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On vertex-disjoint cycles and degree sum conditions

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ABSTRACT

This paper considers a degree sum condition sufficient to imply the existence of k vertex-disjoint cycles in a graph G. For an integer $t \geq 1$, let $\sigma_t(G)$ be the smallest sum of degrees of t independent vertices of G. We prove that if G has order at least $7k+1$ and $\sigma_4(G) \geq 8k-3$, with $k \geq 2$, then G contains k vertex-disjoint cycles. We also show that the degree sum condition on $\sigma_4(G)$ is sharp and conjecture a degree sum condition on $\sigma_t(G)$ sufficient to imply G contains k vertex-disjoint cycles for $k \geq 2$.

1. Introduction

In this paper, all graphs are simple. Let G be a graph. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$. Let H be a subgraph of G, and let $S \subseteq V(G)$. For $u \in V(G)-V(H)$, we denote $N_H(u) = N_G(u) \cap V(H)$ and $d_H(u) = |N_H(u)|$. For $u \in V(G) - S$, $N_S(u) = N_G(u) \cap S$. Furthermore, $N_G(S) = \cup_{w \in S} N_G(w)$ and $N_H(S) = N_G(S) \cap V(H)$. Let A, B be two disjoint subgraphs of G. Then $N_G(A) = N_G(V(A))$ and $N_B(A) = N_G(A) \cap V(B)$. The subgraph of G induced by S is denoted by $\langle S \rangle$. And let $G - S = \langle V(G) - S \rangle$ and $G - H = \langle V(G) - V(H) \rangle$. If $S = \{u\}$, then we write $G - u$ for $G - S$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For graphs G_1 and G_2, $G_1 \cup G_2$ denotes the union of G_1 and G_2, $G_1 + G_2$ denotes the join of G_1 and G_2, and mG denotes the union of m copies of G. If Q is a path or a cycle with a given orientation and $x \in V(Q)$, then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of Q from x to y (including x and y) in the given direction. The notation $Q^-[x, y]$ denotes the path from y to x in the opposite direction. We also write $Q(x, y] = Q[x^+, y]$, $Q[x, y) = Q[x, y^-]$ and $Q(x, y) = Q[x^+, y^-]$. If Q is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t(, x_1)$, then we assume that an orientation of Q is given from x_1 to x_t. We say that x_i *precedes* x_j on Q if $i \leq j$. For $u, v \in V(Q)$, we define the path $Q^{\pm}[u, v]$ as follows; if u precedes v on Q, then $Q^{\pm}[u, v] = Q[u, v]$, and if v precedes u on Q, then $Q^{\pm}[u, v] = Q^-[u, v]$. If T is a tree with at least one branch and $x, y \in V(T)$, where a branch vertex of a tree is a vertex of degree at least three, then we denote the path from x to y as $T[x, y]$. For $X \subseteq V(G)$, let $d_H(X) = \sum_{x \in X} d_H(x)$. If $H = G$, then we denote $d_G(X) = d_H(X)$. For a graph G, $|G|$ is the order of G, $\delta(G)$ is the minimum degree of G, $\omega(G)$ is the number of components of G, $\alpha(G)$ is the independence number of G. If G is one vertex, that is, $V(G) = \{x\}$, then we simply write x instead of G. For an integer $t \geq 1$, let

$$\sigma_t(G) = \min \left\{ \sum_{v \in X} d_G(v) \,|\, X \text{ is an independent set of } G \text{ with } |X| = t. \right\},$$

and $\sigma_t(G) = \infty$ when $\alpha(G) < t$. Note that if $t = 1$, then $\sigma_1(G) = \delta(G)$. For an integer $r \geq 1$ and two disjoint subgraphs A, B of G, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from A to B such that $d_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. In

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this paper, since it is sufficient to consider the case of equality in the above inequality, when we write $(d_1, d_2, \ldots, d_r)$, we assume that $d_B(v_i) = d_i$ for each $1 \le i \le r$. For $X, Y \subseteq V(G)$, $E(X, Y)$ denote the set of edges of G joining a vertex in X and a vertex in Y. For vertex-disjoint subgraphs H_1, H_2 of G, we simply write $E(H_1, H_2)$ instead of $E(V(H_1), V(H_2))$. A *forest* is a graph whose components are trees, and a *leaf* is a vertex of a forest whose degree is at most one. A cycle of length ℓ is called an *ℓ-cycle*. For terminology and notation not defined here, see [4].

The study of cycles in graphs is an important and rich area. In this paper, "disjoint" means "vertex-disjoint". One of the more interesting questions is to find conditions that insure the existence of k $(k \ge 2)$ disjoint cycles. A number of such results exist. Corrádi and Hajnal [1] proved that if a graph G has order at least $3k$ and $\delta(G) \ge 2k$, then G contains k disjoint cycles. Justesen [5] proved the same result from the condition $\sigma_2(G) \ge 4k$. Enomoto [2] and Wang [6] independently improved Justesen's bound to $\sigma_2(G) \ge 4k - 1$. Fujita et al. [3] proved that if $|G| \ge 3k+2$ and $\sigma_3(G) \ge 6k-2$, then G contains k disjoint cycles. The purpose of this paper is to further extend these results. We also conjecture the following:

Conjecture. *Let G be a graph of sufficiently large order. If $\sigma_t(G) \ge 2kt - (t-1)$ for any two integers $k \ge 2$ and $t \ge 1$, then G contains k disjoint cycles.*

The cases for $t = 1, 2, 3$ have already been shown. We add to the evidence for this conjecture by showing the following:

Theorem 1. *Let G be a graph of order $n \ge 7k + 1$ for an integer $k \ge 2$. If $\sigma_4(G) \ge 8k - 3$, then G contains k disjoint cycles.*

The degree sum condition conjectured above would be sharp. And in particular, the degree sum condition of Theorem 1 is sharp. Sharpness is given by $G = K_{2k-1} + mK_1$. The only independent vertices in G are those in mK_1. Each of these vertices has degree $2k - 1$. Thus, for any t with $1 \le t \le m$, $\sigma_t(G) = t(2k-1) = 2kt - t$, and G fails to contain k disjoint cycles as any such cycle must contain two vertices of K_{2k-1}.

2. Lemmas

In the proof of Theorem 1, we make use of the following Lemmas A, B and C that were proved by Fujita, Matsumura, Tsugaki and Yamashita in [3]. Proofs omitted in Chapter 2 appear after the proof of Theorem 1, that is, in Chapter 4.

Let $C_1, \ldots, C_r$ be r disjoint cycles of a graph G. If $C'_1, \ldots, C'_r$ are r disjoint cycles of G and $|\cup_{i=1}^r V(C'_i)| < |\cup_{i=1}^r V(C_i)|$, then we call $C'_1, \ldots, C'_r$ *shorter cycles* than $C_1, \ldots, C_r$. We also call $\{C_1, \ldots, C_r\}$ *minimal* if G does not contain shorter r disjoint cycles than $C_1, \ldots, C_r$.

Lemma A (*Fujita et al. [3]*). *Let r be a positive integer and $C_1, \ldots, C_r$ be r minimal disjoint cycles of a graph G. Then $d_{C_i}(x) \le 3$ for any $x \in V(G) - \cup_{i=1}^r V(C_i)$ and for any $1 \le i \le r$. Furthermore, $d_{C_i}(x) = 3$ implies $|C_i| = 3$, and $d_{C_i}(x) = 2$ implies $|C_i| \le 4$.*

Lemma B (*Fujita et al. [3]*). *Suppose that F is a forest with at least two components and C is a triangle. Let x_1, x_2, x_3 be leaves of F from at least two components. If $d_C(\{x_1, x_2, x_3\}) \ge 7$, then there exist two disjoint cycles in $\langle F \cup C\rangle$ or there exists a triangle C' in $\langle F \cup C\rangle$ such that $\omega(\langle F \cup C\rangle - C') < \omega(F)$.*

Lemma 1. *Suppose that F is a forest with at least two components and C is a triangle. Let x_1, x_2, x_3, x_4 be leaves of F from at least two components. If $d_C(\{x_1, x_2, x_3, x_4\}) \ge 9$, then there exist two disjoint cycles in $\langle F \cup C\rangle$ or there exists a triangle C' in $\langle F \cup C\rangle$ such that $\omega(\langle F \cup C\rangle - C') < \omega(F)$.*

Lemma C (*Fujita et al. [3]*). *Let C be a cycle and T be a tree with three leaves x_1, x_2, x_3. If $d_C(\{x_1, x_2, x_3\}) \ge 7$, then there exist two disjoint cycles in $\langle C \cup T\rangle$ or there exists a cycle C' in $\langle C \cup T\rangle$ such that $|C'| < |C|$.*

Lemma 2. *Let C be a cycle and T be a tree with four leaves x_1, x_2, x_3, x_4. If $d_C(\{x_1, x_2, x_3, x_4\}) \ge 9$, then there exist two disjoint cycles in $\langle C \cup T\rangle$ or there exists a cycle C' in $\langle C \cup T\rangle$ such that $|C'| < |C|$.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$. If $d_C(x_{i_0}) \le 2$ for some $1 \le i_0 \le 4$, then $d_C(X - \{x_{i_0}\}) \ge 7$, and we apply Lemma C to $X - \{x_{i_0}\}$. Otherwise, $d_C(x_i) \ge 3$ for each $1 \le i \le 4$, and we apply Lemma C to any three vertices in X. □

Lemma 3. *Let G be a graph satisfying the assumption of Theorem 1, and let $C_1, \ldots, C_{k-1}$ be $k-1$ minimal disjoint cycles of G. Suppose that there exists a tree T with at least four leaves, which is a component of $G - \cup_{i=1}^{k-1} C_i$. Then G contains k disjoint cycles.*

Proof. Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, and let $X = \{x_1, x_2, x_3, x_4\}$ be a set of leaves of T. Since X is an independent set, $d_{\mathscr{C}}(X) \ge (8k-3)-4 = 8(k-1)+1$. Then there exists a cycle C_i for some $1 \le i \le k-1$ such that $d_{C_i}(X) \ge 9$. Since $\{C_1, \ldots, C_{k-1}\}$ is minimal, there exist two disjoint cycles in $\langle C_i \cup T\rangle$ by Lemma 2. Thus G contains k disjoint cycles. □

Lemma 4. *Let G be a graph satisfying the assumption of Theorem 1, and let $C_1, \ldots, C_{k-1}$ be $k-1$ minimal disjoint cycles of G. Suppose that $H = G - \cup_{i=1}^{k-1} C_i$ has at least two components at least one of which is a tree T with at least three leaves. Then there exist two disjoint cycles in $\langle C_i \cup T\rangle$ for some $1 \le i \le k-1$ or there exists a triangle C in $\langle H \cup C_i\rangle$ such that $\omega(\langle H \cup C_i\rangle - C) < \omega(H)$.*

Proof. Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$. Let x_1, x_2, x_3 be three leaves of the tree T, and let x_4 be a leaf from another component, and $X = \{x_1, x_2, x_3, x_4\}$. Since X is an independent set, $d_{\mathscr{C}}(X) \geq (8k-3) - 4 = 8(k-1) + 1$. Then there exists a cycle C_i for some $1 \leq i \leq k-1$ such that $d_{C_i}(X) \geq 9$. If $d_{C_i}(x_4) \leq 2$, then $d_C(\{x_1, x_2, x_3\}) \geq 7$. By Lemma C, there exist two disjoint cycles in $\langle C_i \cup T\rangle$ or there exists a cycle C in $\langle C_i \cup T\rangle$ such that $|C| < |C_i|$. Since $\{C_1, \dots, C_{k-1}\}$ is minimal, the lemma holds. If $d_{C_i}(x_4) \geq 3$, then C_i is a triangle by Lemma A. Thus the lemma holds by Lemma 1. □

Lemma 5. *Let C_1 and C_2 be two disjoint cycles such that $|C_2| \geq 6$. Suppose that C_2 contains vertices with at least one of the following degree sequences from C_2 to C_1. Then $\langle C_1 \cup C_2\rangle$ contains two disjoint cycles C_1' and C_2' such that $|C_1'| + |C_2'| < |C_1| + |C_2|$.*
(i) (2, 2, 2, 2, 2)
(ii) (5, 3)
(iii) (3, 1, 1, 1, 1, 1)
(iv) (3, 2, 1, 1)
(v) (3, 3, 1)

Lemma 6. *Let H be a graph with two components H_1, H_2, where $H_1 = x_1, \dots, x_s$ $(s \geq 1)$ is a path and $H_2 = y_1, \dots, y_t$ $(t \geq 3)$ is a path. Let $W = \{x_1, y_1, y_i, y_t\}$ for any $2 \leq i \leq t-1$, and let C be a triangle. If there exists a degree sequence $(3, 3, 2, 0)$ or $(3, 3, 1, 1)$ from W to C, then $\langle H \cup C\rangle$ contains two disjoint cycles.*

3. Proof of Theorem 1

Suppose that the theorem does not hold. Let G be an edge-maximal counter-example. If G is a complete graph, then G contains k disjoint cycles. Thus we may assume that G is not a complete graph. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$. Since G' is not a counter-example by the maximality of G, G' contains k disjoint cycles $C_1, \dots, C_k$. Without loss of generality, we may assume that $xy \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ disjoint cycles $C_1, \dots, C_{k-1}$. Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$ and $H = G - \mathscr{C}$. Choose $C_1, \dots, C_{k-1}$ such that
(1) $\sum_{i=1}^{k-1} |C_i|$ is minimal, and
(2) subject to (1), $\omega(H)$ is minimal.

Note that any cycle C in $\mathscr{C}$ has no chords by (1). Clearly, H is a forest, otherwise, since H contains a cycle, G contains k disjoint cycles, a contradiction. If H contains at least two components at least one of which is a tree with at least three leaves, then by Lemma 4, G contains k disjoint cycles, or contradicting (2). Thus if H contains at least two components, then H must be a collection of paths. If H has only one component, then it is a tree. If H is a tree with at least four leaves, then the theorem holds by Lemma 3. Thus if H has only one component, then H is a tree with at most three leaves.

Now, we consider two cases on $|H|$.

Case 1. $|H| \leq 7$.

Let C be a longest cycle in $\mathscr{C}$. Suppose that $|C| \leq 7$. Then $|C'| \leq 7$ for any cycle C' in $\mathscr{C}$, and $|\mathscr{C}| \leq 7(k-1)$. Since $|G| \geq 7k+1$, $|H| = |G| - |\mathscr{C}| \geq (7k+1) - 7(k-1) = 8$, contradicting the assumption of this case. Thus $|C| \geq 8$. Let $|C| = 4t + r$, $t \geq 2$ and $0 \leq r \leq 3$. Then there exist at least t disjoint independent sets in $V(C)$ each of which has four vertices. By (1) and $|C| \geq 8$, $d_C(v) \leq 1$ for any $v \in V(H)$. Thus $|E(H, C)| \leq 7$.

Suppose that $k = 2$. Then $\mathscr{C}$ has only one cycle C, and $H = G - C$. Since $|C| \geq 8$, C contains at least two independent sets each of which has four vertices. Let X_1 and X_2 be such sets. Since $d_C(X_i) = 8$ for each $i \in \{1, 2\}$, $d_H(X_i) \geq (8k-3) - 8 = 8k - 11$. Then $d_H(X_1 \cup X_2) \geq 16k - 22 \geq 10$, since $k \geq 2$. Thus $|E(C, H)| \geq 10$, a contradiction.

Suppose that $k \geq 3$. We claim that $|E(C, C')| \geq 8t$ for some cycle C' in $\mathscr{C} - C$. Note that each of t disjoint independent sets in $V(C)$ sends at least $(8k-3) - 8 = 8k - 11$ edges out of C. Since $|E(C, H)| \leq 7$ and $t \geq 2$, $|E(C, \mathscr{C} - C)| \geq t(8k-11) - 7 > 8t(k-2)$. Thus the claim holds. Since $|C| = 4t + r \leq 4t + 3$ and $|E(C, C')|/|C| \geq 8t/(4t+3) > 8t(4t+4) = 2t/(t+1) > 1$, $d_{C'}(v) \geq 2$ for some $v \in V(C)$.

Suppose that $\max\{d_{C'}(v) | v \in V(C)\} = 2$. Let $X = \{v \in V(C) | d_{C'}(v) \leq 1\}$ and $Y = V(C) - X$. Then noting that $t \geq 2$ and $r \leq 3$,

$$\begin{aligned} 8t &\leq |E(C, C')| \leq |X| + 2|Y| = (|C| - |Y|) + 2|Y| = |C| + |Y| \\ &\Rightarrow |Y| \geq 8t - |C| = 8t - (4t + r) = 4t - r \\ &\geq 8 - 3 = 5. \end{aligned}$$

Thus we have the degree sequence (2, 2, 2, 2, 2) from C to C'. By Lemma 5(i), $\langle C \cup C'\rangle$ contains two shorter disjoint cycles, contradicting (1).

Suppose that $h = \max\{d_{C'}(v) | v \in V(C)\} \geq 3$. Let $d_{C'}(v^*) = h$ for some $v^* \in V(C)$. Since $|C'| \leq |C| = 4t + r$ by the choice of C, $d_{C'}(v^*) \leq |C'| \leq 4t + r$. Then since $t \geq 2$ and $r \leq 3$, $|E(C - v^*, C')| \geq 8t - (4t + r) = 4t - r \geq 5$. This implies that $N_{C'}(C - v^*) \neq \emptyset$. Let $Z = \{v \in V(C) | N_{C'}(v) \neq \emptyset\}$. Then $|Z| \geq 2$.

Suppose that $|Z| = 2$. Then $d_{C'}(v) \geq 5$ for any $v \in Z$ by the above observations. By Lemma 5(ii), $\langle C \cup C'\rangle$ contains two shorter disjoint cycles, contradicting (1).

Suppose that $|Z| \geq 3$. Since $|E(C - v^*, C')| \geq 5$, we may assume that the minimum degree sequence S from vertices of C to C' is at least one of $(h, 4, 1), (h, 3, 2), (h, 3, 1, 1), (h, 2, 2, 1), (h, 2, 1, 1, 1)$, or $(h, 1, 1, 1, 1, 1)$, where by the definition of h, if $S = (h, 4, 1)$, then $h \geq 4$, and if S is the other degree sequence, then $h \geq 3$. If $S = (h, 4, 1)$ or $(h, 3, 2)$, then by Lemma 5(v), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles. If $S = (h, 3, 1, 1)$, $(h, 2, 2, 1)$ or $(h, 2, 1, 1, 1)$, then by Lemma 5(iv), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles. If $S = (h, 1, 1, 1, 1, 1)$, then by Lemma 5(iii), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles.

Case 2. $|H| \geq 8$.

Claim 1. *H is connected.*

Proof. Suppose to the contrary that H is disconnected. Then note that H is a collection of paths. Suppose that X is an independent set that consists of four leaves from at least two components in H such that $d_H(X) \leq 4$. Then $d_{\mathscr{C}}(X) \geq (8k-3) - 4 = 8(k-1) + 1$, and $d_{C_{i_0}}(X) \geq 9$ for some $1 \leq i_0 \leq k-1$. Thus $d_{C_{i_0}}(x) \geq 3$ for some $x \in X$, and $|C_{i_0}| = 3$ by Lemma A. By Lemma 1 and (2), $\langle H \cup C_{i_0} \rangle$ contains two disjoint cycles, and G contains k disjoint cycles, a contradiction. Thus H does not contain such an independent set.

Now, we consider three cases on $\omega(H)$.

Case 1. $\omega(H) \geq 4$.

We take four leaves x_1, x_2, x_3, x_4, one from each component of H. Then $X = \{x_1, x_2, x_3, x_4\}$ is an independent set such that $d_H(X) \leq 4$, a contradiction.

Case 2. $\omega(H) = 3$.

We take three leaves x_1, x_2, x_3, one from each component of H. Since $|H| \geq 8$, some component of H, say H_1, has the order at least 3. Now, we take the other leaf from H_1, call it x_4. Then $X = \{x_1, x_2, x_3, x_4\}$ is an independent set such that $d_H(X) \leq 4$, a contradiction.

Case 3. $\omega(H) = 2$.

Let H_1, H_2 be two distinct components in H. Without loss of generality, we may assume that $|H_1| \leq |H_2|$. Suppose that $|H_1| \geq 3$. Then we take two leaves from each component of H, yielding a set X of four independent vertices such that $d_H(X) = 4$, a contradiction. Suppose that $|H_1| \in \{1, 2\}$. Since $|H| \geq 8$, $|H_2| \geq 6$. Let $H_1 = x_1, x_s$ $(s \in \{1, 2\})$, $H_2 = y_1, y_2, \ldots, y_t$ $(t \geq 6)$, and let $W = \{x_1, y_1, y_3, y_t\}$. Since W is an independent set and $d_H(W) \leq 5$, $d_{\mathscr{C}}(W) \geq (8k-3) - 5 = 8(k-1)$. Then there is a cycle C_0 in $\mathscr{C}$ such that $d_{C_0}(W) \geq 8$. By Lemma A, $d_{C_0}(u) \leq 3$ for any $u \in W$, and $|C_0| \leq 4$. Then the minimum possible degree sequence S from W to C_0 is $(3, 3, 2, 0)$, $(3, 3, 1, 1)$, $(3, 2, 2, 1)$ or $(2, 2, 2, 2)$.

Suppose that $|C_0| = 4$. Let $C_0 = v_1, v_2, v_3, v_4, v_1$. Then $d_{C_0}(u) \leq 2$ for any $u \in W$ by Lemma A. Thus we must have degree sequence $(2, 2, 2, 2)$. If some $u \in W$ has consecutive neighbors in C_0, then u and these two neighbors form a 3-cycle, contradicting (1). Thus for any $u \in W$, its neighbors in C_0 are not consecutive. It follows that for any $u \in W$, either $N_{C_0}(u) = \{v_1, v_3\}$ or $N_{C_0}(u) = \{v_2, v_4\}$. Without loss of generality, we may assume that $N_{C_0}(x_1) = \{v_1, v_3\}$. If y_{i_0}, y_{j_0} with some $i_0, j_0 \in \{1, 3, t\}$ and $i_0 < j_0$ do not share neighbors in C_0 with x_1, then we can easily find two disjoint cycles, as follows. Since $N_{C_0}(y_m) = \{v_2, v_4\}$ for each $m \in \{i_0, j_0\}$, $H_2[y_{i_0}, y_{j_0}], v_4, y_{i_0}$ is a cycle, and x_1, v_3, v_2, v_1, x_1 is the other disjoint cycle. Thus at most one vertex in $\{y_1, y_3, y_t\}$ does not share neighbors in C_0 with x_1. Suppose that some vertex in $\{y_1, y_3, y_t\}$ does not share neighbors in C_0 with x_1. First, suppose that such a vertex is y_1, that is, $N_{C_0}(y_1) = \{v_2, v_4\}$. Then y_1, v_4, v_3, v_2, y_1 is a cycle. Since $v_1 \in N_{C_0}(y_i)$ for each $i \in \{3, t\}$, $H_2[y_3, y_t], v_1, y_3$ is the other disjoint cycle. If $N_{C_0}(y_t) = \{v_2, v_4\}$, then y_t, v_4, v_3, v_2, y_t and $H_2[y_1, y_3], v_1, y_1$ are two disjoint cycles. Suppose that $N_{C_0}(y_3) = \{v_2, v_4\}$. Then we form a 4-cycle $C_0' = y_3, v_4, v_3, v_2, y_3$. Since $v_1 \in N_{C_0}(y_i)$ for each $i \in \{1, t\}$, $\langle H \cup C_0 \rangle - C_0'$ is connected, contradicting (2). Thus $N_{C_0}(x_1) = N_{C_0}(y_i)$ for each $i \in \{1, 3, t\}$. Then $C_0' = H_2[y_1, y_3], v_1, y_1$ is a 4-cycle. Since $v_3 \in N_{C_0}(u)$ for each $u \in \{x_1, y_t\}$, $\langle H \cup C_0 \rangle - C_0'$ is connected, contradicting (2). Thus if there exists a 4-cycle in $\mathscr{C}$, we get a contradiction.

Suppose that $|C_0| = 3$. Let $C_0 = v_1, v_2, v_3, v_1$.

Subcase 1. $S = (3, 3, 2, 0)$ or $S = (3, 3, 1, 1)$.

By Lemma 6, we can find two disjoint cycles in $\langle C_0 \cup H \rangle$, a contradiction.

Subcase 2. $S = (3, 2, 2, 1)$.

If $d_{C_0}(y_3) = 1$, then since $\{x_1, y_1, y_t\}$ satisfies the conditions of Lemma B, we get a contradiction. Thus $d_{C_0}(y_3) \in \{2, 3\}$.

First, suppose that $d_{C_0}(x_1) = 1$. Let $v_1 \in N_{C_0}(x_1)$. Note that $d_{C_0}(y_i) \geq 2$ for each $i \in \{1, 3, t\}$. If $v_1 \notin N_{C_0}(y_{i_0})$ for some $i_0 \in \{1, t\}$, then $d_{C_0}(y_{i_0}) = 2$, and $C_0' = y_{i_0}, v_3, v_2, y_{i_0}$ is a 3-cycle. Since $d_{C_0}(y_{i_1}) = 3$ for some $i_1 \in \{1, 3, t\} - \{i_0\}$, $v_1 \in N_{C_0}(y_{i_1})$. Then $\langle C_0 \cup H \rangle - C_0'$ is connected, contradicting (2) (see Fig. 1). Thus $v_1 \in N_{C_0}(y_i)$ for each $i \in \{1, t\}$. Since $d_{C_0}(y_{i_2}) = 3$ for some $i_2 \in \{1, 3, t\}$, $C_0'' = y_{i_2}, v_3, v_2, y_{i_2}$ is a 3-cycle. Then $\langle C_0 \cup H \rangle - C_0''$ is connected, contradicting (2).

Next, suppose that $d_{C_0}(x_1) = 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_{C_0}(x_1)$. Suppose that $d_{C_0}(y_3) = 2$. Since $|C_0| = 3$, we may assume that $v_1 \in N_{C_0}(x_1) \cap N_{C_0}(y_3)$. Since $d_{C_0}(y_{j_0}) = 3$ for some $j_0 \in \{1, t\}$, $C_0' = y_{j_0}, v_3, v_2, y_{j_0}$ is a 3-cycle. Then $\langle C_0 \cup H \rangle - C_0'$ is connected, contradicting (2). Suppose that $d_{C_0}(y_3) = 3$. If $v_3 \in N_{C_0}(y_{m_0})$ for some $m_0 \in \{1, t\}$, then $H_2^{\pm}[y_3, y_{m_0}], v_3, y_3$ and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus $v_3 \notin N_{C_0}(y_m)$ for each $m \in \{1, t\}$, that is, $N_{C_0}(y_m) \subseteq \{v_1, v_2\}$. Since one of y_1 and y_t has the degree 1 and the other has the degree 2, without loss of generality, we may assume that $v_1 \in N_{C_0}(y_1) \cap N_{C_0}(y_t)$. Since $d_{C_0}(y_3) = 3$, $C_0'' = y_3, v_3, v_2, y_3$ is a 3-cycle, and $\langle C_0 \cup H \rangle - C_0''$ is connected, contradicting (2) (see Fig. 2).

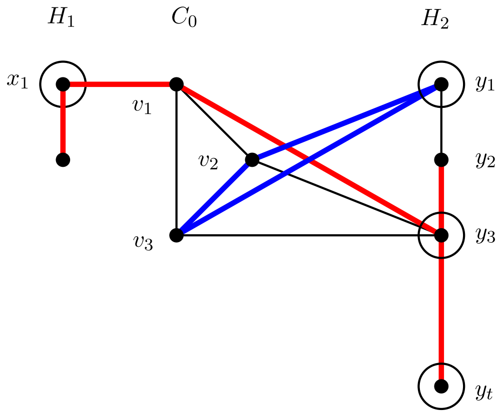

Fig. 1. The case when $i_0 = 1$ and $i_1 = 3$.

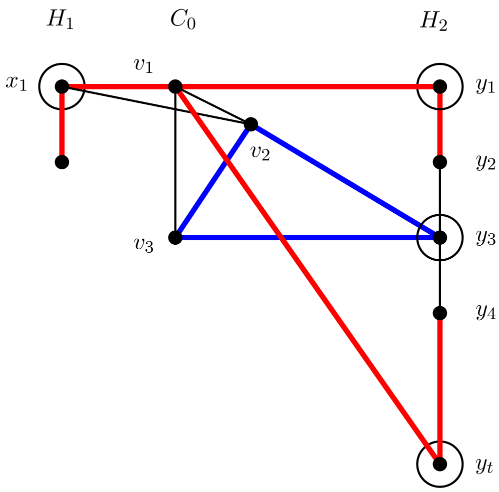

Fig. 2. The case when $v_1 \in N_{C_0}(y_1) \cap N_{C_0}(y_t)$.

Finally, suppose that $d_{C_0}(x_1) = 3$. Since $d_{C_0}(y_{i_0}) = d_{C_0}(y_{j_0}) = 2$ for some $i_0, j_0 \in \{1, 3, t\}$ with $i_0 < j_0$, we may assume that $v_1 \in N_{C_0}(y_{i_0}) \cap N_{C_0}(y_{j_0})$. Then $H_2[y_{i_0}, y_{j_0}], v_1, y_{i_0}$ is a cycle. Since $d_{C_0}(x_1) = 3$, x_1, v_3, v_2, x_1 is the other disjoint cycle.

Subcase 3. $S = (2, 2, 2, 2)$.

Without loss of generality, we may assume that $N_{C_0}(x_1) = \{v_1, v_2\}$. If $v_3 \in N_{C_0}(y_{i_0}) \cap N_{C_0}(y_{j_0})$ for some $i_0, j_0 \in \{1, 3, t\}$ with $i_0 < j_0$, then $H_2[y_{i_0}, y_{j_0}], v_3, y_{i_0}$ and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus at most one in $\{y_1, y_3, y_t\}$ can be adjacent to v_3. Suppose that $v_3 \in N_{C_0}(y_{i_0})$ for some $i_0 \in \{1, 3, t\}$. Since $d_{C_0}(y_{i_0}) = 2$, we may assume that $v_2 \in N_{C_0}(y_{i_0})$. Then $C'_0 = y_{i_0}, v_3, v_2, y_{i_0}$ is a 3-cycle. For each $i \in \{1, 3, t\} - \{i_0\}$, $N_{C_0}(y_i) = \{v_1, v_2\}$. Then $\langle C_0 \cup H\rangle - C'_0$ is connected, contradicting (2). Thus $v_3 \notin N_{C_0}(y_i)$ for each $i \in \{1, 3, t\}$, that is, $N_{C_0}(y_i) = \{v_1, v_2\}$. Then $C''_0 = H_2[y_1, y_3], v_2, y_1$ is a 3-cycle, and $\langle C_0 \cup H\rangle - C''_0$ is connected, contradicting (2). This completes the proof of Claim 1. □

Claim 2. *H is a path.*

Proof. Suppose that H is not a path. Then recall that H is a tree with one branch vertex of degree 3 in H. Then H has three leaves, say x_1, x_2, x_3. Removing the branch vertex in H, there exist three disjoint paths each of which has one in $\{x_1, x_2, x_3\}$ as an endpoint. Also, some path has a length at least two, say P, since there exist at least seven vertices distributed over three paths. Without loss of generality, we may assume that x_1 is one of the endpoints of P, and let the other endpoint be x_4. Let $X = \{x_1, x_2, x_3, x_4\}$ (see Fig. 3). Then X is an independent set. Since $d_H(X) = 5$, $d_{\mathscr{C}}(X) \geq (8k - 3) - 5 = 8(k - 1)$. Thus there exists a cycle C_{i_0} in $\mathscr{C}$ such that $d_{C_{i_0}}(X) \geq 8$ for some $1 \leq i_0 \leq k - 1$. Then $d_{C_{i_0}}(x) \geq 2$ for some $x \in X$. By Lemma A, $d_{C_{i_0}}(x) \leq 3$ and $|C_{i_0}| \leq 4$.

Case 1. $|C_{i_0}| = 3$.

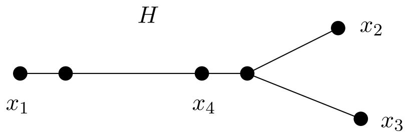

Fig. 3. The graph H and an independent set $X = \{x_1, x_2, x_3, x_4\}$.

Let $C_{i_0} = v_1, v_2, v_3, v_1$. Suppose that $d_{C_{i_0}}(x) = 2$ for each $x \in X$. Let $v_1, v_2 \in N_{C_{i_0}}(x_1)$. Since $|C_{i_0}| = 3$, $N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3) \neq \emptyset$. If $v_3 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3)$, then $H[x_2, x_3], v_3, x_2$ and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus without loss of generality, we may assume that $v_1 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3)$. Then $H[x_2, x_3], v_1, x_2$ is a cycle. Since $d_{C_{i_0}}(x_4) = 2$, $N_{C_{i_0}-v_1}(x_4) \neq \emptyset$. If $v_2 \in N_{C_{i_0}}(x_4)$, then $H[x_1, x_4], v_2, x_1$ is the other disjoint cycle, and if $v_3 \in N_{C_{i_0}}(x_4)$, then $H[x_1, x_4], v_3, v_2, x_1$ is the other disjoint cycle. Thus there exists at least one vertex $x \in X$ such that $d_{C_{i_0}}(x) = 3$. Then the minimum possible degree sequences from X to C_{i_0} are (3, 3, 2, 0), (3, 3, 1, 1) or (3, 2, 2, 1).

We claim that if there exists a degree sequence (3, 3, 1, 0) from X to C_{i_0}, then there exist two disjoint cycles in $\langle H \cup C_{i_0} \rangle$.

First, suppose that $d_{C_{i_0}}(x_{j_0}) = 1$ for some $1 \leq j_0 \leq 3$. Let $v_1 \in N_{C_{i_0}}(x_{j_0})$. If $d_{C_{i_0}}(x_4) = 0$, then since $d_{C_{i_0}}(x_m) = 3$ for each $m \in \{1, 2, 3\} - \{j_0\}$, $H[x_{j_0}, x_m], v_1, x_{j_0}$ is a cycle. Since $d_{C_{i_0}}(x_{m'}) = 3$ for $m' \in \{1, 2, 3\} - \{j_0, m\}$, $x_{m'}, v_3, v_2, x_{m'}$ is the other disjoint cycle. If $d_{C_{i_0}}(x_4) = 3$, then $H[x_{j_0}, x_4], v_1, x_{j_0}$ is a cycle, and since $d_{C_{i_0}}(x_{m_0}) = 3$ for some $m_0 \in \{1, 2, 3\} - \{j_0\}$, $x_{m_0}, v_3, v_2, x_{m_0}$ is the other disjoint cycle. Next, suppose that $d_{C_{i_0}}(x_4) = 1$. Let $v_1 \in N_{C_{i_0}}(x_4)$. Then $d_{C_{i_0}}(x_{m_1}) = 3$ and $d_{C_{i_0}}(x_{m_2}) = 3$ for some $1 \leq m_1 < m_2 \leq 3$, and $H[x_{m_1}, x_4], v_1, x_{m_1}$ and $x_{m_2}, v_3, v_2, x_{m_2}$ are two disjoint cycles.

Thus by the claim, we have only to consider the degree sequence (3, 2, 2, 1). If the degree 3 vertex does not lie on the path connecting the degree 2 vertices, then since the two vertices with degree 2 must have a common neighbor by $|C_{i_0}| = 3$, we can easily find two disjoint cycles. Thus the degree 3 vertex does lie on the path connecting the two vertices with degree 2. Then $d_{C_{i_0}}(x_4) = 3$, $d_{C_{i_0}}(x_1) = 2$, and we may assume that $d_{C_{i_0}}(x_2) = 1$ and $d_{C_{i_0}}(x_3) = 2$. Let $v_1 \in N_{C_{i_0}}(x_2)$. Since $|N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)| = 2$, there exists $v_{h_0} \in N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)$ for some $h_0 \in \{2, 3\}$. Then $H[x_1, x_4], v_{h_0}, x_1$ is a cycle. Since $d_{C_{i_0}}(x_3) = 2$, there exists $v_{h_1} \in N_{C_{i_0}}(x_3)$ for some $h_1 \in \{1, 2, 3\} - \{h_0\}$. If $h_1 = 1$, then $H[x_2, x_3], v_1, x_2$ is the other disjoint cycle, and if $h_1 \in \{2, 3\}$, then $H[x_2, x_3], v_{h_1}, v_1, x_2$ is the other disjoint cycle.

Case 2. $|C_{i_0}| = 4$.

Let $C_{i_0} = v_1, v_2, v_3, v_4, v_1$. By Lemma A, $d_{C_{i_0}}(x) \leq 2$ for each $x \in X$. Since $d_{C_{i_0}}(X) \geq 8$, $d_{C_{i_0}}(x) = 2$ for each $x \in X$. Any vertex in X does not have consecutive neighbors in C_{i_0}, otherwise, we can immediately find a 3-cycle, contradicting (1). Thus for each $x \in X$, either $N_{C_{i_0}}(x) = \{v_1, v_3\}$ or $N_{C_{i_0}}(x) = \{v_2, v_4\}$.

Subcase 1. All four vertices in X have the same two neighbors in C_{i_0}.

We may assume that $N_{C_{i_0}}(X) = \{v_1, v_3\}$. Then $H[x_1, x_4], v_1, x_1$ and $H[x_2, x_3], v_3, x_2$ are two disjoint cycles.

Subcase 2. Three vertices in X have the same two neighbors in C_{i_0}.

Suppose that x_1, x_4 have the same two neighbors in C_{i_0}. Then we may assume that $v_1 \in N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)$, and $H[x_1, x_4], v_1, x_1$ is a cycle. Since $d_{C_{i_0}}(x_j) = 2$ for each $j \in \{2, 3\}$, $N_{C_{i_0}-v_1}(x_j) \neq \emptyset$. Then $\langle H[x_2, x_3] \cup (C_{i_0} - v_1) \rangle$ contains the other disjoint cycle. Suppose that x_1, x_4 do not have the same two neighbors in C_{i_0}. Since x_2, x_3 have the same two neighbors in C_{i_0}, we repeat the above arguments, replacing x_1, x_4 with x_2, x_3.

Subcase 3. Two vertices of X have the same two neighbors in C_{i_0}, and the other two vertices of X have the same two neighbors, different from the neighbors of the first two.

Suppose that x_1, x_4 have the same two neighbors. We may assume that $v_1 \in N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)$. Then $H[x_1, x_4], v_1, x_1$ is a cycle. Since x_2, x_3 have the same two neighbors, different from the neighbors of x_1 and x_4, $H[x_2, x_3], v_2, x_2$ is the other disjoint cycle. Suppose that x_1, x_4 have different neighbors. We may assume that $v_1 \in N_{C_{i_0}}(x_1)$ and $v_2 \in N_{C_{i_0}}(x_4)$. Then $H[x_1, x_4], v_2, v_1, x_1$ is a cycle. Since x_2, x_3 have the neighbors, different from v_1, v_2, $\langle H[x_2, x_3] \cup \{v_3, v_4\} \rangle$ contains the other disjoint cycle. □

Since H is a path by Claim 2, let $H = x_1, x_2, \ldots, x_t$ ($t \geq 8$). Let $X = \{x_1, x_3, x_5, x_t\}$. Then X is an independent set with $d_H(X) = 6$, and $d_{\mathscr{C}}(X) \geq (8k - 3) - 6 = 8k - 9 \geq 7(k - 1)$, since $k \geq 2$. Thus either $d_{C_0}(X) \geq 8$ for some cycle C_0 in $\mathscr{C}$, or $d_C(X) = 7$ for every cycle C in $\mathscr{C}$. If $d_C(X) \geq 8$ for some cycle C in $\mathscr{C}$, then we have the minimum possible degree sequences (3, 3, 2, 0), (3, 3, 1, 1), (3, 2, 2, 1) or (2, 2, 2, 2) from X to C. If $d_C(X) = 7$ for some cycle C in $\mathscr{C}$, then we have the minimum possible degree sequences (3, 3, 1, 0), (3, 2, 1, 1), (3, 2, 2, 0) or (2, 2, 2, 1) from X to C.

Subclaim 1. *If there exists a degree sequence* (3, 3, 1, 0) *from X to C, then there exist two disjoint cycles in $\langle H \cup C \rangle$.*

Proof. By Lemma A, $|C| = 3$. Let $C = v_1, v_2, v_3, v_1$. We may assume that $d_C(x_{i_0}) = 1$ for some $i_0 \in \{1, 3\}$, otherwise, $i_0 \in \{5, t\}$, and we may argue in a similar manner from the other end of the path H. Let $v_1 \in N_C(x_{i_0})$. First, suppose that $i_0 = 1$, that is, $d_C(x_1) = 1$. Then $d_C(x_{j_1}) = d_C(x_{j_2}) = 3$ for some $j_1, j_2 \in \{3, 5, t\}$ with $j_1 < j_2$. Thus $H[x_1, x_{j_1}], v_1, x_1$ and

$x_{j_2}, v_3, v_2, x_{j_2}$ are two disjoint cycles. Next, suppose that $i_0 = 3$, that is, $d_C(x_3) = 1$. If $d_C(x_1) = 0$, then since $d_C(x_j) = 3$ for each $j \in \{5, t\}$, x_3, x_4, x_5, v_1, x_3 and x_t, v_3, v_2, x_t are two disjoint cycles. If $d_C(x_1) = 3$, then x_1, x_2, x_3, v_1, x_1 is a cycle, and since $d_C(x_{j_0}) = 3$ for some $j_0 \in \{5, t\}$, $x_{j_0}, v_3, v_2, x_{j_0}$ is the other disjoint cycle. □

Subclaim 2. *If there exists a degree sequence* $(2, 2, 2, 1)$ *from* X *to* C*, then there exist two disjoint cycles in* $\langle H \cup C \rangle$.

Proof. By Lemma A, $|C| \le 4$. Let $C = v_1, v_2, \ldots, v_q, v_1$, where $q = |C|$. We may assume that $d_C(x_{i_0}) = 1$ for some $i_0 \in \{5, t\}$, otherwise, $i_0 \in \{1, 3\}$, and we may argue in a similar manner from the other end of the path H. Let $v_1 \in N_C(x_{i_0})$.

Case 1. $N_C(x_1) \cap N_C(x_3) \ne \emptyset$.

First, suppose that $v_{j_0} \in N_{C-v_1}(x_1) \cap N_{C-v_1}(x_3)$ for some $2 \le j_0 \le q$. Then $x_1, x_2, x_3, v_{j_0}, x_1$ is a cycle. Since $d_C(x_r) = 2$ for $r \in \{5, t\} - \{i_0\}$, $N_{C-v_{j_0}}(x_r) \ne \emptyset$. Then $\langle H[x_5, x_t] \cup (C - v_{j_0}) \rangle$ contains the other disjoint cycle. Next, suppose that $v_1 \in N_C(x_1) \cap N_C(x_3)$. Then x_1, x_2, x_3, v_1, x_1 is a cycle. Since $d_C(x_r) = 2$ for $r \in \{5, t\} - \{i_0\}$, if $v_1 \notin N_C(x_r)$, then $\langle x_r \cup (C - v_1) \rangle$ contains the other disjoint cycle. Thus we may assume that $v_1 \in N_C(x_r)$. Then $H[x_5, x_t], v_1, x_5$ is a cycle. Since $d_C(x_i) = 2$ for each $i \in \{1, 3\}$, $N_{C-v_1}(x_i) \ne \emptyset$, and $\langle H[x_1, x_3] \cup (C - v_1) \rangle$ contains the other disjoint cycle.

Case 2. $N_C(x_1) \cap N_C(x_3) = \emptyset$.

In this case, if $|C| = 3$, then since $d_C(x_i) = 2$ for each $i \in \{1, 3\}$, $N_C(x_1) \cap N_C(x_3) \ne \emptyset$, contradicting our assumption. Thus $|C| = 4$, and either $N_C(x_1) = \{v_1, v_3\}$ and $N_C(x_3) = \{v_2, v_4\}$ or $N_C(x_1) = \{v_2, v_4\}$ and $N_C(x_3) = \{v_1, v_3\}$.

Suppose that $N_C(x_1) = \{v_1, v_3\}$ and $N_C(x_3) = \{v_2, v_4\}$. Suppose that $d_C(x_5) = 1$. Then $x_5v_1 \in E(G)$ by our earlier assumption, and $d_C(x_t) = 2$. If $x_tv_1 \in E(G)$, then $H[x_5, x_t], v_1, x_5$ is a cycle, and x_3, v_4, v_3, v_2, x_3 is the other disjoint cycle. Thus $N_C(x_t) = \{v_2, v_4\}$. Then $H[x_3, x_t], v_4, x_3$ and x_1, v_3, v_2, v_1, x_1 are two disjoint cycles. Suppose that $d_C(x_t) = 1$. Then we can find two disjoint cycles in $\langle H \cup C \rangle$ similar to the case where $d_C(x_5) = 1$.

Suppose that $N_C(x_1) = \{v_2, v_4\}$ and $N_C(x_3) = \{v_1, v_3\}$. Then x_1, v_4, v_3, v_2, x_1 is a cycle, and since $d_C(x_{i_0}) = 1$ for some $i_0 \in \{5, t\}$ and $x_{i_0}v_1 \in E(G)$, $H[x_3, x_{i_0}], v_1, x_3$ is the other disjoint cycle. □

By Subclaims 1 and 2, if $d_C(X) \ge 8$ for some cycle C in $\mathscr{C}$, noting the minimum possible degree sequences, then $\langle H \cup C \rangle$ contains two disjoint cycles. Thus we may assume that $d_C(X) = 7$ for every cycle C in $\mathscr{C}$. Let $X' = \{x_2, x_4, x_6, x_t\}$. Then X' is an independent set with $d_H(X') = 7$, and $d_{\mathscr{C}}(X') \ge (8k - 3) - 7 = 8k - 10 \ge 6(k - 1)$, since $k \ge 2$. Thus we choose some cycle C in $\mathscr{C}$ such that $d_C(X') \ge 6$. Since $d_C(x_t) \le 3$ by Lemma A, note that $d_C(X' - \{x_t\}) \ge 6 - 3 = 3$. Now, we have only to consider degree sequences $(3, 2, 1, 1)$ and $(3, 2, 2, 0)$ from X to C by Subclaims 1 and 2. Since both degree sequences contain degree 3, $|C| = 3$ by Lemma A. Let $C = v_1, v_2, v_3, v_1$.

Case 1. The sequence is $(3, 2, 1, 1)$.

Suppose that $d_C(x_1) = 3$. By the degree sequence of this case and $|C| = 3$, there are distinct integers $i_1, i_2 \in \{3, 5, t\}$ with $i_1 < i_2$ such that $N_C(x_{i_1}) \cap N_C(x_{i_2}) \ne \emptyset$. Without loss of generality, we may assume that $v_1 \in N_C(x_{i_1}) \cap N_C(x_{i_2})$. Then $H[x_{i_1}, x_{i_2}], v_1, x_{i_1}$ is a cycle. Since $d_C(x_1) = 3$, x_1, v_3, v_2, x_1 is the other disjoint cycle. If $d_C(x_t) = 3$, then we can find two disjoint cycles similar to the case where $d_C(x_1) = 3$. Thus we may assume that $d_C(x_{i_0}) = 3$ for some $i_0 \in \{3, 5\}$.

Suppose that $d_C(x_1) = 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_1)$. First, suppose that $d_C(x_3) = 1$. Then $d_C(x_5) = 3$. If $x_3v_1 \in E(G)$, then x_1, x_2, x_3, v_1, x_1 and x_5, v_3, v_2, x_5 are two disjoint cycles. If $x_3v_2 \in E(G)$, then we can find two disjoint cycles similar to the case where $x_3v_1 \in E(G)$, replacing v_1 with v_2. If $x_3v_3 \in E(G)$, then x_3, x_4, x_5, v_3, x_3 and x_1, v_2, v_1, x_1 are two disjoint cycles. Next, suppose that $d_C(x_3) = 3$. If $x_5v_3 \in E(G)$, then x_3, x_4, x_5, v_3, x_3 and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus $x_5v_{j_0} \in E(G)$ for some $j_0 \in \{1, 2\}$. If $j_0 = 1$, that is, $x_5v_1 \in E(G)$, then x_3, v_3, v_2, x_3 is a 3-cycle, and $\langle (H - x_3) \cup v_1 \rangle$ is connected and not a path. Thus we can find two disjoint cycles in $\langle H \cup C \rangle$ as in the proof of Claim 2. Similarly, we can prove the case where $j_0 = 2$.

If $d_C(x_t) = 2$, then we can find two disjoint cycles similar to the case where $d_C(x_1) = 2$. Thus we may assume that $d_C(x_{m_0}) = 2$ for some $m_0 \in \{3, 5\}$.

Then $d_C(x_i) = 1$ for each $i \in \{1, t\}$. Let $x_1v_1 \in E(G)$. Then we may assume that $d_C(x_3) = 2$ and $d_C(x_5) = 3$, otherwise, $d_C(x_3) = 3$ and $d_C(x_5) = 2$, and we may argue in a similar manner from the other end of the path H. If $x_3v_1 \in E(G)$, then $H[x_1, x_3], v_1, x_1$ and x_5, v_3, v_2, x_5 are two disjoint cycles. Thus $x_3v_i \in E(G)$ for each $i \in \{2, 3\}$. If $x_tv_1 \in E(G)$, then $H[x_5, x_t], v_1, x_5$ and x_3, v_3, v_2, x_3 are two disjoint cycles. If $x_tv_2 \in E(G)$, then $H[x_5, x_t], v_2, x_5$ and $H[x_1, x_3], v_3, v_1, x_1$ are two disjoint cycles. If $x_tv_3 \in E(G)$, then $H[x_5, x_t], v_3, x_5$ and $H[x_1, x_3], v_2, v_1, x_1$ are two disjoint cycles.

Case 2. The sequence is $(3, 2, 2, 0)$.

We may assume that $d_C(x_{i_0}) = 0$ for some $i_0 \in \{1, 3\}$, otherwise, $i_0 \in \{5, t\}$, and we may argue in a similar manner from the other end of the path H. Let $j_0 \in \{1, 3\} - \{i_0\}$. Then $d_C(x_{j_0}) \ge 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_{j_0})$.

Suppose that $d_C(x_5) = 2$. If $d_C(x_{j_0}) = 2$, then $N_C(x_{j_0}) \cap N_C(x_5) \ne \emptyset$, say v, and $H[x_{j_0}, x_5], v, x_{j_0}$ is a cycle. Since $d_C(x_t) = 3$, $\langle x_t \cup (C - v) \rangle$ contains the other disjoint cycle. If $d_C(x_{j_0}) = 3$, then $d_C(x_j) = 2$ for each $j \in \{5, t\}$. Since $N_C(x_5) \cap N_C(x_t) \ne \emptyset$, say v, $H[x_5, x_t], v, x_5$ is a cycle. Since $d_C(x_{j_0}) = 3$, $\langle x_{j_0} \cup (C - v) \rangle$ contains the other disjoint cycle.

Suppose that $d_C(x_5) = 3$. If $|N_C(x_{j_0}) \cap N_C(x_t)| = 1$, then let $v \in N_C(x_{j_0}) - N_C(x_t)$. Then $H[x_{j_0}, x_5], v, x_{j_0}$ is a cycle, and $\langle x_t \cup (C - v) \rangle$ contains the other cycle. Thus x_{j_0}, x_t have all the same neighbors in C, say v_1, v_2. Suppose that $N_C(x_6) \ne \emptyset$. If $N_C(x_6) \cap N_C(x_t) \ne \emptyset$, say v, then $H[x_6, x_t], v, x_6$ is a cycle, and $\langle x_5 \cup (C - v) \rangle$ contains the other disjoint cycle. If $N_C(x_6) \cap N_C(x_t) = \emptyset$, then $x_6v_3 \in E(G)$. Thus x_5, x_6, v_3, x_5 and x_t, v_2, v_1, x_t are two disjoint cycles.

Suppose that $N_C(x_4) \neq \emptyset$. Then replacing x_6 in the above argument with x_4 and x_t with x_1, we can prove this case by the same arguments above. Thus $N_C(x_i) = \emptyset$ for each $i \in \{4, 6\}$. This implies that $d_C(x_2) = 3$. Then $x_{j_0}, x_2, v_1, x_{j_0}$ and x_5, v_3, v_2, x_5 are two disjoint cycles. □

4. Proofs of Lemmas

4.1. Proof of Lemma 1

Let $F, C, x_i\,(1 \le i \le 4)$ be as in Lemma 1. Let F_1, F_2 be two components of F, $C = v_1, v_2, v_3, v_1$, and $X = \{x_1, x_2, x_3, x_4\}$. Now, we consider two cases.

Case 1. At most two vertices of X lie in the same component of F.

Since $d_C(X) \ge 9$, $d_C(x_{i_0}) \ge 3$ for some $1 \le i_0 \le 4$. By $|C| = 3$, $d_C(x_i) \le 3$ for each $1 \le i \le 4$. Thus $d_C(x_{i_0}) = 3$. Without loss of generality, we may assume that $i_0 = 1$, that is, $d_C(x_1) = 3$. Then $d_C(\{x_2, x_3, x_4\}) \ge 6$. Also, we may assume that $d_C(x_2) \ge d_C(x_3) \ge d_C(x_4)$. Now, we claim that $d_C(\{x_2, x_3\}) \ge 4$. Otherwise, if $d_C(\{x_2, x_3\}) \le 3$, then $d_C(x_{j_0}) \le 1$ for some $j_0 \in \{2, 3\}$. That implies that $d_C(x_4) \le 1$, since $d_C(x_4)$ is the smallest degree in $\{x_2, x_3, x_4\}$. Then $d_C(\{x_2, x_3, x_4\}) \le 3 + 1 = 4$, a contradiction. Thus the claim holds. Noting our assumption of this case, $\{x_1, x_2, x_3\}$ is a set of leaves from at least two components of F. Since $d_C(\{x_1, x_2, x_3\}) \ge 3 + 4 = 7$, Lemma B applies, completing this case.

Case 2. Three vertices of X lie in the same component of F.

Without loss of generality, we may assume that $x_1, x_2, x_3 \in V(F_1)$, $x_4 \in V(F_2)$, and $d_C(x_1) \ge d_C(x_2) \ge d_C(x_3)$. Recall that $d_C(X) \ge 9$. It follows that the minimum possible degree sequence S from X to C is $(3, 3, 3, 0)$, $(3, 3, 2, 1)$ or $(3, 2, 2, 2)$.

Subcase 1. $S = (3, 3, 3, 0)$.

If $d_C(x_{i_0}) = 0$ for some $1 \le i_0 \le 3$, then $i_0 = 3$, that is, $d_C(x_3) = 0$. Now, we take $\{x_1, x_2, x_4\}$ that is a set of leaves from at least two components of F. Since $d_C(\{x_1, x_2, x_4\}) = 9$, Lemma B applies. If $d_C(x_4) = 0$, then $d_C(x_i) = 3$ for each $1 \le i \le 3$. Since all the x_is are leaves, x_3 does not lie on the path in F_1 connecting x_1 and x_2. Then $F_1[x_1, x_2], v_1, x_1$ and x_3, v_3, v_2, x_3 are two disjoint cycles in $\langle F \cup C\rangle$.

Subcase 2. $S = (3, 3, 2, 1)$.

Take $\{x_1, x_2, x_4\}$. If $d_C(x_4) \in \{1, 2\}$, then $d_C(\{x_1, x_2\}) \ge 6$. If $d_C(x_4) = 3$, then $d_C(\{x_1, x_2\}) \ge 5$. Since $d_C(\{x_1, x_2, x_4\}) \ge 7$ for all cases, Lemma B applies.

Subcase 3. $S = (3, 2, 2, 2)$.

Take $\{x_1, x_2, x_4\}$. If $d_C(x_4) = 2$, then $d_C(\{x_1, x_2\}) \ge 5$. If $d_C(x_4) = 3$, then $d_C(\{x_1, x_2\}) \ge 4$. Since $d_C(\{x_1, x_2, x_4\}) \ge 7$ for all cases, Lemma B applies. □

4.2. Proof of Lemma 5

Proof of (i). Let v_1, v_2, v_3, v_4, v_5 be the vertices such that $d_{C_1}(v_i) = 2$ for each $1 \le i \le 5$, appearing in this order on C_2. Let $w_1, w_2 \in N_{C_1}(v_1)$ appear in this order on C_1. The neighbors of v_1 partition C_1 into two intervals $C_1(w_1, w_2]$ and $C_1(w_2, w_1]$. We claim that each of v_2, v_3, v_4, v_5 has one neighbor in different interval of C_1.

First, suppose that $v_{i_1}, v_{i_2}, v_{i_3}$ for some $2 \le i_1 < i_2 < i_3 \le 5$ have both their neighbors in a common interval of C_1, say $C_1(w_1, w_2]$. We may assume that at least one of their neighbors is not w_2. Let $z_{i_1} \in N_{C_1(w_1,w_2)}(v_{i_1})$ and $z_{i_2} \in N_{C_1(w_1,w_2)}(v_{i_2})$. Then $C_1^{\pm}[z_{i_1}, z_{i_2}], C_2^{-}[v_{i_2}, v_{i_1}], z_{i_1}$ and $C_1[w_2, w_1], v_1, w_2$ are shorter two disjoint cycles, since v_{i_3} is not used.

Next, suppose that v_{i_1}, v_{i_2} for some $2 \le i_1 < i_2 \le 5$ have both their neighbors in a common interval of C_1, say $C_1(w_1, w_2]$. Then we may assume that $i_1 = 2$ and $i_2 = 5$, otherwise, we can prove the other pairs of i_1 and i_2 by the same arguments above. Let $z_{i_1} \in N_{C_1(w_1,w_2)}(v_2)$ and $z_{i_2} \in N_{C_1(w_1,w_2)}(v_5)$. If $N_{C_1(w_1,w_2)}(v_{j_0}) \neq \emptyset$ for some $j_0 \in \{3, 4\}$, then there exist shorter two disjoint cycles. Thus $N_{C_1(w_1,w_2)}(v_j) = \emptyset$ for each $j \in \{3, 4\}$. Since $d_{C_1}(v_j) = 2$ for each $j \in \{3, 4\}$, $N_{C_1(w_2,w_1]}(v_j) \neq \emptyset$. Let $z_{i_3} \in N_{C_1(w_2,w_1]}(v_3)$ and $z_{i_4} \in N_{C_1(w_2,w_1]}(v_4)$. Then $C_1^{\pm}[z_{i_3}, z_{i_4}], C_2^{-}[v_4, v_3], z_{i_3}$ and $C_1^{\pm}[z_{i_1}, z_{i_2}], C_2[v_5, v_2], z_{i_1}$ are shorter two disjoint cycles, since w_2 is not used.

Finally, suppose that v_{i_0} for some $2 \le i_0 \le 5$ has both the neighbors in an interval of C_1, say $C_1(w_1, w_2]$. Then we have only to consider $i_0 = 2$ or $i_0 = 3$, otherwise, we take a cycle from v_1 in the opposite direction. First, suppose that $i_0 = 2$. Let $x_1, x_2 \in N_{C_1(w_1,w_2]}(v_2)$, appearing in this order on C_1. If $x_2 \neq w_2$, then $C_1[x_1, x_2], v_2, x_1$ and $C_1[w_2, w_1], v_1, w_2$ are shorter two disjoint cycles, since v_3 is not used. Thus $x_2 = w_2$. Let $y_1, y_2 \in N_{C_1}(v_3)$, appearing in this order on C_1. Suppose that $y_1 \in C_1(w_1, w_2)$. Then $C_1^{\pm}[x_1, y_1], C_2^{-}[v_3, v_2], x_1$ and $C_1[w_2, w_1], v_1, w_2$ are shorter two disjoint cycles, since v_4 is not used. Thus $y_1 \notin C_1(w_1, w_2)$, that is, $y_1 \in C_1[w_2, w_1]$. Note that $y_2 \in C_1(w_2, w_1]$. If $y_1 \neq w_2$, then $C_1[x_1, w_2], v_2, x_1$ and $C_1[y_1, y_2], v_3, y_1$ are shorter two disjoint cycles, since v_1 is not used. Thus $y_1 = w_2$. If $y_2 \neq w_1$, then $C_1[w_2, y_2], v_3, w_2$ and $C_1[w_1, x_1], C_2^{-}[v_2, v_1], w_1$ are shorter two disjoint cycles, since v_4 is not used. Thus $y_2 = w_1$. Let $z_1, z_2 \in N_{C_1}(v_4)$, appearing in this order on C_1. Suppose that $z_1 \in C_1[w_1, w_2)$. Then $C_1[w_1, z_1], C_2^{-}[v_4, v_3], w_1$ and $C_2[v_1, v_2], w_2, v_1$ are shorter two disjoint cycles, since v_5 is not used. Suppose that $z_1 \in C_1[w_2, w_1)$. Then $C_1[w_1, x_1], C_2^{-}[v_2, v_1], w_1$ and $C_1[w_2, z_1], C_2^{-}[v_4, v_3], w_2$ are shorter two disjoint cycles, since v_5 is not used. Next, suppose that $i_0 = 3$. Then, by the same arguments as the case where $i_0 = 2$, we have shorter two disjoint cycles, replacing v_2 with v_3.

Thus each of v_2, v_3, v_4, v_5 has one neighbor in each interval of C_1. Let $x \in N_{C_1(w_1,w_2]}(v_2)$, $y \in N_{C_1(w_1,w_2]}(v_3)$, $z \in N_{C_1(w_2,w_1]}(v_4)$, $u \in N_{C_1(w_2,w_1]}(v_5)$. Then $C_1^{\pm}[x, y], C_2^{-}[v_3, v_2], x$ and $C_1^{\pm}[z, u], C_2^{-}[v_5, v_4], z$ are shorter two disjoint cycles, since v_1 is not used. □

Proof of (ii). Let $v_1, v_2 \in V(C_2)$ such that $d_{C_1}(v_1) = 5$ and $d_{C_1}(v_2) = 3$, appearing in this order on C_2. Let $w_1, w_2, w_3, w_4, w_5 \in N_{C_1}(v_1)$, appearing in this order on C_1, and let $u_1, u_2, u_3 \in N_{C_1}(v_2)$, appearing in this order on C_1. The neighbors of v_1 partition C_1 into five intervals $C_1(w_i, w_{i+1}]$, $1 \le i \le 5$ (mod 5). Suppose that $u_{i_0}, u_{j_0} \in C_1(w_{m_0}, w_{m_0+1}]$ (mod 5) for some $1 \le i_0 < j_0 \le 3$ and for some $1 \le m_0 \le 5$. Without loss of generality, we may assume that $i_0 = 1, j_0 = 2$ and $m_0 = 1$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_4], v_1, w_3$ are shorter two disjoint cycles, since w_1 is not used. Thus neighbors of v_2 are contained in different intervals. Since C_1 is partitioned into five intervals, some two neighbors of v_2 lie in neighboring intervals, say $u_1 \in (w_1, w_2]$ and $u_2 \in C_1(w_2, w_3]$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_4, w_5], v_1, w_4$ are shorter two disjoint cycles, since w_1 is not used. □

Proof of (iii). Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the vertices on C_2 with the degree sequence $(3, 1, 1, 1, 1, 1)$, appearing in this order on C_2. Without loss of generality, we may assume that $d_{C_1}(v_1) = 3$ and $d_{C_1}(v_i) = 1$ for each $2 \le i \le 6$. Let $w_1, w_2, w_3 \in N_{C_1}(v_1)$, appearing in this order on C_1. The neighbors of v_1 partition C_1 into three intervals: $C_1(w_1, w_2], C_1(w_2, w_3], C_1(w_3, w_1]$. Then there exist some integer $1 \le i_0 \le 3$ and distinct integers $2 \le j_1 < j_2 \le 5$ such that $N_{C_1(w_{i_0},w_{i_0+1}]}(v_{j_1}) \ne \emptyset$ and $N_{C_1(w_{i_0},w_{i_0+1}]}(v_{j_2}) \ne \emptyset$. Without loss of generality, we may assume that $i_0 = 1$. Let $u_1 \in N_{C_1(w_1,w_2]}(v_{j_1})$ and $u_2 \in N_{C_1(w_1,w_2]}(v_{j_2})$. Then $C_1^{\pm}[u_1, u_2], C_2^{-}[v_{j_2}, v_{j_1}], u_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since v_6 is not used. □

Proof of (iv). Let v_1, v_2, v_3, v_4 be the vertices on C_2 with the degree sequence $(3, 2, 1, 1)$, say $d_{C_1}(v_1) = 3$, $d_{C_1}(v_2) = 2$ and $d_{C_1}(v_i) = 1$ for each $i \in \{3, 4\}$. Suppose that v_1, v_2 are in this order on C_2. Let $w_1, w_2, w_3 \in N_{C_1}(v_1)$ be in this order on C_1, and let $u_1, u_2 \in N_{C_1}(v_2)$ be in this order on C_1. Let v_3, v_4 be in this order on C_2. Let $z_1 \in N_{C_1}(v_3)$, and let $z_2 \in N_{C_1}(v_4)$. The neighbors of v_1 partition C_1 into three intervals: $C_1(w_1, w_2], C_1(w_2, w_3], C_1(w_3, w_1]$. If v_2 has both its neighbors in the same interval in C_1, then we can find shorter two disjoint cycles. If the neighbors of v_2 are into two different intervals of C_1 and neither is in $\{w_1, w_2, w_3\}$, then we can also find shorter two disjoint cycles. Thus the neighbors of v_2 are into two different intervals of C_1 and at least one of them is at an endpoint of these intervals. Without loss of generality, we may assume that $u_1 \in C_1(w_1, w_2]$ and $u_2 \in C_1(w_2, w_3]$. Now, we consider two cases.

Case 1. $v_3, v_4 \in C_2(v_1, v_2)$ or $v_3, v_4 \in C_2(v_2, v_1)$.

Without loss of generality, we may assume that $v_3, v_4 \in C_2(v_1, v_2)$. If $z_2 \in C_1(w_1, w_3)$, then $C_1^{\pm}[u_1, z_2], C_2[v_4, v_2], u_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since v_3 is not used. If $z_2 \in C_1[w_3, w_1)$, then $C_1[u_2, z_2], C_2[v_4, v_2], u_2$ and $C_1[w_1, w_2], v_1, w_1$ are shorter two disjoint cycles, since v_3 is not used. Thus $z_2 = w_1$.

If $u_2 \in C_1(w_2, w_3)$, then $C_1[u_1, u_2], v_2, u_1$ and $C_2[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since v_3 is not used. Thus $u_2 = w_3$.

If $z_1 \in C_1(w_3, u_1)$, then $C_1^{\pm}[z_1, w_1], C_2[v_1, v_3], z_1$ and $C_1[u_1, w_3], v_2, u_1$ are shorter two disjoint cycles, since v_4 is not used. Thus $z_1 \in C_1[u_1, w_3]$.

Suppose that $u_1 \in C_1(w_1, w_2)$. If $z_1 \in C_1[u_1, w_2)$, then $C_1[w_1, z_1], C_2[v_3, v_4], w_1$ and $C_1[w_2, w_3], v_1, w_2$ are shorter two disjoint cycles, since v_2 is not used. If $z_1 = w_2$, then $C_2[v_1, v_3], w_2, v_1$ and $C_1[w_1, u_1], C_2^{-}[v_2, v_4], w_1$ are shorter two disjoint cycles, since w_3 is not used. If $z_1 \in C_1(w_2, w_3]$, then $C_1[z_1, w_3], C_2[v_1, v_3], z_1$ and $C_1[w_1, u_1], C_2^{-}[v_2, v_4], w_1$ are shorter two disjoint cycles, since w_2 is not used. Thus $u_1 = w_2$.

Now, we consider two disjoint cycles $C' = w_1, C_2[v_1, v_4], w_1$ and $C'' = C_1[w_2, w_3], v_2, w_2$. Note that $|C_2| \ge 6$. If $C_2(v_4, v_2) \ne \emptyset$ or $C_2(v_2, v_1) \ne \emptyset$, then C' and C'' are shorter two disjoint cycles. Thus $C_2(v_4, v_2) = \emptyset$ and $C_2(v_2, v_1) = \emptyset$. First, suppose that $z_1 \in C_1[w_2, w_3)$. If $C_2(v_1, v_3) \ne \emptyset$, then $C_1[w_3, w_1], v_1, w_3$ and $C_2[v_3, v_2], C_1[w_2, z_1], v_3$ are shorter two disjoint cycles. If $C_2(v_3, v_4) \ne \emptyset$, then $C_1[w_2, z_1], C_2^{-}[v_3, v_1], w_2$ and $C_1[w_3, w_1], C_2[v_4, v_2], w_3$ are shorter two disjoint cycles. Next, suppose that $z_1 = w_3$. If $C_2(v_1, v_3) \ne \emptyset$, then $C_1[w_1, w_2], v_1, w_1$ and $C_2[v_3, v_2], w_3, v_3$ are shorter two disjoint cycles. If $C_2(v_3, v_4) \ne \emptyset$, then $C_2[v_1, v_3], w_3, v_1$ and $C_1[w_1, w_2], C_2^{-}[v_2, v_4], w_1$ are shorter two disjoint cycles.

Case 2. $v_3 \in C_2(v_1, v_2)$ and $v_4 \in C_2(v_2, v_1)$.

If $z_1 \in C_1(w_1, w_3)$, then $C_1^{\pm}[u_1, z_1], C_2[v_3, v_2], u_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since v_4 is not used. If $z_1 \in C_1[w_3, w_1)$, then $C_1[u_2, z_1], C_2[v_3, v_2], u_2$ and $C_1[w_1, w_2], v_1, w_1$ are shorter two disjoint cycles, since v_4 is not used. Thus $z_1 = w_1$. Then $C_2[v_1, v_3], w_1, v_1$ and $C_1[u_1, u_2], v_2, u_1$ are shorter two disjoint cycles, since v_4 is not used. □

Proof of (v). Let v_1, v_2, v_3 be the vertices on C_2 with the degree sequence $(3, 3, 1)$. Suppose that v_1, v_2, v_3 exist in this order on C_2. Without loss of generality, we may assume that $d_{C_1}(v_i) = 3$ each $i \in \{1, 2\}$ and $d_{C_1}(v_3) = 1$. Suppose that $w_1, w_2, w_3 \in N_{C_1}(v_1)$ exist in this order on C_1. Let $W = \{w_1, w_2, w_3\}$. These neighbors of v_1 partition C_1 into three intervals: $C_1(w_1, w_2], C_1(w_2, w_3], C_1(w_3, w_1]$. Let $u_1, u_2, u_3 \in N_{C_1}(v_2)$, and suppose that u_1, u_2, u_3 are in this order on C_1.

Case 1. Some two neighbors of v_2 are in the same interval of C_1.

Without loss of generality, we may assume that $u_1, u_2 \in C_1(w_1, w_2]$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since v_3 is not used.

Case 2. No two neighbors of v_2 are in the same interval of C_1.

Then $u_1 \in C_1(w_1, w_2]$, $u_2 \in C_1(w_2, w_3]$, and $u_3 \in C_1(w_3, w_1]$. First, suppose that $u_{i_0}, u_{j_0} \notin W$ for some $1 \le i_0 < j_0 \le 3$. Without loss of generality, we may assume that $i_0 = 1$ and $j_0 = 2$, that is, $u_1 \in C_1(w_1, w_2)$ and $u_2 \in C_1(w_2, w_3)$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since v_3 is not used.

Next, suppose that $u_{i_0} \notin W$ for only some $1 \le i_0 \le 3$. Without loss of generality, we may assume that $i_0 = 1$, that is, $u_1 \in C_1(w_1, w_2)$. Then note that $u_3 = w_1$, $C_1[w_1, u_1], v_2, w_1$ and $C_1[w_2, w_3], v_1, w_2$ are shorter two disjoint cycles, since v_3 is not used.

Finally, suppose that $u_i = w_{i+1} \pmod 3$ for each $1 \le i \le 3$. Without loss of generality, we may assume that $v_3z_1 \in E(G)$ for $z_1 \in (w_2, w_3]$. Now, we have two choices for constructing shorter two disjoint cycles. We may construct $C_1[w_1, w_2], v_2, w_1$ and $C_1[z_1, w_3], C_2^-[v_1, v_3], z_1$, or $C_1[w_1, w_2], v_1, w_1$ and $C_1[z_1, w_3], C_2[v_2, v_3], z_1$. Since $|C_2| \ge 6$, one of these two choices must leave out a vertex of C_2, and hence we may form shorter two disjoint cycles. □

4.3. *Proof of Lemma 6*

Let $C = v_1, v_2, v_3, v_1$.

Case 1. The sequence is $(3, 3, 2, 0)$.

Suppose that $d_C(x_1) = 0$. Then $d_C(y_{i_0}) = 3$ for some $i_0 \in \{1, i, t\}$, and we may assume that $i_0 = 1$, that is, $d_C(y_1) = 3$. Since $d_C(y_r) \ge 2$ for each $r \in \{i, t\}$ and $|C| = 3$, $v_{m_0} \in N_C(y_i) \cap N_C(y_t)$ for some $1 \le m_0 \le 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_i, y_t], v_1, y_i$ and y_1, v_3, v_2, y_1 are two disjoint cycles.

Suppose that $d_C(x_1) = 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_1)$. Then x_1, v_2, v_1, x_1 is a cycle. Since $d_C(y_{i_0}) = d_C(y_{j_0}) = 3$ for some $i_0, j_0 \in \{1, i, t\}$ with $i_0 < j_0$ and $|C| = 3$, $v_3 \in N_C(y_{i_0}) \cap N_C(y_{j_0})$. Then $H_2[y_{i_0}, y_{j_0}], v_3, y_{i_0}$ is the other disjoint cycle.

Suppose that $d_C(x_1) = 3$. Since $d_C(y_{i_0}) \ge 2$ and $d_C(y_{j_0}) \ge 2$ for some $i_0, j_0 \in \{1, i, t\}$ with $i_0 < j_0$ and $|C| = 3$, $v_{m_0} \in N_C(y_{i_0}) \cap N_C(y_{j_0})$ for some $1 \le m_0 \le 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_{i_0}, y_{j_0}], v_1, y_{i_0}$ and x_1, v_3, v_2, x_1 are two disjoint cycles.

Case 2. The sequence is $(3, 3, 1, 1)$.

Suppose that $d_C(x_1) = 1$. Then $d_C(y_{i_0}) = 3$ for some $i_0 \in \{1, i, t\}$, and we may assume that $i_0 = 1$, that is, $d_C(y_1) = 3$. Since one of y_i and y_t has degree 3 to C and the other one of them has degree 1 to C, noting that $|C| = 3$, $v_{m_0} \in N_C(y_i) \cap N_C(y_t)$ for some $1 \le m_0 \le 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_i, y_t], v_1, y_i$ and y_1, v_3, v_2, y_1 are two disjoint cycles.

Suppose that $d_C(x_1) = 3$. Since one of y_1, y_i, y_t has degree 3 to C and the others of them have degree 1 to C, $d_C(y_{i_0}) = 3$ and $d_C(y_{j_0}) = 1$ for some distinct $i_0, j_0 \in \{1, i, t\}$. Then note that either $i_0 < j_0$ or $i_0 > j_0$. Since $|C| = 3$, $v_{m_0} \in N_C(y_{i_0}) \cap N_C(y_{j_0})$ for some $1 \le m_0 \le 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2^{\pm}[y_{i_0}, y_{j_0}], v_1, y_{i_0}$ and x_1, v_3, v_2, x_1 are two disjoint cycles. □

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