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Forbidden subgraphs for chorded pancyclicity

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ABSTRACT

We call a graph G *pancyclic* if it contains at least one cycle of every possible length m, for $3 \leq m \leq |V(G)|$. In this paper, we define a new property called *chorded pancyclicity*. We explore forbidden subgraphs in claw-free graphs sufficient to imply that the graph contains at least one chorded cycle of every possible length $4, 5, \ldots, |V(G)|$. In particular, certain paths and triangles with pendant paths are forbidden.

1. Introduction

In the past, forbidden subgraphs for Hamiltonian properties in graphs have been widely studied (for an overview, see [1]). A graph containing a cycle of every possible length from three to the order of the graph is called pancyclic. The property of pancyclicity is well-studied. In this paper, we define the notion of chorded pancyclicity, and study forbidden subgraph results for chorded pancyclicity. We consider only $K_{1,3}$-free (or *claw*-free) graphs, and we forbid certain paths and triangles with pendant paths.

Further, we consider only simple claw-free graphs. In this paper we let G be a graph and P_t be a path on t vertices. Let Z_i be a triangle with a pendant P_i adjacent to one of the vertices of the triangle. In particular, we will be considering the graphs Z_1 and Z_2, shown in Fig. 1. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. Let $N_G(u)$ denote the set of neighbors of the vertex u, that is, the vertices adjacent to u in the graph G. Let $N_G[u] = N_G(u) \cup \{u\}$. A graph is called *traceable* if it contains a Hamiltonian path. We use $H \,\square\, G$ to denote the Cartesian product of H and G. For terms not defined here see [4]. We will first note well-known results on forbidden subgraphs for pancyclicity.

Theorem 1. *Let R, S be connected graphs and let G be a 2-connected graph of order $n \geq 10$ such that $G \neq C_n$. Then if G is $\{R, S\}$-free then G is pancyclic for $R = K_{1,3}$ when S is either P_4, P_5, P_6, Z_1, or Z_2.*

The proof of a theorem (Theorem 4) in [3] yields the following result.

Theorem 2 (*[3]*). *If G is a 2-connected graph of order $n \geq 10$ that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_1, then G is either a cycle or G is pancyclic.*

Gould and Jacobson proved a similar result for Z_2 in [5].

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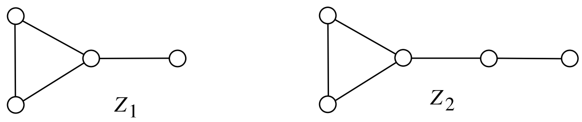

Fig. 1. The graphs Z_1 and Z_2.

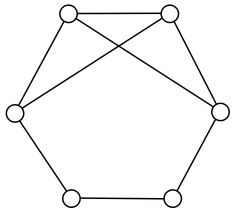

Fig. 2. A pancyclic graph with no chorded 5-cycle.

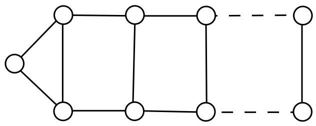

Fig. 3. An infinite family of pancyclic graphs with no chorded C_4.

Theorem 3 ([5]). *If G is a 2-connected graph of order $n \geq 10$ that contains no induced subgraph isomorphic to $K_{1,3}$ or Z_2, then G is either a cycle or G is pancyclic.*

Faudree, Gould, Ryjacek, and Schiermeyer proved a similar result for certain paths in [2].

Theorem 4 ([2]). *If G is a 2-connected graph of order $n \geq 6$ that is $\{K_{1,3}, P_5\}$-free, then G is either a cycle or G is pancyclic.*

Theorem 5 ([2]). *If G is a 2-connected graph of order $n \geq 10$ that is $\{K_{1,3}, P_6\}$-free, then G is either a cycle or G is pancyclic.*

Theorem 4 implies the following result for P_4.

Theorem 6 ([2]). *If G is a 2-connected graph of order $n \geq 6$ that is $\{K_{1,3}, P_4\}$-free, then G is either a cycle or G is pancyclic.*

In this paper, we will extend each of these theorems to analogous results on chorded pancyclicity.

2. Results

Definition 1. A graph G of order n is called *chorded pancyclic* if it contains a chorded cycle of every length m, $4 \leq m \leq n$.

Note first that not all pancyclic graphs are chorded pancyclic. The graph in Fig. 2 is pancyclic, but contains no chorded 5-cycle. Further, the graph in Fig. 3 represents an infinite family of pancyclic graphs that do not contain a chorded 4-cycle.

An important tool used in the proofs of this paper is a k-tab, which we define here.

Definition 2. Given a subgraph H of G, a *k-tab on H* is a path $a_0 a_1 \cdots a_{k+1}$ with k internal vertices so that $\{a_1, \ldots, a_k\} \subseteq V(G) - V(H)$ and there exist distinct vertices $u, v \in V(H)$ where $a_0 = u$ and $a_{k+1} = v$.

Note that, because there are no cut-vertices in a 2-connected graph, every proper subgraph must have a k-tab for some $k > 0$. Another tool we use in proofs, to better highlight when a subgraph is induced in G, is a *frozen set* F. Depending upon particular cases, the frozen set may be modified during the course of a proof. If a vertex is in the frozen set, it has no neighbors other than those already given. In particular, if a subgraph contains only frozen vertices, then it must be induced. Also, if every vertex of G is in F, then we have completely described the graph G. In figures, frozen vertices will be indicated by a solid vertex.

We now turn our attention to extending the results for forbidden subgraphs in Theorem 1. The following simple lemma has appeared many times in the literature.

Lemma 1. *Let G be claw-free. For any $x \in V(G)$, $N_G(x)$ is either connected and traceable, or two disjoint cliques.*

Using Lemma 1 we prove the following useful lemma.

Lemma 2. *Let G be a $K_{1,3}$-free graph. For any $x \in V(G)$ and any integer $k \geq 3$, if $\deg_G(x) \geq 2k - 1$ then there is a chorded $(k+1)$-cycle in G.*

Proof. Consider a vertex $x \in V(G)$ such that $\deg_G(x) = m \geq 2k - 1$. Lemma 1 implies that $N_G(x)$ is either connected and traceable, or two disjoint cliques.

Case 1: Suppose that $N_G(x)$ is connected and traceable.

Let $v_1 v_2 \cdots v_k v_{k+1} \cdots v_m$ be a Hamiltonian path in $N_G(x)$. Then $x v_1 v_2 \cdots v_k x$ is a $(k+1)$-cycle in G with chord $x v_2$ (in fact, there are $k-2$ chords in this $(k+1)$-cycle).

Case 2: Suppose that $N_G(x)$ is two disjoint cliques.

Partition the m vertices into two cliques. Then at least one clique has at least k vertices. If the vertices $\{v_1, v_2, \ldots, v_k\}$ form a clique in $N_G(x)$, then $x v_1 v_2 \cdots v_k x$ is a $(k+1)$-cycle in G with chord $x v_2$. Thus, the lemma is proven. □

Theorem 7. *Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, Z_2\}$-free, then $G = C_n$ or G is chorded pancyclic.*

Proof. Suppose that G is a 2-connected graph of order $n \geq 10$ that is $\{K_{1,3}, Z_2\}$-free and that G is not C_n. By Theorem 3, we know G must be pancyclic. For the sake of contradiction, suppose that G is not chorded pancyclic. Let m be the largest value with $4 \leq m < n$ such that every m-cycle in G does not contain a chord. First we show that m must be 4, and then we show that there is a chorded 4-cycle in G.

Suppose that $m \geq 5$ and consider an m-cycle $C = v_1 v_2 v_3 \cdots v_m v_1$ in G. Since G is 2-connected and $m < n$, there exists a vertex $x \in V(G) - V(C)$ such that $xv \in E(G)$ for some $v \in V(C)$. Without loss of generality, we may assume that $x v_1 \in E(G)$. Then $\{v_1, x, v_2, v_m\}$ induces a claw in G unless $x v_2$, $x v_m$ or $v_2 v_m$ is an edge. If $v_2 v_m$ is added then C is a chorded m-cycle and we are done. By symmetry, adding either $x v_2$ or $x v_m$ as an edge is equivalent so, without loss of generality, we assume that $x v_2 \in E(G)$. Now $\{x, v_1, v_2, v_3, v_4\}$ induces a Z_2 in G. The only two edges that can eliminate this induced Z_2 without adding a chord to the m-cycle C are $x v_3$ and $x v_4$.

If $x v_4 \in E(G)$ then $v_1 v_2 x v_4 v_5 \cdots v_m v_1$ is an m-cycle with chord $v_1 x$. Thus we may assume that $x v_3 \in E(G)$. Now $\{x, v_2, v_3, v_4, v_5\}$ induces Z_2 in G. The only edges that will eliminate this induced Z_2, but will not add a chord to C are $x v_4$ and $x v_5$. If $x v_4 \in E(G)$ then $v_1 v_2 x v_4 v_5 \cdots v_m v_1$ is an m-cycle with chord $x v_1$. If instead $x v_5 \in E(G)$, then $v_1 v_2 v_3 x v_5 \cdots v_m v_1$ is an m-cycle with chord $x v_1$.

Therefore, it follows that $m = 4$. By Lemma 2, it follows that $\Delta(G) \leq 4$. Let $D = vwxyzv$ be a 5-cycle in G with chord vx.

Suppose that both v and x can be added to F; that is, suppose that v and x have degree 3 in G. Because G is 2-connected and $n > 5$, D has a k-tab for some k with two distinct endpoints in $\{w, z, y\}$. By symmetry, we may assume that y is an endpoint of the k-tab and thus ay is an edge for $a \in V(G) - V(D)$. But now $\{y, a, x, z\}$ induces a claw, unless za is an edge (recall that x is frozen). Further, $\{w, v, x, y, a\}$ induces a Z_2 unless one of wy or wa is an edge. Because the edge wy gives a 4-cycle $wyxvw$ in G with chord wx, it follows that za and wa are both edges. But now $G[V(D) \cup \{a\}]$ is isomorphic to $K_3 \,\square\, K_2$, which is vertex-transitive. Therefore, all vertices of G can be added to F, as vertices can be relabeled so that any edge from $V(D) \cup \{a\}$ to $V(G) - (V(D) \cup \{a\})$ is a new edge using the vertex x. As a side note, this symmetry argument is used many times in further proofs to add vertices to F.

Thus it follows that, without loss of generality, ax is an edge for $a \in V(G) - V(D)$. Now $\{x, a, w, y\}$ induces a claw unless one of $\{wy, wa, ay\}$ is an edge. Because wy gives a 4-cycle $wyxvw$ with chord wx and wa gives a 4-cycle $waxvw$ with chord wx, it must be the case that ay is an edge. Note that $\deg_G(x) = 4$ so we may add x to F (now $F = \{x\}$).

Because $n > 6$ we look for a neighbor for a vertex $b \in V(G) - (V(D) \cup \{a\})$. Suppose that both v and y can be added to F. Then z must also be added to F, as the edge bz would give an induced claw $\{z, b, v, y\}$. Now w and a are the only unfrozen vertices, so we may assume by symmetry that ba is an edge. Now $\{w, v, x, a, b\}$ induces a Z_2 unless wb is an edge, as wa has already been ruled out and v and x are frozen by supposition. If wb is an edge, then note that swapping the labels of v, z, y with w, b, a, respectively, does not change the graph. Thus an additional edge to w, b, or a is equivalent to an additional edge to v, z, or y which are all frozen. Therefore all vertices can be added to F. This implies that $n = 7$, which is a contradiction.

Thus it follows that, without loss of generality, by is an edge. To avoid an induced claw on $\{y, b, z, x\}$ we must have the edge bz. Now $\{w, v, x, y, b\}$ induces a Z_2. Because the edges yw and yv create chorded 4-cycles on $\{w, v, x, y\}$ and the edge bv would give the 4-cycle $bvzyb$ with chord bz, it follows that bw must be an edge to eliminate this induced Z_2. Note that $\deg_G(y) = 4$, so we also add y to F (now $F = \{x, y\}$).

Let $R = V(D) \cup \{a, b\}$. Because G is 2-connected and $n > 7$, there exists a vertex $c \in V(G) - R$ with a neighbor in $R - \{a\}$. Because x and y are frozen, and $G[R - \{a\}]$ is isomorphic to $K_3 \,\square\, K_2$, the edges cw, cv, cz, and cb are all equivalent. We assume, without loss of generality, that cz is an edge (see Fig. 4).

Now to prevent $\{z, c, v, y\}$ from inducing a claw we must have the edge cv. Because the degree of v and z are now 4, we add them to F (so $F = \{x, y, v, z\}$). To prevent $\{c, v, z, x, a\}$ from inducing a Z_2 we must have the edge ca as these are the only unfrozen vertices.

Because $n > 8$ there exists another vertex $d \in V(G) - (R \cup \{c\})$. By symmetry any edge from d to one of the unfrozen vertices $\{w, a, b, c\}$ is equivalent so we may assume without loss of generality that dw is an edge. To prevent $\{w, x, d, b\}$ from

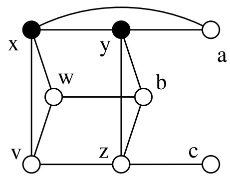

Fig. 4. Adding c to $G[R]$ in the proof of Theorem 7.

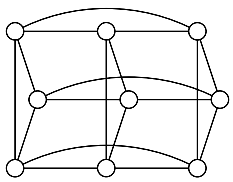

Fig. 5. $K_3 \square K_3$ contains no chorded C_4.

inducing a claw db must be an edge. Now to prevent $\{w, b, d, y, a\}$ from inducing a Z_2 we must have the edge da because wa gives a 4-cycle $waxvw$ with chord wx and ba gives a 4-cycle $bayzb$ with chord by. Now that da is an edge, we must also have the edge dc to prevent $\{a, d, c, x\}$ from inducing a claw. But now G is isomorphic to $K_3 \square K_3$ (see Fig. 5), all vertices have degree 4 and are thus frozen, and $n = 9$. This is a contradiction, so the theorem is proven. □

As Z_1 is an induced subgraph of Z_2, Theorem 2 implies the following corollary.

Corollary 8. *Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, Z_1\}$-free, then $G = C_n$ or G is chorded pancyclic.*

We now turn our attention to forbidden paths; many of these theorems will be proven using a collection of lemmas.

Theorem 9. *Let G be a 2-connected graph of order $n \geq 5$. If G is $\{K_{1,3}, P_4\}$-free, then G is chorded pancyclic.*

Proof. Let G be a 2-connected graph of order $n \geq 5$ that is $\{K_{1,3}, P_5\}$-free. One can easily verify that G is chorded pancyclic if $n = 5$. Therefore we assume that $n \geq 6$. By Theorem 6, we know that G is pancyclic. Suppose, for the sake of contradiction, that G is not chorded pancyclic. Let m be the largest number with $4 \leq m \leq n$ such that every m-cycle in G is chordless. Any chordless m-cycle for $m > 4$ contains an induced P_4, so it follows that $m = 4$.

Consider a 4-cycle $C = v_1v_2v_3v_4v_1$ in G. Since $n \geq 5$ and G is 2-connected, there exists a vertex $x \notin V(C)$ such that $xv \in E(G)$ for some $v \in V(G)$. Without loss of generality, we may assume that $v_1x \in E(G)$. To avoid an induced claw on $\{v_1, x, v_2, v_4\}$, one of v_2v_4, xv_4, or xv_2 must be in $E(G)$. If v_2v_4 is an edge, then C is a chorded 4-cycle, which is a contradiction.

Then by symmetry, we may assume without loss of generality that xv_2 is an edge. Because $xv_1v_4v_3$ cannot be an induced P_4 subgraph of G, G must contain either xv_4, xv_3, or v_1v_3 as an edge. The edge v_1v_3 creates a chord of the 4-cycle C, a contradiction. The edge v_3x yields the 4-cycle $v_1xv_3v_2v_1$ with chord v_2x, a contradiction. Similarly, the edge v_4x yields the 4-cycle $v_1v_2xv_4v_1$ with chord v_1x, again a contradiction. Thus, every 4-cycle in G must be chorded. □

Lemma 3. *Let G be a 2-connected graph of order $n \geq 8$. If G is $\{K_{1,3}, P_5\}$-free and G contains a C_4, then G contains a chorded C_5.*

Proof. Let G be a 2-connected graph of order $n \geq 8$ that is $\{K_{1,3}, P_5\}$-free, and let $C = wxyzw$ be a 4-cycle in G. Suppose, for the sake of contradiction, that G does not have a chorded C_5. Because $n > 4$ and G is 2-connected, there is a k-tab on the cycle C. Choose a k-tab $Q = a_0 \cdots a_{k+1}$ which minimizes k. Over all minimal k-tabs, choose one where a_0a_{k+1} is an edge of C if possible. Without loss of generality, we may assume that $x = a_0$.

Suppose Q is a 1-tab. If a_2 is y (or w by symmetry), then xa_1yzwx is a 5-cycle with chord xy. If $a_2 = z$ then $\{x, a_1, y, w\}$ cannot be an induced claw. But a_1w and a_1y contradict our choice of Q, so it follows that wy is an edge. Now $wyxa_1zw$ is a 5-cycle with chord xw.

Suppose that $k \geq 3$. Then let a'_0 be a neighbor of $x = a_0$ on C which is not a_{k+1}. Now $a'_0xa_1a_2a_3$ is an induced P_5 unless there is another edge among $\{a'_0, x, a_1, a_2, a_3\}$. However, each of these edges creates either a 1-tab or a 2-tab on C, which contradicts the minimality of Q.

Therefore, it follows that $k = 2$. If a_3 is y (or w by symmetry), then to avoid an induced claw on $\{x, a_1, y, w\}$ one of wy, ya_1, wa_1 must be an edge. However, each of these creates a chorded 5-cycle. Thus $a_3 = z$. To prevent $\{x, a_1, y, w\}$ from inducing a claw without creating a 1-tab, we must have the edge yw. One can check that any further edge among $R = \{x, y, w, z, a_1, a_2\}$ creates a chorded 5-cycle. However, $n > 6$, so there exists some $b \in V(G) - R$ that is adjacent to a vertex in R.

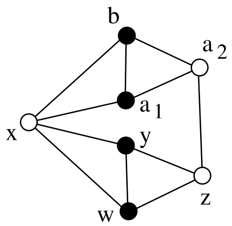

Fig. 6. The graph $G[R']$ from Lemma 3.

If by is an edge (or bw by symmetry) then to avoid an induced claw $\{y, b, x, z\}$, either bx, bz, or xz is an edge. All three give chorded 5-cycles, so we may add y and w to the frozen set F. Now $F = \{y, w\}$.

Suppose that we can now add both x and z to F so that $F = \{y, w, x, z\}$. Then, without loss of generality, we may assume that ba_1 is an edge. But now ba_1xwz is an induced P_5 because x, w, z are all frozen. This is a contradiction, and thus at least one of $\{x, z\}$ cannot be added to F. In particular, let bx be an edge. To avoid an induced claw $\{x, b, a_1, w\}$ it follows that ba_1 is an edge. Also, $bxwza_2$ cannot be an induced P_5, so there is another edge among these vertices. Because bz creates a shorter k-tab on C, it follows that ba_2 is an edge. Let $R' = R \cup \{b\}$ and note that any additional edge in $G[R']$ creates a chorded 5-cycle. By symmetry with $\{y, w\}$ we can now add b and a_1 to F (see Fig. 6).

However, $n > 7$, so there exists some $c \in V(G) - R'$ that is adjacent to a vertex in R'. If cx is an edge for $c \in V(G) - R'$ then $\{x, c, b, w\}$ induces a claw as b and w are frozen. Therefore, we can also add x to F and, without loss of generality, cz is an edge. But now $czwxb$ is an induced P_5, which is a contradiction. □

Lemma 4. *Let G be a 2-connected graph of order $n \geq 7$. If G is $\{K_{1,3}, P_5\}$-free and G contains a chorded C_5, then it also contains a chorded C_4.*

Proof. Let G be a 2-connected graph of order $n \geq 7$ that is $\{K_{1,3}, P_5\}$-free, and let $C = vwxyzv$ be a 5-cycle with chord vx. Suppose, for the sake of contradiction, that G has no chorded C_4.

We show that both x and v can be added to F. If $ax \in E(G)$ for some $a \in V(G) - V(C)$, then to avoid a claw there must be some edge among $\{w, a, y\}$. The edge aw gives a 4-cycle $vwaxv$ with chord wx and the edge wy gives a 4-cycle $wyxvw$ with chord wx. Thus ay must be an edge of G. Any further edge among $\{v, w, x, y, z, a\}$ yields a chorded 4-cycle, but such an edge must exist because otherwise $ayzvw$ is an induced P_5. Therefore, we conclude that x (and v by symmetry) can be added to F.

Because G is 2-connected, there is a tab on C. Therefore we may assume that ay (or az by symmetry) is an edge for some $a \in V(G) - V(C)$. To avoid an induced claw on $\{y, a, x, z\}$, we must have the edge az. By symmetry with $\{v, x\}$, both z and y can now be added to F. Because $n > 6$, there is a vertex b adjacent to either a or w. If ba is an edge then $bazvx$ is an induced P_5, and if bw is an edge then $bwvzy$ is an induced P_5. In either case we arrive at a contradiction, so we have proven the lemma. □

Theorem 10. *Let G be a 2-connected graph of order $n \geq 8$. If G is $\{K_{1,3}, P_5\}$-free, then G is chorded pancyclic.*

Proof. From Theorem 4 we know G must be pancyclic. Then by applying Lemma 3 followed by Lemma 4, we find a chorded m-cycle in G for $m = 4, 5$. Any chordless m-cycle for $m > 5$ contains an induced P_5. Therefore G contains a chorded m-cycle for $4 \leq m \leq n$. □

Note that the graph in Fig. 6 shows that Theorem 10 is sharp. We now turn to P_6 as a forbidden subgraph. We will prove our result using three lemmas.

Lemma 5. *Let G be a 2-connected graph of order $n \geq 11$. If G is $\{K_{1,3}, P_6\}$-free and G contains a C_4, then G contains a chorded C_5.*

Proof. Let G be a 2-connected graph of order $n \geq 11$ that is $\{K_{1,3}, P_6\}$-free, and let $C = vwxyv$ be a 4-cycle in G. Suppose, for the sake of contradiction, that G does not have a chorded C_5. Because G is 2-connected and $n > 5$, there is a k-tab on the cycle C. We choose a k-tab $Q = a_0a_1 \cdots a_{k+1}$ that minimizes k. Over all minimal k-tabs, choose one where a_0a_{k+1} is an edge of C if possible. If $k \geq 4$, then let a_0' be a neighbor of a_0 on C which is not a_{k+1} and now $a_0'a_0a_1 \cdots a_4$ is an induced P_6. Therefore we may assume that $1 \leq k \leq 3$.

Case 1. Suppose that Q is a 1-tab.

If a_0a_2 is an edge of C then $G[\{v, w, x, y, a_1\}]$ contains a chorded 5-cycle. Otherwise, we may assume $a_0 = v$ and $a_2 = x$. To avoid a claw centered at v or a 1-tab with endpoints adjacent in C it follows that wy is an edge. Now va_1xywv is a 5-cycle with chord wx.

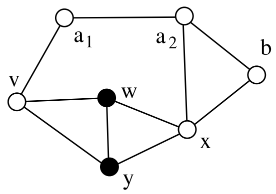

Fig. 7. The graph $G[R]$ is the starting point for Subcase 2.2 of Lemma 5.

Case 2. Suppose that Q is a 2-tab.

First, we show that a_0a_3 cannot be an edge in C. If a_0a_3 is an edge, we may assume it is vw. To avoid a claw induced by $\{v, w, y, a_1\}$ or a 1-tab on C, we must have wy as an edge. But now va_1a_2wyv is a 5-cycle with chord vw. Therefore we will assume that $v = a_0$ and $x = a_3$.

Note that wy must be an edge to avoid an induced claw $\{v, w, y, a_1\}$ or a 1-tab on C. If w (or y by symmetry) has another neighbor $z \in V(G) - V(C)$ then to avoid an induced claw $\{w, z, v, x\}$ or a 1-tab on C we must have the edge vx. But now va_1a_2xwv is a 5-cycle with chord vx. Therefore, w and y are added to F for the remainder of this case.

Subcase 2.1. Suppose that v and x can be added to F so that $F = \{w, y, v, x\}$.

Because G is 2-connected and $n \geq 8$, there exist distinct vertices b_1 and b_2 so that b_1a_1 and b_2a_2 are edges. To avoid an induced claw on either $\{a_1, b_1, v, a_2\}$ or $\{a_2, b_2, v, a_1\}$ we must also have the edges b_1a_2 and b_2a_1. Also $\{a_1, b_1, b_2, v\}$ cannot be an induced claw so b_1b_2 is an edge.

Because $n > 8$ there is another vertex c. If ca_1 (or ca_2 by symmetry) is an edge, then either $\{a_1, c, b_1, v\}$ is an induced claw or $a_1cb_1b_2a_2a_1$ is a 5-cycle with chord a_1b_1. Thus, without loss of generality we let cb_1 be an edge. But now cba_1vyx is an induced P_6, which is a contradiction.

Subcase 2.2. Suppose that at least one of v and x cannot be added to F.

In particular, we will assume that bx is an edge for a vertex $b \notin V(C) \cup V(Q)$. The current state of G (except for the edge ba_2 which we show next) is given in Fig. 7. Let $R = \{v, w, x, y, a_1, a_2, b\}$. Because $w, y \in F$, the neighborhood of x is not traceable and, by Lemma 1, $N_G(x)$ must be two disjoint cliques. Therefore ba_2 is an edge.

First, we show that ba_1 cannot be an edge. If ba_1 is an edge, then $G[V(C)]$ and $G[\{x, a_2, b, a_1\}]$ are both isomorphic to $K_4 - e$. By symmetry, we may add a_2 and b to F. Because $N_G(x)$ is two disjoint cliques, we may also add x to F. Because G is 2-connected and $n \geq 9$, there exist distinct vertices $c, c' \in V(G) - R$ so that cv and $c'a_1$ are edges. To avoid the induced claw $\{a_1, c', v, b\}$ we also have the edge $c'v$. By Lemma 1 the neighborhood of v is two disjoint cliques, as it is not traceable. Thus cc' and ca_1 are also edges. If $N_G(v)$ contains a K_4, then G has a chorded 5-cycle, so v must now be added to F. By symmetry, a_1 is also added to F. If G has another vertex d then either $dcvyxb$ or $dc'vyxb$ is an induced P_6, so $n = 9$ which is a contradiction. Therefore ba_1 is not an edge, and to avoid any chorded 5-cycle it follows that the induced subgraph $G[R]$ is exactly the graph shown in Fig. 7.

Now we show that b cannot be added to F yet. Suppose that b is added to F, so that $F = \{w, y, b\}$. Because $N_G(x)$ is two disjoint cliques, we may also add x to F. Either v can now be added to F, or v has another neighbor c. If vc is an edge then, because $N_G(v)$ is two disjoint cliques, ca_1 is an edge. By symmetry with b, x we can add c and v to F. However, regardless of whether v has degree 3 or 4 in G, since G is 2-connected and $n > 8$ we may assume that a_1 has a neighbor $d \notin R$. But now da_1vyxb is an induced P_6, which is a contradiction.

Because b cannot be added to F, it follows that b has a neighbor $b' \in V(G) - R$. Now $b'bxyva_1$ cannot induce a P_6 so b' has another neighbor among these vertices. Note that $b'a_1$ creates a 5-cycle $b'a_1a_2xbb'$ with chord ba_2 so this cannot be an edge. Thus, the edge $b'v$ would create an induced claw $\{v, b', a_1, y\}$, and it follows that $b'x$ must be an edge. Because $N_G(x)$ is two disjoint cliques, $b'a_2$ is an edge. We can now add x to F, because otherwise one of the cliques in $N_G(x)$ would be a K_4 which gives a chorded 5-cycle in $N_G[x]$. If b has a neighbor $c \notin R \cup \{b'\}$, then to prevent $cbxyva_1$ being an induced P_6 either cv or ca_1 is an edge. When ca_1 is an edge then $ca_1a_2b'bc$ is a 5-cycle with chord a_2b, and when ca_1 is not an edge then cv is an edge and $\{v, c, a_1, y\}$ induces a claw. Therefore b (and b' by symmetry) is also added to F.

Now the only unfrozen vertices are v, a_1, and a_2. We claim that a_2 can also be added to F. Suppose instead that a_2 has a neighbor $c \notin R \cup \{b'\}$. Then ca_1 is also an edge to prevent $\{a_2, c, a_1, x\}$ from inducing a claw. Also, ca_1vyxb is not an induced P_6 so cv is an edge. Now $G[\{v, a_1, a_2, c\}]$ is isomorphic to $G[V(C)]$, so by symmetry $\{c, a_1, a_2\}$ are added to F. Because G has no cut vertex, v is also added to F. All vertices are frozen, so it follows that $n = 9$ which is a contradiction.

Therefore a_2 is added to F, as seen in Fig. 8. Because G is 2-connected and $n \geq 10$, there exist distinct vertices c and c' so that cv and $c'a_1$ are edges. To avoid the induced claw $\{a_1, c', v, a_2\}$ we also have the edge $c'v$. But $N_G(v)$ is two disjoint cliques, so cc' and ca_1 are also edges. By symmetry with $G[x, b, b', a_2]$, the vertices v, c, c', a_1 are added to F. Thus $F = V(G)$, and it follows that $n = 10$ which is a contradiction.

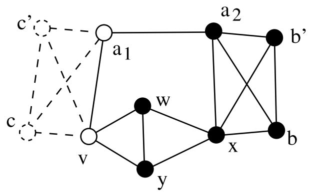

Fig. 8. The graph resulting from Subcase 2.2 in Lemma 5.

Case 3. Suppose that Q is a 3-tab.

Recall that $C = vwxyv$, and that no vertices are in F yet. Let $R = \{v, w, x, y, a_1, a_2, a_3\}$. Note that no 4-cycle can have a 1-tab or a 2-tab, or we reduce to an earlier case.

Subcase 3.1. Suppose that a_0a_4 is an edge of C.

In particular, we may assume that $a_0 = v$ and $a_4 = w$. To prevent $\{v, w, y, a_1\}$ and $\{w, v, x, a_3\}$ from inducing claws without introducing a chorded 5-cycle, vx and wy must be edges. Note that if R induces any other edge in G then there is a 1-tab or 2-tab on C, which is a contradiction. We add y (and x by symmetry) to F by the following argument. If by is an edge, then to prevent $byva_1a_2a_3$ being an induced P_6 there must be another neighbor of b. Because bv, ba_1, and ba_3 all create 1- or 2-tabs it follows that only ba_2 can be an edge. But then $\{a_2, a_1, a_3, b\}$ induces a claw, because a_1a_3 would make a 2-tab on C.

We now show that v and w can also be added to F. If v has a neighbor $b \in V(G) - R$ then, to prevent the induced claw $\{v, b, a_1, y\}$, ba_1 is also an edge. To prevent $ba_1a_2a_3wx$ being an induced P_6 there must be another neighbor of b among those vertices. However, bw and ba_3 create 1- and 2-tabs on C. Therefore ba_2 is an edge, but now a_2a_3wv is a 2-tab on the 4-cycle $a_2a_1vba_2$, which is a contradiction. Therefore $F = \{x, y, v, w\}$.

Because G is 2-connected and $n > 7$ there is a tab on $G[R]$. Because only a_1, a_2, a_3 are unfrozen we may assume that ba_1 (or ba_3 by symmetry) is an edge for some $b \in V(G) - R$. To prevent an induced claw on $\{a_1, b, a_2, v\}$, we must also have the edge ba_2. If ba_3 were also an edge then a_1vwa_3 would be a 2-tab on the 4-cycle $a_1a_2a_3ba_1$, so ba_3 cannot be an edge. But now a_1 can be added to F by the following argument. If a_1 has another neighbor $c \in V(G) - R$ distinct from b then by the same reasoning, we must have the edge ca_2 and we cannot have the edge ca_3. By Lemma 1, the neighborhood of a_1 is two disjoint cliques so bc is also an edge. But now $G[\{a_1, a_2, b, c\}]$ is isomorphic to $G[V(C)]$ and by symmetry all of a_1, a_2, b, c are added to F. Because a_3 is not a cut-vertex in G it follows that $n = 9$, which is a contradiction. Therefore, once we have the edge ba_1 we can immediately add a_1 to F. Hence $F = \{x, y, v, w, a_1\}$.

Now b, a_2, and a_3 are the only unfrozen vertices. Suppose that a_3 has a neighbor $c \notin R \cup \{b\}$. Then if cb is not an edge then ca_3wva_1b induces a P_6, and if cb is an edge then $cba_1a_2a_3c$ is a 5-cycle with chord ba_2. Both are contradictions, so $\deg_G(a_3) = 2$ and a_3 is added to F. Because $n > 8$ there must be another vertex c, and now b and a_2 are the only unfrozen vertices. However, if cb is an edge then cba_1vwa_3 is an induced P_6 and if ca_2 is an edge then $\{a_2, c, a_1, a_3\}$ induces a claw. Both are contradictions, so the lemma is proven in this case.

Subcase 3.2. Suppose that a_0 and a_4 are not adjacent in C.

In particular, let $a_0 = v$ and $a_4 = x$. Recall that F is empty for now. To prevent $\{v, a_1, w, y\}$ from inducing a claw, and to avoid chorded 5-cycles, wy must be an edge. Note that now vx cannot be an edge, or we reduce to Subcase 3.1. Therefore w (and y by symmetry) cannot have any neighbor in $V(G) - R$ without inducing a claw or creating a chorded 5-cycle. Thus w and y are added to F.

Now we show that at least one element of $\{x, v\}$ cannot be added to F yet. Suppose instead that $F = \{w, y, x, v\}$. Then, because G is 2-connected, there is a tab on $G[R]$ and we may assume that a_1 (or a_3 by symmetry) has a neighbor $b \in V(G) - R$. To prevent the induced claw $\{a_1, b, a_2, v\}$ we must also have the edge ba_2. To prevent ba_1vyxa_3 from being an induced P_6 we must have the edge ba_3. But now $G[\{a_1, a_2, a_3, b\}]$ is isomorphic to $G[\{v, w, x, y\}]$, so by symmetry all vertices are added to F. This implies that $n = 8$, which is a contradiction. Therefore, at least one of x or v has a neighbor in $V(G) - R$.

Suppose, without loss of generality, that x has a neighbor $b \in V(G) - R$. To prevent the induced claw $\{x, b, a_3, y\}$ we must also have the edge ba_3. Also, $bxyva_1a_2$ cannot be an induced P_6, so b must have another neighbor among these vertices. The edges bv and ba_1 create shorter tabs on C, so it follows that ba_2 must be an edge (see Fig. 9).

By symmetry with w and y, the vertices a_3 and b are also added to F, so $F = \{w, y, a_3, b\}$. Because the neighborhood of x is two disjoint frozen cliques, x can also be added to F by Lemma 1. Because $n \geq 9$, there is a tab with endpoints in $\{v, a_1, a_2\}$. Without loss of generality, we may assume that cv is an edge. Now ca_1 must also be an edge to prevent $\{v, c, a_1, y\}$ inducing a claw, and ca_2 must be an edge to prevent $cvyxba_2$ from being an induced P_6. But now $G[\{a_1, a_2, v, c\}]$ is isomorphic to $G[V(C)]$ and all vertices can be added to F. Therefore $n = 9$, which is a contradiction. □

Lemma 6. *Let G be a 2-connected graph of order $n \geq 10$. If G is $\{K_{1,3}, P_6\}$-free and G contains a chorded C_5, then it also contains a chorded C_4.*

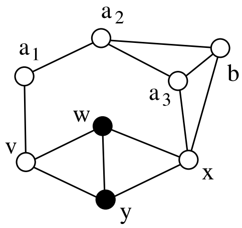

Fig. 9. A graph for Subcase 3.2 of Lemma 5.

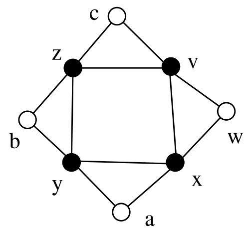

Fig. 10. For Subcase 2.1 of Lemma 6.

Proof. Let G be a 2-connected graph of order $n \geq 10$ that is $\{K_{1,3}, P_6\}$-free, and let $C = vwxyzv$ be a 5-cycle with chord vx. Suppose, for the sake of contradiction, that G does not have a chorded C_4. Then there are no additional edges in $G[V(C)]$ and, by Lemma 2, $\Delta(G) \leq 4$. Because $n > 5$, there is a vertex $a \in V(G) - V(C)$ adjacent to a vertex of C.

Case 1. Suppose that $F = \{x, v\}$.

Because there is a k-tab on C, we may assume without loss of generality that ay is an edge. To avoid an induced claw on $\{y, x, z, a\}$ it follows that az must also be an edge of G. By symmetry we may now add z and y to F. Because $n > 7$, to avoid a cut-vertex in G there must be two distinct vertices b and c where ab and wc are edges in G.

If aw is an edge, then to avoid a claw on $\{a, w, b, y\}$, we must also have the edge wb. But then $\deg_G(w) \geq 5$ which contradicts $\Delta(G) \leq 4$. Therefore, aw is not an edge in G.

If ac is an edge, then to avoid a claw on $\{a, b, c, z\}$, we must also have the edge bc. Now $bcwxyz$ is an induced P_6 unless bw is an edge. However, if bw is an edge then $bwcab$ is a 4-cycle with chord bc. Therefore ac (and by symmetry bw) is not an edge in G.

Now $bayxwc$ is an induced P_6 if bc is not an edge, and $cbayxv$ is an induced P_6 if bc is an edge.

Case 2. Suppose that at least one of v and x cannot be added to F.

In particular, we will assume without loss of generality that ax is an edge in G. To avoid chorded 4-cycles or $\{x, y, a, w\}$ inducing a claw, ay must also be an edge in G. Because $\Delta(G) \leq 4$ we can now let $F = \{x\}$. Let $R = \{a, v, w, x, y, z\}$ and note that any additional edge in $G[R]$ creates a chorded C_4. Because $n > 6$ there must be a vertex $b \in V(G) - R$ adjacent to a vertex of R.

We cannot add both v and y to F by the following argument. If v and y are added to F then z must also be added to F, as the edge zb would create an induced claw on $\{z, b, y, v\}$. Then only w and a are unfrozen, so we may assume that ba is an edge. Now $bayzvw$ is an induced P_6 unless bw is also an edge. By symmetry with $\{v, y, z\}$, the vertices a, w, b are added to F. Thus $|V(G)| = 7$ which is a contradiction.

Therefore we may assume, without loss of generality, that by is an edge. To avoid $\{y, b, a, z\}$ inducing a claw or a chorded C_4, bz must also be an edge. Because $\Delta(G) \leq 4$, we add y to F so that $F = \{x, y\}$. Because $n > 7$ there is another vertex c adjacent to a vertex in $R \cup \{b\}$.

Subcase 2.1. Suppose that at least one of z and v cannot be added to F.

Then we may assume, without loss of generality, that cz is an edge. To avoid $\{z, c, b, v\}$ being a claw or a chorded C_4 on $\{z, c, b, y\}$, cv must also be an edge. By symmetry, we can add z and v to F (see Fig. 10). Because $n > 8$, there is an unpictured vertex d and, without loss of generality, we assume that cd is an edge.

Since G does not contain a chorded C_4, cw is not an edge. But then dw is an edge, because otherwise $dczyxw$ is an induced P_6. Similarly, db must be an edge to avoid $dcvxyb$ being an induced P_6. Now ad is also an edge, because otherwise $axvzbd$ is an induced P_6. We add d to F because $\Delta(G) \leq 4$. Thus $F = \{x, y, z, v, d\}$. Now $\{d, b, c, w\}$ induces a claw, and because $dbzcd$ and $dcvwd$ are 4-cycles that are not chorded, it follows that bw is an edge. Similarly, to prevent $\{d, a, b, c\}$ from inducing a claw, we must have the edge ac. But now every vertex of G has 4 neighbors, so $F = V(G)$. Thus $n = 9$, which is a contradiction.

Subcase 2.2. Suppose that both z and v can be added to F.

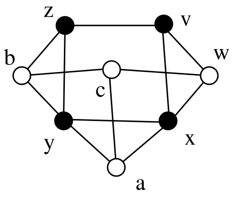

Fig. 11. For Subcase 2.2 of Lemma 6.

The current state of G (except for the vertex c, which we describe next) is given in Fig. 11. Because G is 2-connected, there is a k-tab with endpoints in the set $\{b, a, w\}$. Thus, without loss of generality, we may assume that cb is an edge. To avoid $cbzvxa$ inducing a P_6, either ba or ca must be an edge. However, ba gives a chorded C_4 so we may assume that ca is an edge. Because $cayzvw$ cannot be an induced P_6, and aw creates a chorded C_4, it follows that cw must also be an edge (see Fig. 11).

But now $\{c, a, b, w\}$ cannot induce a claw, so G contains at least one of $\{ab, aw, bw\}$. Because $cbyac$ and $caxwc$ are 4-cycles that are not chorded, it follows that bw is an edge. Since $\Delta(G) \leq 4$, we add b and w to F. The only unfrozen vertices are c and a. If c has a neighbor d then $dcbzvx$ is an induced P_6, and if a has a neighbor d then $dayzvw$ is an induced P_6. Therefore it follows that $n = 8$, which is a contradiction. □

Lemma 7. *Let G be a 2-connected graph of order $n \geq 13$. If G is $\{K_{1,3}, P_6\}$-free and G contains a chorded C_5, then it also contains a chorded C_6.*

Proof. Let G be a 2-connected graph of order $n \geq 13$ that is $\{K_{1,3}, P_6\}$-free, and let $C = vwxyzv$ be a C_5 with chord vx. Suppose, for the sake of contradiction, that G does not contain a chorded C_6. Because G is 2-connected and $n > 5$, there is a k-tab on the cycle C. We choose a tab $Q = a_0a_1 \cdots a_{k+1}$ that minimizes k. Over all minimal k-tabs, choose one which minimizes the distance from a_0 to a_{k+1} on C. If $k \geq 4$ then let a_0' be a neighbor of a_0 on C which is not a_{k+1}; now $a_0'a_0a_1 \cdots a_4$ is an induced P_6. Therefore $1 \leq k \leq 3$.

Case 1. Suppose that Q is a 1-tab.

If a_0a_2 is an edge of C, then this edge is the chord of a 6-cycle. Therefore a_0 and a_2 are not neighbors on C. If $a_0 = w$ then by symmetry we may assume that $a_2 = y$. Now wa_1yzvxw is a C_6 with chord vw. Therefore, we may assume without loss of generality that $a_0 = x$ and $a_2 = z$. By minimality of Q, neither of va_1 and ya_1 can be edges. Therefore vy is an edge to avoid $\{z, a_1, v, y\}$ inducing a claw. But now $vwxa_1zyv$ is a C_6 with chord vz.

Case 2. Suppose that Q is a 2-tab.

We first consider when one endpoint of Q, say a_0 is y (or z by symmetry). Because $\{y, a_1, z, x\}$ cannot induce a claw, zx is an edge by minimality of Q. If $a_3 = x$ or if $a_3 = z$ then we have a 6-cycle on the vertices $\{y, a_1, a_2, x, v, z\}$ with chord zx. If $a_3 = w$ then ya_1a_2wvxy is a 6-cycle with chord wx, and if $a_3 = v$ then ya_1a_2vwxy is a 6-cycle with chord vx. Therefore both a_0 and a_3 must be in $\{v, w, x\}$.

We may assume without loss of generality that a_0 is x (or v by symmetry). Now if $a_3 = v$ then xa_1a_2vzyx is a 6-cycle with chord vx, so it follows that $a_3 = w$. Because $\{x, y, v, a_1\}$ cannot induce a claw in G, and both ya_1 and va_1 create 1-tabs on C, it follows that vy must be an edge. But now xa_1a_2wvyx is a 6-cycle with chord vx.

Case 3. Suppose that Q is a 3-tab.

Let $R = \{v, w, x, y, z, a_1, a_2, a_3\}$. We first show that a_0 and a_4 cannot be neighbors in C. Suppose that a_0 and a_4 are neighbors in C. If $a_0 = y$ then to avoid a claw at y or a 1-tab on C, zx must be an edge. But now the edges $\{yx, zy, xz\}$ along with Q form a chorded 6-cycle. The same contradiction arises if $a_0 = z$. So if a_0 and a_4 are neighbors in C then they both come from the set $\{v, w, x\}$. But now the edges $\{vw, wx, xv\}$ along with Q form a chorded 6-cycle. Therefore we may assume that a_0 and a_4 are not neighbors in C.

Up to symmetry, the 3-tab Q has $a_4 = y$ and either $a_0 = w$ or $a_0 = v$. Note that any other edge between C and $\{a_1, a_2, a_3\}$ will create a 1- or 2-tab and contradict minimality. Suppose first that $a_0 = w$. To avoid an induced claw centered at y we must have the edge xz. Also, $vwa_1a_2a_3y$ cannot be an induced P_6 so there must be another edge among these vertices. Because wy creates a chorded 6-cycle $wa_1a_2a_3yxw$, we must have the edge vy. As wy is not an edge and $wa_1a_2a_3yz$ cannot be an induced P_6, we also have the edge zw. Now $G[\{v, w, x, y, z\}] = K_5 - wy$ where w and y are the endpoints of Q (see Fig. 12). If R induces any other edge in G, then there is a chorded 6-cycle.

We suppose instead that $a_0 = v$ and show that this gives an isomorphic graph to the graph in Fig. 12. Note that the edge yv would create a 6-cycle $yxva_1a_2a_3y$ with chord yv. To avoid 1-tabs or an induced claw on $\{y, a_3, x, z\}$ we must have the edge xz. To avoid $wva_1a_2a_3y$ being an induced P_6, we must have the edge wy (other edges would contradict the minimality of Q). Further, zw is an edge because $\{v, a_1, z, w\}$ cannot be an induced claw. Now $G[V(C)] = K_5 - e$, so we may continue using the notation used in Fig. 12.

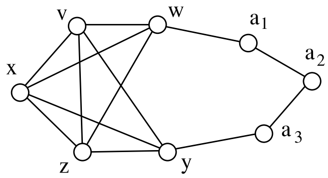

Fig. 12. The graph $G[R]$ in Case 3 of Lemma 7. Note that $G[V(C)] = K_5 - wy$.

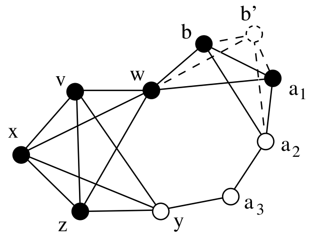

Fig. 13. This is $G[R']$. Only y, a_2, or a_3 can have neighbors in $G - R'$.

We claim that x can be added to F (and v and z by symmetry). Suppose instead that there is a vertex $b \in V(G) - R$ with edge bx. Because $\{x, b, w, y\}$ cannot be an induced claw, either bw or by is an edge. In the first case $xbwzvyx$ is a 6-cycle with chord xw, and in the second case $xbyvzwx$ is a 6-cycle with chord xy. Thus, x, v, z are all added to F.

Subcase 3.1. Suppose that w and y can be added to F.

Now $F = \{x, v, z, w, y\}$. There is a tab on $G[R]$ because G is 2-connected and $n > 8$. Thus, without loss of generality, there is a vertex b such that ba_1 (or ba_3 by symmetry) is an edge. To avoid an induced claw $\{a_1, w, a_2, b\}$, we must also have the edge ba_2. Because ba_2a_3yxw cannot be an induced P_6 and $y, x, w \in F$, we also have the edge ba_3. Because there is no 2-tab on C, $G[R \cup \{b\}]$ induces no other edges. However $n > 9$, so there must be another vertex c. If a_1 and a_3 can now be added to F, then $\{a_1, a_3, c\}$ and their common neighbor induce a claw. So we may assume that ca_1 is an edge (or ca_3 by symmetry). To avoid inducing claws with $\{a_1, c, w, a_2\}$ or $\{a_1, c, w, b\}$ both ca_2 and cb must be edges. Because cba_3yxw cannot be an induced P_6, ca_3 must also be an edge. Now $G[\{a_1, a_2, a_3, b, c\}] = K_5 - e$ and by symmetry, $F = V(G)$. Therefore $n = 10$, which is a contradiction.

Subcase 3.2. Suppose that at least one of w and y cannot be added to F.

Recall that the current state of G is described in Fig. 12 and that $F = \{v, x, z\}$. We may assume without loss of generality that bw is an edge in G for $b \in V(G) - R$. To avoid $\{w, b, v, a_1\}$ inducing a claw, ba_1 must also be an edge. Now $ba_1a_2a_3yz$ is an induced P_6 unless b has another neighbor in R. Because by and ba_3 create 1- and 2-tabs respectively, we must have the edge ba_2. Note that b has no other neighbors in R.

We claim that, if b has a neighbor $b' \in V(G) - R$, then $N_G[b] = N_G[b']$. Suppose that b has a neighbor $b' \in V(G) - R$. Note that $b'y$ makes $wbb'y$ a 2-tab and that $b'a_3$ gives a 6-cycle $b'a_3a_2a_1wbb'$ with chord a_1b, so neither of these can be edges. Thus $b'bwxya_3$ is an induced P_6 unless $b'w$ is an edge. But now $\{w, v, a_1, b'\}$ is an induced claw unless $b'a_1$ is an edge, and also $b'wxya_3a_2$ is an induced P_6 unless $b'a_2$ is an edge. Therefore $N_G[b] = N_G[b']$.

If $\deg_G(b) = 3$ then b (and a_1 by symmetry) is now added to F. If b has a neighbor b' in $V(G) - R$ then $G[w, b, b', a_1, a_2] = K_5 - wa_2$. Thus, by symmetry with $G[V(C)]$ we conclude that b, b', and a_1 are all added to F now. Whether or not b' exists, call $R' = R \cup N_G[b]$. If w has a neighbor $c \in V(G) - R'$ then $\{w, c, v, b\}$ is an induced claw because both v and b are in F. Therefore w is added to F now and the graph $G[R']$ is shown in Fig. 13, where the solid vertices (and b', if it exists) are all in F.

Because $n > 10$ there must be a vertex $c \in V(G) - R'$. The only unfrozen vertices are a_2, a_3 and y. If ca_2 is not an edge, then cy and ca_3 must both be edges because G is claw-free. But then $cyxwba_2$ would be an induced P_6. Therefore, ca_2 must be an edge. To avoid $\{a_2, c, a_3, b\}$ inducing a claw, ca_3 is also an edge. Also cy is an edge, because ca_2bwxy cannot be an induced P_6.

Suppose that c can be added to F (and a_3 as well, by symmetry). Because $n > 11$ there is another vertex d in G. However the edge yd makes $\{y, d, z, c\}$ induce a claw, and the edge ya_2 makes $\{a_2, d, a_1, c\}$ induce a claw. Therefore c cannot be added to F, and must have a neighbor $c' \in V(G) - R'$. Because $c'cyxwb$ and $c'ca_2bwx$ cannot be induced P_6 subgraphs, both $c'y$ and $c'a_2$ must be edges. Now $\{y, c', z, a_3\}$ cannot induce a claw, so $c'a_3$ is also an edge. However, we now have $G[y, c, c', a_3, a_2] = K_5 - ya_2$ and by symmetry $V(G) = F$. It follows that $n = 12$ (or $= 11$ if b' does not exist). This is a contradiction because $n \geq 13$. □

Theorem 11. *Let G be a 2-connected graph of order $n \geq 13$. If G is $\{K_{1,3}, P_6\}$-free then G is chorded pancyclic.*

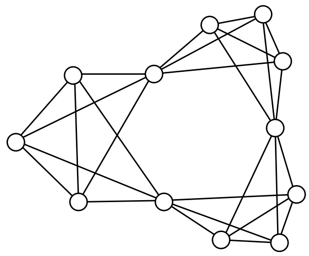

Fig. 14. This graph has no chorded C_6, which shows that Theorem 11 is sharp.

Proof. By Theorem 5, we know that G is pancyclic. Then by applying Lemma 5, followed by Lemmas 6 and 7, we find a chorded m-cycle in G for $4 \leq m \leq 6$. Any chordless m-cycle for $m > 6$ contains an induced P_6. Therefore G contains a chorded m-cycle for $4 \leq m \leq n$. □

Fig. 14 shows a 12-vertex, 2-connected, claw-free, and P_6-free graph which is not chorded pancyclic, because there is no chorded 6-cycle. This proves that Theorem 11 is sharp.

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