac{1}{2} + \epsilon)|V(H_1)|$ for some positive ϵ. Hence for sufficiently large n we can (iteratively) find distinct common neighbors $y_1, \ldots, y_{k-1}$ for the pairs $\{w_1, w_2\}, \ldots, \{w_{k-1}, w_k\}$. We thus have a w_1, w_k-path P_1 with vertex set W, where $W = \{w_1, \ldots, w_k\} \cup \{y_1, \ldots, y_{k-1}\}$. This path dominates Q.

Let $H_1' = H_1 - (W - \{w_k\})$; note that $\delta(H_1') \geq \delta(H_1) - 3k$. Let $n_1 = |V(H_1')|$ and $n_2 = |V(H_2)|$. Since $a > \frac{1}{3}$, we can choose a constant a' with $\frac{1}{2} < a' < \frac{3}{2}a$; in particular, let $a' = b - (1-b)(b - \frac{1}{2})$, where $b = \frac{3}{2}a$. Since $n_i < \frac{2}{3}n$, both H_1' and H_2 have minimum degree at least $a'n_i + \log_{a'/(1-a')} n_i$ when n is sufficiently large.

By Lemma 3.1, H_1' has a dominating path P_1' with at most $\log_{a'/(1-a')} n$ vertices, starting from w_k. Moving in the other direction, extend $P_1 \cup P_1'$ back past its beginning at w_1 to visit v_1, then a neighbor u_1 of v_1 in $V(H_2)$, then follow a dominating path P_2 in H_2 starting from u_1. Again Lemma 3.1 guarantees P_2 with at most $\log_{a'/(1-a')} n$ vertices.

The resulting dominating path in G has at most $2\log_{a'/(1-a')} n + 2\log_{1/(1-a)} n$ vertices. Let $\alpha = \log_{1/(1-a)} \frac{a'}{1-a'}$. Since $\log_{a'/(1-a')} n = (1/\alpha)\log_{1/(1-a)} n$, our dominating path has at most $(2 + 2/\alpha)\log_{1/(1-a)} n$ vertices.

Case 2. $k > \log_{1/(1-a)} n$. We first obtain a dominating r-set, where $r = \lceil \log_{1/(1-a)} n \rceil$. Choose any $x_1 \in V(G)$. Having chosen $S_i = \{x_1, \ldots, x_i\}$, let B be the set of vertices not dominated by S_i. We seek $x_{i+1} \in V(G) - S_i$ such that $N_G(x_{i+1}) \cap B > a|B|$. If no such vertex exists, then each vertex of G has at least $(1-a)|B|$ nonneighbors in B (vertices of S_i have no neighbors in B). By averaging, some vertex of B has at least $(1-a)n$ nonneighbors and hence degree less than an, a contradiction. Iterating yields S_r of size r

leaving fewer than $(1-a)^r n$ undominated vertices. Since $r \geq \log_{1/(1-a)} n$, in fact S_r is a dominating set.

Dirac [7] observed that an r-connected graph has a cycle through any r vertices, by Menger's Theorem. Let C be a shortest such cycle in G containing S_r. We claim that at most four vertices not in S_r separate consecutive vertices of S_r on C. Otherwise, let x and z be two vertices of S_r connected along C by a path P with at least five internal vertices but no other vertex of S_r. Choose $y \in V(P)$ with distance at least 3 along P from both x and z.

Since $3\delta(G) > n$, the neighborhoods of x, y, and z cannot be disjoint. Two of them having a common neighbor outside $V(C)$ would contradict the choice of C, yielding a shorter cycle. Hence the three vertices are incident to at least $3an - b - 6$ chords of C, where C has length $n - b$. Let c be the number of vertices of C at the other ends of these chords. The choice of C again implies that none of these vertices lie in P, so $b + 6 + c < n$.

After allowing for one chord to each of c vertices, they receive at least $3an - b - 6 - c$ "excess" chords, at most two at each vertex. If some segment of C joining vertices of S_r receives three excess chords, then it has three vertices each receiving at least two chords, or it has two vertices with one receiving at least two chords and one receiving three chords. In either case, this segment receives crossing chords from two of $\{x, y, z\}$. That yields a shorter cycle without losing any vertices in S_r, contradicting the choice of C.

Hence we seek $3an - b - 6 - c > 2r$. Since $b + 6 + c < n$, it suffices to have $(3a - 1)n > 2r$. Since $3a - 1 > 0$ and $r = \lceil \log_{1/(1-a)} n \rceil$, this holds for sufficiently large n, and hence $|V(C)| \leq 5r$.

In Case 1 or Case 2, we obtain a dominating path with at most $c_a \log_{1/(1-a)} n$ vertices when n is sufficiently large, where $c_a = \max\{2 + 2/\alpha, 5\}$. □

4. DOMINATION WITH CONSTANT SIZE

The minimum degree needed to guarantee dominating paths with constant size is very large. As n grows, this threshold is asymptotic to n. Nevertheless, one can study the lower-order terms to understand how sparse the complement must be.

We begin with a minimum-degree threshold in terms of n and s that when n is sufficiently large guarantees a dominating path with at most s vertices. We then provide a probabilistic construction to show that for fixed s a somewhat smaller value cannot even guarantee a dominating set with at most s vertices, let alone a dominating path. This is analogous to the work of Alon and Wormald [2], who for fixed k and large n gave a probabilistic construction of k-regular n-vertex graphs with no dominating set of size less than $\frac{\ln(k+1)}{k+1} n$. By their result, minimum degree $k - 1$ guaranteeing an s-vertex dominating path requires $s > \frac{\ln k}{k} n$, or $k > \frac{n}{s}(\ln \frac{n}{s} + \ln\ln \frac{n}{s})$. When s is constant, a direct argument gives a stronger result, requiring minimum degree asymptotic to n to guarantee a dominating s–set.

Although the upper and lower bounds are close to optimal, there remains a gap between the sufficient degree and necessary degree, as noted in Question 3. We begin with the sufficient condition, since the approach is analogous to that of Lemma 3.1.

Theorem 4.1. *Fix a positive integer s and a real constant c less than* 1. *For n sufficiently large in terms of s, each connected n-vertex graph G satisfying* $\delta(G) \geq n - 1 - cn^{1-1/s}$ *contains an s-vertex dominating path starting at any vertex.*

Proof. Given a starting vertex x_1, we prove for $1 \leq j \leq s$ that G has a path through vertices $x_1, \ldots, x_j$ that dominate all but at most $cn^{1-j/s}$ vertices in G. When $j = s$, the resulting number of undominated vertices is less than 1, so $\{x_1, \ldots, x_s\}$ is then a dominating set.

For $j = 1$, the claim is immediate from the condition on $\delta(G)$. For $1 < j \leq k$, suppose that $x_1, \ldots, x_{j-1}$ have already been chosen; name this set S. Let B be the set of vertices not dominated by S; we are given $|B| \leq cn^{1-(j-1)/s}$. Let $A = N_G(x_{j-1}) - S$. If some vertex of A dominates all but at most $cn^{1-j/s}$ vertices in B, then we can choose this vertex as x_j.

Otherwise, each vertex of A has more that $cn^{1-j/s}$ nonneighbors in B. By the pigeonhole principle, some vertex $y \in B$ has more than $cn^{1-j/s}|A|/|B|$ nonneighbors in A. We compute

$$\begin{aligned} cn^{1-j/s}\frac{|A|}{|B|} \geq cn^{1-j/s}\frac{n-j-cn^{1-1/s}}{cn^{1-(j-1)/s}} &= n^{-1/s}\left(n-j-cn^{1-1/s}\right) \\ = n^{1-1/s}-jn^{-1/s}-cn^{1-2/s} &> cn^{1-1/s}. \end{aligned}$$

The last inequality holds for sufficiently large n because $c < 1$. We obtain $d_G(y) < \delta(G)$, a contradiction. Hence y does not exist, and x_j can be chosen as desired. □

With a somewhat smaller minimum degree, one cannot guarantee a dominating $s-$set.

Theorem 4.2. *Fix a positive integer s and a real constant c greater than* 1. *For n sufficiently large in terms of s and c, there is an n-vertex graph H with* $\Delta(H) \leq c(s \ln n)^{1/s} n^{1-1/s}$ *whose complement* $\overline{H}$ *has no dominating set of size at most s.*

Proof. Form a random graph H with n vertices by letting each pair of vertices be an edge with probability p, independently. We use $p = \frac{c+1}{2}(\frac{s \ln n}{n})^{1/s}$. We claim that for n sufficiently large, $\Pr[\Delta(H) > c(s \ln n)^{1/s} n^{1-1/s}] < 1/2$ and $\Pr[\overline{H}$ has a dominating $s-$set$] < 1/2$. This implies that when n is sufficiently large, some graph with sufficiently small maximum degree has no dominating $s-$set in its complement.

The degree distribution for a single vertex v is a binomial distribution with success probability p; the expectation is $(n-1)p$. The probability that $d_H(v)/np > \frac{2c}{c+1}$ is exponentially small, by the Chernoff bound. Multiplying by n, the probability that some vertex has degree exceeding $c(s \ln n)^{1/s} n^{1-1/s}$ remains less than $1/2$ when n is sufficiently large.

A set S of s vertices is a dominating set in $\overline{H}$ if and only if every vertex outside S is nonadjacent in H to some vertex of S. For $x \in V(H) - S$, the probability that x is adjacent to all of S is p^s. Hence the probability that S dominates $\overline{H}$ is $(1-p^s)^{n-s}$. Since we may use any $s-$set, the probability that $\overline{H}$ has a dominating s–set is bounded by $\binom{n}{s}(1-p^s)^{n-s}$. Since $\binom{n}{s} < n^s$ and $1 - p^s < e^{-p^s}$ and s is fixed, the probability is bounded by $e^{s \ln n} e^{-np^s(1+o(1))}$. This quantity tends to 0 when $np^s = c^s s \ln n$ with $c > 1$, which is precisely how we chose p. Thus, the probability that $\overline{H}$ has a dominating s–set is less than $1/2$ when n is sufficiently large. □

An immediate corollary of Theorem 4.2 is the following.

Corollary 4.3. *For any constant c with $c > 1$, when n is sufficiently large there is an n-vertex graph G having minimum degree at least $n - c(s \ln n)^{1/s} n^{1-1/s}$ and no dominating set of size at most s. In particular, G has no s-vertex dominating path.* □

Of course, for $s = 1$ we can strengthen the lower bound on $\Delta(G)$ to $n - 2$.

5. BALANCED SPANNING CATERPILLARS

Recall that a caterpillar is *balanced* if all spine vertices have the same number of leaf neighbors. For example, $K_{4,5}$ has a spanning balannced caterpillar with three spine vertices, where each spine vertex has two leaf neighbors.

We next prove that high minimum degree guarantees balanced caterpillars when obvious necessary divisibility conditions hold. The case $p = 1$ states that if n is even and sufficiently large, then $\delta(G) \geq 3n/4$ guarantees that G contains a balanced spanning caterpillar consisting of an $\frac{n}{2}$-vertex path P and a matching joining $V(P)$ to $V(G) - V(P)$.

Theorem 5.1. *Let G be an n-vertex graph, and let p be a positive integer. If n is sufficiently large and is divisible by $p + 1$, then $\delta(G) \geq (1 - \frac{p}{(p+1)^2})n$ implies that G contains a balanced spanning caterpillar with $\frac{n}{p+1}$ spine vertices.*

Proof. Note that $\delta(G) > n/2$. A nonextendible path has more than $\delta(G)$ vertices, so we may choose a path Q with $\frac{n}{p+1}$ vertices. Let $R = V(G) - V(Q)$. Given Q, let a *p-packing* be a subgraph consisting of disjoint stars with centers in $V(Q)$ and leaves in R, each having at most p edges. A vertex of Q is *saturated* by a p-packing if its star has p edges. Let M be a p-packing with the most edges. If all vertices of Q are saturated by M, then $Q \cup M$ is the desired balanced spanning caterpillar. Otherwise, we seek a larger p-packing.

Let x be a vertex of Q not saturated by M; there is thus also a vertex z in R not covered by M. By the maximality of M, every neighbor w of z in Q must be saturated. In addition, each leaf y of the star saturating w must not be adjacent to x, since otherwise we replace yw with xy and wz to obtain a larger p-packing.

The number of nonneighbors of x is at most $n - 1 - \delta(G)$, and hence M has at most $\frac{1}{p}(n - 1 - \delta(G))$ stars with p edges whose leaves are all nonadjacent to x. On the other hand, z has at least $\delta(G) - |R| + 1$ neighbors in $V(Q)$, all of which must be centers of those stars. Therefore, to avoid having a larger p-packing, we must have $\delta(G) - |R| + 1 \leq \frac{1}{p}(n - 1 - \delta(G))$. With $|R| = \frac{pn}{p+1}$, the inequality simplifies to $\delta(G) \leq (1 - \frac{p}{(p+1)^2})n - 1$. Hence the assumed lower bound on $\delta(G)$ implies that a largest p-packing saturates all of $V(Q)$. □

Recall that a caterpillar is *nearly balanced* if the numbers of leaf neighbors of vertices of the spine differ by at most 1.

Theorem 5.2. *Fix a positive integer s and a real constant c less than 1. Let G be a connected n-vertex graph such that $\delta(G) \geq n - cn^{1-1/s}$. If n is sufficiently large, then G contains a nearly balanced spanning caterpillar with k spine vertices for each k such that $s \leq k < 0.5\frac{\log n}{\log \log n}$.*

Proof. Theorem 4.1 provides a dominating s-vertex path. By Lemma 2.3, G has a dominating k-vertex path P and a spanning caterpillar T with k spine vertices. Let X be the set of vertices outside P not adjacent to all of $V(P)$. Since $\delta(G) \geq n - cn^{1-1/k}$, we have $|X| \leq kcn^{1-1/k}$.

We obtain a new caterpillar. Use P again as the spine. Let the vertices in X have the same neighbors as in T, contributing n_i leaf neighbors to the ith vertex v_i of P. If $n_i < n/k$ for all i, then since all of $V(P)$ is adjacent to all of $V(G) - V(P) - X$, the vertices of $V(G) - V(P) - X$ can be distributed as leaf neighbors of vertices on P arbitrarily so that the numbers of leaf neighbors of the vertices of P differ by at most 1.

To ensure $n_i < n/k$, it suffices to have $|X| < n/k$. For this we require $kcn^{1-1/k} < n/k$, or $n > c^k k^{2k}$. With $c < 1$, it suffices to have $n > k^{2k}$, which holds when $k < 0.5\frac{\log n}{\log \log n}$.□

One may also ask how sharp the minimum-degree thresholds in Theorems 5.1 and 5.2 are.

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