On the Ramsey Number of Trees Versus Graphs with Large Clique Number

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ABSTRACT

Chvátal established that \( r(T_m, K_n) = (m - 1)(n - 1) + 1 \), where \( T_m \) is an arbitrary tree of order \( m \) and \( K_n \) is the complete graph of order \( n \). This result was extended by Chartrand, Gould, and Polimeni who showed \( K_n \) could be replaced by a graph with clique number \( n \) and order \( n + 1 \) provided \( n \geq 3 \) and \( m \geq 3 \). We further extend these results to show that \( K_n \) can be replaced by any graph on \( n + 2 \) vertices with clique number \( n \), provided \( n \geq 5 \) and \( m \geq 4 \). We then show that further extensions, in particular to graphs on \( n + 3 \) vertices with clique number \( n \) are impossible. We also investigate the Ramsey number of trees versus complete graphs minus sets of independent edges. We show that \( r(T_m, K_n - tK_2) = (m - 1)(n - t - 1) + 1 \) for \( m \geq 3, n \geq 6 \), where \( T_m \) is any tree of order \( m \) except the star, and for each \( t, 0 \leq t \leq (n - 2)/2 \).

INTRODUCTION

For graphs \( G \) and \( H \), the Ramsey number \( r(G, H) \) is the smallest positive integer \( p \) such that if every edge of the complete graph \( K_p \) is arbitrarily colored red or blue, then there exists either a red \( G \) (a subgraph isomorphic to \( G \), all of whose edges are colored red) or a blue \( H \). Equivalently, \( r(G, H) \) is the smallest positive integer \( p \) such that if \( K_p = R \oplus B \) is an arbitrary factorization of \( K_p \) (i.e., \( R \) and \( B \) have order \( p \) and \( E(R) \cup E(B) \) partitions \( E(K_p) \)) then \( G \subseteq R \) or \( H \subseteq B \). A \( (G, H) \)-blocking pattern of \( K_p \) is a factorization \( K_p = R \oplus B \) such that \( G \not\subseteq R \) and \( H \not\subseteq B \). The clique number of a graph \( G \) is the maximum order of a complete subgraph of \( G \).

\[ \text{Research supported by a grant from Emory University.} \]
\[ \text{Research supported by a grant from the University of Louisville.} \]

Journal of Graph Theory, Vol. 7 (1983) 71–78
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Theorem A. (Chvátal [1]). If $T_m$ is a tree of order $m$ and $n$ is a positive integer then $r(T_m, K_n) = (m - 1)(n - 1) + 1$.

A result related to Theorem A was given in [3].

Theorem B [3]. For each tree $T_m$ of order $m \geq 3$ and each integer $n \geq 4$, $r(T_m, K_n - e) = (m - 1)(n - 2) + 1$ and hence, $r(T_m, G) = (m - 1)(n - 2) + 1$ for each graph $G$ of order $n$ and clique number $n - 1$.

We shall also require the following results:

Theorem C [2]. Let $G$ be a graph of order $n$. Then

$$r(P_3, G) = \begin{cases} n, & \text{if } \overline{G} \text{ has a 1-factor,} \\ 2n - 2\beta_1(\overline{G}) - 1, & \text{otherwise,} \end{cases}$$

where $\beta_1(\overline{G})$ denotes the edge independence number of the complement of $G$.

Theorem D [7]. If $P_m$ is a path of order $m \geq 4$ and $G_n$ is a graph of order $n + 2$ with clique number $n(n \geq 3)$ then

$$r(P_m, G_n) = (m - 1)(n - 1) + 1$$

Theorem E [5]. If $l, t \geq 1, m \geq 2$, and

$$l > (t - 1) - \left\lfloor \frac{t - 1}{m - 1} \right\rfloor (m - 1)$$

then

$$r(T_m, K_l + K_t) = \left( l + \left\lfloor \frac{t - 1}{m - 1} \right\rfloor \right)(m - 1) + 1.$$

We note that Theorem E is an extension of Theorem B in that we may consider $K_l + K_t$ to be $K_{l+t} - K_t$. 

$S \subseteq V(G)$, the subgraph induced by $S$, denoted $\langle S \rangle$, is the subgraph with vertex set $S$ and whose edge set consists of those edges of $G$ incident with two elements of $S$. We denote by $G_1 - G_2$ the graph obtained by deleting the edges of $G_2$ from the graph $G_1$. Note that $G - K_2$ is also denoted $G - e$. 

A well known result is the following:
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Theorem F [6]. If \( g_1, g_2, \ldots, g_t \in \mathcal{S}(H) \) and \( G = \bigcup_{i=1}^t g_i \), where \( \mathcal{S}(H) = \{g | g \text{ is connected and } r(g, H) = (|V(g)| - 1)(\chi(H) - 1) + 1 \} \), then,

\[
r(G, H) = \max_{i \leq c(G)} \left\{ (j - 1)(\chi(H) - 2) + \sum_{i=1}^{c(G)} ik_i \right\},
\]

where \( c(G) \) denotes the order of the largest component of \( G \) and \( k_i \) is the number of components of \( G \) of order \( i \).

The purpose of this article is to investigate the Ramsey number of trees versus complete graphs minus a set of independent edges. We also further extend Theorem B, and show that in general this extension cannot be improved.

Theorem 1. If \( m \geq 3 \), \( n \geq 3 \), and \( T_m \) is any tree of order \( m \) other than \( K_{1,m-1} \) (\( m \geq 4 \)) then

\[
r(T_m, K_{2n-1} - (n-2)K_2) = (m - 1)n + 1.
\]

Proof. By Theorem A, \( r(T_m, K_{2n-1} - (n-2)K_2) \geq (m - 1)n + 1 \). We prove the reverse inequality by induction on \( n \) and \( m \). The case for \( n = 3 \) follows from Theorem B. The case \( m = 3 \) follows from Theorem C while the case for \( m = 4 \) is a simple induction on \( n \) with the anchor cases of \( n = 3 \) and \( n = 4 \) following from Theorems B and D, respectively.

Assume \( r(T_m, K_{2n-1} - (n-2)K_2) = (m - 1)n + 1 \) for a fixed but arbitrary integer \( n \geq 3 \) and for each \( m \geq 3 \). We prove \( r(T_m, K_{2m+1} - (n-1)K_2) = (m - 1)(n + 1) + 1 \) for every \( m \geq 3 \). As previously noted, this is true if \( m = 3 \) and \( 4 \). Hence, we assume \( r(T_m, K_{2m+1} - (n-1)K_2) = (m' - 1) \) \( (n + 1) + 1 \) for all \( m \geq m' \geq 3 \) for a fixed \( m \geq 4 \) and show \( r(T_{m+1}, K_{2m+1} - (n-1)K_2) = m(n + 1) + 1 \).

Let \( T = T_{m+1} \) be an arbitrary tree (not a star) of order \( m + 1 \) and assume that a \( (T, K_{2m+1} - (n-1)K_2) \)-blocking pattern exists. Let \( v \) be an end vertex of \( T \) and \( u \) be the vertex of \( T \) adjacent to \( v \), such that \( T - v \) is a tree of order \( m \) that is not \( K_{1,m-1} \). By the induction hypothesis \( r(T - v, K_{2m+1} - (n-1)K_2) = (m - 1)(n + 1) + 1 \). Let \( S \) denote the set of vertices of \( K_{m(n+1)+1} \) that do not belong to the red \( T - v \). Since \( |S| = mn + 1 \) and \( r(T, K_{2m-1} - (n-2)K_2) = mn + 1 \), we see that \( \langle S \rangle \cap B \supseteq K_{2m-1} - (n-2)K_2 \).

If any edge joining \( u \) to \( S \) is red, a red \( T \) results, hence \( u \) is blue adjacent to \( S \). Now consider \( H = K_{2m} - (n-2)K_2 \) (formed with the \( K_{2m-1} - (n-2)K_2 \) contained in \( \langle S \rangle \cap B \) and the vertex \( u \)) and let \( S' = V(K_{m(n+1)+1}) - H \).
Then \(|S'| = (m - 2)(n + 1) + 3\) and since \(r(T_{m-1}, K_{2n+1} - (n - 1)K_2) = (m - 2)(n + 1) + 1\), we see that \(T_{m-1} \subseteq \langle S' \rangle \cap R\), where \(T_{m-1} \neq K_{1,m-2}\) except if \(m = 4\). (We note that since \(T_{m+1}(m \geq 5)\) is not a star, it is possible to find vertices \(u\) and \(v\) such that \(T_{m+1} - u - v\) is a tree other than a star, that is \(T_{m+1} - u - v = T_{m-1}\).)

Case 1. Suppose \(u\) and \(v\) (as noted above) are adjacent to distinct vertices \(x\) and \(y\) in \(T_{m-1}\). If either \(x\) or \(y\) is blue adjacent to \(H\) we are done. So each must be red adjacent to some vertex of \(H\). If they are red adjacent to distinct vertices, a red \(T_{m+1}\) results. Thus \(x\) and \(y\) must be red adjacent to the same vertex of \(H\). Let this vertex be \(p\). If \(p\) is not an end vertex of a red edge in the coloring of the vertices of \(H\), then \(\langle \langle H \cup \{x\} \rangle \cap B \supseteq K_{2n+1} - (n - 1)K_2\). Thus \(p\) is an end vertex of an independent red edge in \(H\). But then \(\langle \langle V(H) - p \rangle \cup \{x, y\} \rangle \cap B \supseteq K_{2n+1} - (n - 2)K_2 \supseteq K_{2n+1} - (n - 1)K_2\).

Case 2. Suppose \(u\) and \(v\) (as noted above) are adjacent to the same vertex \(w\) in \(T_{m+1}\). As above, the remaining vertices contain a blue \(K_{2n} - (n - 2)K_2\). Call this set of vertices \(H\). The vertex \(w\) has at most one red edge to \(H\) for otherwise a red \(T_{m+1}\) results. Further, this edge is not incident with any vertex \(p \in H\), where \(p\) is not an end vertex of a red edge in \(\langle H \rangle \cap R\). Let \(uv \in E(R)\), \(v \in H\). Let \(H' = (H - \{v\}) \cup \{w\}\). Clearly, \(\langle H' \rangle \cap B \supseteq K_{2n} - (n - 3)K_2\). Consider \(S = V(K_{m(n+1)+1}) - H'\). Since \(|S'| = (m - 2)(n + 1) + 3\), it follows from the induction hypothesis that \(T_{m-1} \subseteq \langle S' \rangle \cap R\). Repeat the above process. As in the argument above, there exists a vertex \(w'\) in the red \(T_{m-1}\) such that \(w'\) were adjacent to two distinct vertices \(a, b \in V(T_{m-1})\). Then a red \(T_{m+1}\) would result. Thus, either a red \(T_{m+1}\) results or we find a set of vertices \(H'' = (H' - \{w'\}) \cup \{w'\}\) with \(\langle H'' \rangle \cap B \supseteq K_{2n} - (n - 4)K_2\). We proceed with this process until a red \(T_{m+1}\), a blue \(K_{2n+1} - (n - 1)K_2\), or a blue \(K_{2n}\) results. If a blue \(K_{2n}\) results, we are guaranteed that a red \(T_{m+1}\) or a blue \(K_{2n+1} - (n - 1)K_2\) will result on the next repetition of the process.

**Theorem 2.** If \(m \geq 3, n \geq 3\), and \(T_m\) is any tree of order \(m\), except \(K_{1,m-1}\) when \(m \geq 4\), then

\[r(T_m, K_{2n} - (n - 1)K_2) = (m - 1)(n + 1) + 1.\]

The proof is analogous to that of the previous theorem and is not included. We note that a similar argument holds if we remove \(t\) independent edges from \(K_n (0 \leq t \leq [(n - 2)/2])\). This is summarized in the following theorem.

**Theorem 3.** If \(m \geq 3, n \geq 6\), and \(T_m\) is any tree of order \(m\), except \(K_{1,m-1}\) when \(m \geq 4\), then
$r(T_m, K_n - tK_2) = (m - 1)(n - t - 1) + 1$

for each $t$, $0 \leq t \leq [(n - 2)/2]$.

The following construction shows that stars cannot be included in the set of trees in the previous theorems.

Let $G = kC_n (k \geq 1, m \geq 5)$. It is clear that $K_{1,3} \subseteq G$. If $H = K_n - tK_2$, where

$$n > k \left( 2 \left[ \frac{m}{3} \right] + \left[ \frac{m}{3} - \left[ \frac{m}{3} \right] + \frac{1}{3} \right] \right)$$

and $t \geq [m/3] k$ then $H \subseteq G$. It follows that in these cases

$$r(K_{1,3}, K_n - tK_2) > mk.$$  

This bound is an improvement on the bound obtainable from Chvátal’s formula for many values of $n$. For example, the clique number of $K_{25} - 11K_2$ is 14. From Theorem A we see that $r(K_{1,3}, K_{25} - 11K_2) \geq 40$, but by considering $m = 5$ and $k = 8$ in the above construction, we may conclude that $r(K_{1,3}, K_{25} - 11K_2) > 40$. As a last example, the clique number of $K_{34} - 16K_2$ is 18. Theorem A gives us that $r(K_{1,3}, K_{34} - 16K_2) \geq 52$, but by considering $m = 7$ and $k = 8$ in the above construction, we may conclude that $r(K_{1,3}, K_{34} - 16K_2) > 56$.

We now note another extension of Theorem B. The proof when $T_m = K_{1,n-1}$ follows standard inductive techniques.

**Corollary 4.** If $m \geq 4$, $n \geq 5$, and $T_m$ is any tree of order $m$, then

$$r(T_m, K_{n+2} - 2K_2) = (m - 1)(n - 1) + 1.$$  

**Theorem 5.** If $G$ is a graph of order $n + 2$ ($n \geq 5$) and clique number $n$ and $m \geq 4$, then

$$r(T_m, G) = (m - 1)(n - 1) + 1.$$  

**Proof.** From Theorem E (with $l = n - 1$, $t = 3$, and $m \geq 4$) we see that $r(T_m, K_{n+2} - K_2) = (m - 1)(n - 1) + 1$, when $n \geq 5$. Theorem 3 and its corollary imply $r(T_m, K_{n+2} - 2K_2) = (m - 1)(n - 1) + 1$ for $m \geq 4$ and $n \geq 6$ (while $n = 5$ can be shown by a simple induction on $m$). Now if $G$ is a graph of order $n + 2$ and clique number $n$ then $G \subseteq K_{n+2} - 2K_2$ or $G \subseteq K_{n+2} - K_2$. Theorem A shows that $r(T_m, G) \geq (m - 1)(n - 1) + 1$ and the result follows. ■
Corollary 6. If $F$ is a forest with all components of order at least four and $G_n$ is any graph of order $n + 2$ with clique number $n(n \geq 5)$ then

$$r(F, G_n) = \max_{i \leq j \leq c(F)} \left\{ (j - 1)(n - 2) + \sum_{i=j}^{c(F)} ik_i \right\}.$$  

**Proof.** Theorem 5 implies that $T_n \in \mathscr{G}(G_n)$ when $m \geq 4$ and $n \geq 5$. Since $\chi(G_n) = n$ we may invoke Theorem F. □

It is natural to ask what other graphs $G$ with clique number $n$ follow this pattern? Although a complete solution to this question has eluded us, we offer the following observations.

Let $V_n$ be the graph obtained by joining a single vertex of $K_n$ to $n - 1$ other distinct independent vertices. Let $W_n$ be the graph obtained by joining with an edge each of $n - 1$ distinct independent vertices to distinct vertices in a copy of $K_n$. Let $X_n$ denote the graph obtained by identifying one end vertex of the path $P_n$ with a vertex of $K_n$. Let $Y_n$ denote the graph obtained by identifying each of the end vertices of the path $P_n$ with a distinct vertex of $K_n$. Let $Z_n$ denote the graph obtained by identifying both end vertices of $P_n$ with a single vertex of $K_n$.

**Theorem 7.** If $T_m$ is a tree of order $m \geq 3$, then $r(T_m, G) = (m - 1)(n - 1) + 1$, where $G \in \{V_n, W_n, X_n, Y_n, Z_n \mid n \geq 3 \} \cup \{Z_n \mid n \geq 4\}$.

The proof for any of the classes above follows closely that of Theorem B.

We now show there are some limitations to the graphs with clique number $n$ that follow Chvátal's formula.

**Proposition 8.** If $n \geq 2$ then $r(T_m, K_{2n} - nK_2) \geq (m - 1)(n - 1) + 2$.

**Proof.** The blocking pattern for $K_{(m-1)(n-1)} + 1$ comes from $R = (n - 1)K_{m-1} \cup \{p\}$ for some distinct vertex $p$. □

**Theorem 9.** If $m \geq 3$ and $n \geq 2$, $r(P_m, K_{2n} - nK_2) = (m - 1)(n - 1) + 2$.

**Proof.** As a consequence of a result in [4], $r(P_m, C_4) = m + [4/2] - 1 = m + 1(m \geq 4)$. Further, by Theorem C, since $K_{2n} - nK_2$ has a 1-factor, $r(P_m, K_{2n} - nK_2) = 2n$. So we assume $r(P_m, K_{2(n-1)} - (n - 1)K_2) = (m - 1)(n - 2) + 2$ for a fixed but arbitrary $n \geq 3$ and each $m \geq 3$. We wish to show $r(P_m, K_{2n} - nK_2) = (m - 1)(n - 1) + 2$ by induction on $m$. So suppose there exists a $(P_{m-1}, K_{2n} - nK_2)$-blocking pattern for $K_{m(n-1)+2}$. By the induction hypothesis $F_m \subseteq R$. Further, in the remaining $m(n - 2) + 2$ vertices $K_{2(n-1)} - (n - 1)K_2 \subseteq B$. But then by examining the endpoints of
the red $P_m$ we see that either $P_{m+1} \subseteq R$ or $K_{2n} - nK_2 \subseteq B$. Equality follows from Proposition 8. 

We note that a similar argument shows (see [7])

$$r(P_m, K_{2n-1} - (n-1)K_2) = (m-1)(n-1) + 1.$$ 

Burr, Faudree, Rousseau, and Schelp [private communication] have shown that

$$r(K_{1,m-1}, K_{1,s_2,s_3,\ldots,s_k}) = (k-1)(r(K_{1,m-1}, K_{1,s_2}) - 1) + 1,$$

when $s_2 \leq s_3 \leq \cdots \leq s_k$ and $m$ is sufficiently large. In particular, this says for $m$ large and even that $r(K_{1,m-1}, K_{2n-1} - (n-1)K_2) = m(n-1) + 1$.

Thus, in this case, the Chvátal formula does not hold.

Finally, we show that Theorem 5 is the best extension possible of Theorem B.

Lemma 10. If $G = K_{j+5t} - tC_5$ then $G$ has clique number $j + 2t$ and $\chi(G) = j + 3t$, thus for any connected graph $H$,

$$r(G, H) \geq (|V(H)| - 1)(j + 3t - 1) + 1.$$ 

Observe that if $G = K_{n+3} - C_5$ then $G$ has order $n + 3$ and clique number $n \geq 4$ and by Lemma 10 we see $r(T_m, K_{n+3} - C_5) \geq (m-1)n + 1$. This observation shows that if $G$ is a graph of order $n + 3$ with clique number $n$, then it is not necessarily the case that $r(T_m, G) = (m-1)(n-1) + 1$.

Theorem 10. For $m, n \geq 4$, $r(T_m, K_{n+3} - C_5) = (m-1)n + 1$.

Proof. By Theorem 3, $(m-1)n + 1 = (T_m, K_{n+3} - 2K_2) \geq r(T_m, K_{n+3} - C_5)$. The Theorem follows from the observation above.

Theorem 11. For $m, n \geq 4$, $r(T_m, K_{n+5t} - tC_5) = (m-1)(n + 3t - 1) + 1$.

Proof. Lemma 10 implies that $r(T_m, K_{n+5t} - tC_5) \geq (m-1)(n + 3t - 1) + 1$. Since $K_{n+5t} - tC_5 \subseteq K_{n+5t} - 2tK_2$ it is clear that $r(T_m, K_{n+5t} - tC_5) \leq r(T_m, K_{n+5t} - 2tK_2) = (m-1)(n + 3t - 1) + 1$.

CONCLUSION

The possible directions are numerous. Some interesting problems which occur to us include the following:
(1) Determine all values of \( m, n, \) and \( t \) such that
\[ r(K_{1,m}, K_{n-t}K_2) = m(n-t-1) + 1. \]

(2) Determine for other values of \( m, n, \) and \( t \)
\[ r(K_{1,m}, K_{n-t}K_2). \]

(3) Can similar formulas be found for the removal of multiple copies of other graphs? In particular other complete graphs or paths.

(4) Ultimately, determine all graphs \( G \) and trees \( T_m \) such that
\[ r(T_m, G) = (m-1)(\chi(G) - 1) + 1. \]

References


