On vertex-disjoint cycles and degree sum conditions

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Abstract

This paper considers a degree sum condition sufficient to imply the existence of $k$ vertex-disjoint cycles in a graph $G$. For an integer $t \geq 1$, let $\sigma_t(G)$ be the smallest sum of degrees of $t$ independent vertices of $G$. We prove that if $G$ has order at least $7k + 1$ and $\sigma_t(G) \geq 8k - 3$, with $k \geq 2$, then $G$ contains $k$ vertex-disjoint cycles. We also show that the degree sum condition on $\sigma_t(G)$ is sharp and conjecture a degree sum condition on $\sigma_t(G)$ sufficient to imply $G$ contains $k$ vertex-disjoint cycles for $k \geq 2$.

1. Introduction

In this paper, all graphs are simple. Let $G$ be a graph. For $u \in V(G)$, the set of neighbors of $u$ in $G$ is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$. Let $H$ be a subgraph of $G$, and let $S \subseteq V(G)$. For $u \in V(G) - V(H)$, we denote $N_H(u) = N_G(u) \cap V(H)$ and $d_H(u) = |N_H(u)|$. For $u \in V(G) - S$, $N_G(u) = N_G(u) \cap S$. Furthermore, $N_G(S) = \cup_{u \in S} N_G(u)$ and $N_H(S) = N_G(S) \cap V(H)$. Let $A$, $B$ be two disjoint subgraphs of $G$. Then $N_G(A) = N_G(V(A))$ and $N_B(A) = N_G(A) \cap V(B)$. The subgraph of $G$ induced by $S$ is denoted by $(S)$. And let $G - S = (V(G) - S)$ and $G - H = (V(G) - V(H))$. If $S = \{u\}$, then we write $G - u$ for $G - S$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For graphs $G_1$ and $G_2$, $G_1 \cup G_2$ denotes the union of $G_1$ and $G_2$, and $mG$ denotes the union of $m$ copies of $G$. If $Q$ is a path or a cycle with a given orientation and $x \in V(Q)$, then $x^+$ denotes the first successor of $x$ on $Q$ and $x^-$ denotes the first predecessor of $x$ on $Q$. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of $Q$ from $x$ to $y$ (including $x$ and $y$) in the given direction. The notation $Q^{-x}[x, y]$ denotes the path from $x$ to $y$ in the opposite direction. We also write $Q[x, y] = Q[x^+, y], Q[x, y] = Q[x^-, y^-]$ and $Q(x, y) = Q[x^+, y^-]$. If $Q$ is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t, x_1$, then we assume that an orientation of $Q$ is given from $x_1$ to $x_t$. We say that $x_i$ precedes $x_j$ on $Q$ if $i < j$. For $u, v \in V(Q)$, we define the path $Q[u, v]$ as follows; if $u$ precedes $v$ on $Q$, then $Q[u, v] = Q[u, v]$, and if $v$ precedes $u$ on $Q$, then $Q[u, v] = Q[v, u]$. If $T$ is a tree with at least one branch and $x, y \in V(T)$, where a branch vertex of a tree is a vertex of degree at least three, then we denote the path from $x$ to $y$ as $T[x, y]$. For $X \subseteq V(G)$, let $d_H(X) = \sum_{v \in X} d_H(v)$. If $H = G$, then we denote $d_G(X) = d_H(X)$. For a graph $G$, $|G|$ is the order of $G$, $\delta(G)$ is the minimum degree of $G$, $\omega(G)$ is the number of components of $G$, $\alpha(G)$ is the independence number of $G$. If $G$ is one vertex, that is, $V(G) = \{x\}$, then we simply write $x$ instead of $G$. For an integer $t \geq 1$, let

$$
\sigma_t(G) = \min \left\{ \sum_{v \in X} d_G(v) \mid X \text{ is an independent set of } G \text{ with } |X| = t \right\},
$$

and $\sigma_t(G) = \infty$ when $\alpha(G) < t$. Note that if $t = 1$, then $\sigma_1(G) = \delta(G)$. For an integer $r \geq 1$ and two disjoint subgraphs $A, B$ of $G$, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from $A$ to $B$ such that $d_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. In

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Lemma 2. Let \( G \) be a graph of sufficiently large order. If \( \sigma_t(G) \geq 2kt - (t - 1) \) for any two integers \( k \geq 2 \) and \( t \geq 1 \), then \( G \) contains \( k \) disjoint cycles.

The cases for \( t = 1, 2, 3 \) have already been shown. We add to the evidence for this conjecture by showing the following:

**Theorem 1.** Let \( G \) be a graph of order \( n \geq 7k + 1 \) for an integer \( k \geq 2 \). If \( \sigma_4(G) \geq 8k - 3 \), then \( G \) contains \( k \) disjoint cycles.

The degree sum conjectured above would be sharp. And in particular, the degree sum condition of Theorem 1 is sharp. Sharpness is given by \( G = K_{2k-1} + mk_1 \). The only independent vertices in \( G \) are those in \( mk_1 \). Each of these vertices has degree \( 2k - 1 \).

Thus, for any \( t \) with \( 1 \leq t \leq m \), \( \sigma_t(G) = t(2k - 1) = 2kt - t \), and \( G \) fails to contain \( k \) disjoint cycles as any such cycle must contain two vertices of \( K_{2k-1} \).

2. **Lemma**

In the proof of Theorem 1, we make use of the following Lemmas A, B and C that were proved by Fujita, Matsumura, Tsugaki and Yamashita in [3]. Proofs omitted in Chapter 2 appear after the proof of Theorem 1, that is, in Chapter 4.

Let \( C_1, \ldots, C_r \) be \( r \) disjoint cycles of a graph \( G \). If \( C'_1, \ldots, C'_r \) are \( r \) disjoint cycles of \( G \) and \( |\bigcup_{i=1}^{r} V(C_i')| < |\bigcup_{i=1}^{r} V(C_i)| \), then we call \( C'_1, \ldots, C'_r \) shorter cycles than \( C_1, \ldots, C_r \). We also call \( \{C_1, \ldots, C_r\} \) minimal if \( G \) does not contain shorter \( r \) disjoint cycles than \( C_1, \ldots, C_r \).

**Lemma A** (Fujita et al. [3]). Let \( r \) be a positive integer and \( C_1, \ldots, C_r \) be \( r \) minimal disjoint cycles of a graph \( G \). Then \( d_C(x) \leq 3 \) for any \( x \in V(G) - \bigcup_{i=1}^{r} V(C_i) \) and for any \( 1 \leq i \leq r \). Furthermore, \( d_C(x) = 3 \) implies \( |C_i| = 3 \), and \( d_C(x) = 2 \) implies \( |C_i| \leq 4 \).

**Lemma B** (Fujita et al. [3]). Suppose that \( F \) is a forest with at least two components and \( C \) is a triangle. Let \( x_1, x_2, x_3 \) be leaves of \( F \) from at least two components. If \( d_C(x_1, x_2, x_3) \geq 7 \), then there exist two disjoint cycles in \( \langle F \cup C \rangle \) or there exists a triangle \( C' \) in \( \langle F \cup C \rangle \) such that \( \omega(\langle F \cup C \rangle - C') < \omega(F) \).

**Lemma C** (Fujita et al. [3]). Let \( C \) be a cycle and \( T \) be a tree with three leaves \( x_1, x_2, x_3 \). If \( d_C(x_1, x_2, x_3) \geq 7 \), then there exist two disjoint cycles in \( \langle C \cup T \rangle \) or there exists a cycle \( C' \) in \( \langle C \cup T \rangle \) such that \( |C'| < |C| \).

**Lemma 2.** Let \( C \) be a cycle and \( T \) be a tree with four leaves \( x_1, x_2, x_3, x_4 \). If \( d_C(x_1, x_2, x_3, x_4) \geq 9 \), then there exist two disjoint cycles in \( \langle C \cup T \rangle \) or there exists a cycle \( C' \) in \( \langle C \cup T \rangle \) such that \( |C'| < |C| \).

**Proof.** Let \( X = \{x_1, x_2, x_3, x_4\} \). If \( d_C(x_0) \leq 2 \) for some \( 1 \leq i \leq 4 \), then \( d_C(X - \{x_0\}) \geq 7 \), and we apply Lemma C to \( X - \{x_0\} \). Otherwise, \( d_C(x) \geq 3 \) for each \( 1 \leq i \leq 4 \), and we apply Lemma C to any three vertices in \( X \). □

**Lemma 3.** Let \( G \) be a graph satisfying the assumption of Theorem 1, and let \( C_1, \ldots, C_{k-1} \) be \( k - 1 \) minimal disjoint cycles of \( G \). Suppose that there exists a tree \( T \) with at least four leaves, which is a component of \( G - \bigcup_{i=1}^{k-1} C_i \). Then \( G \) contains \( k \) disjoint cycles.

**Proof.** Let \( \mathcal{C} = \bigcup_{i=1}^{k-1} C_i \), and let \( X = \{x_1, x_2, x_3, x_4\} \) be a set of leaves of \( T \). Since \( X \) is an independent set, \( d_{\mathcal{C}}(X) \geq (8k - 3) - 4 = 8k - 1 \). Then there exists a cycle \( C_i \) for some \( 1 \leq i \leq k - 1 \) such that \( d_C(X) \geq 9 \). Since \( \{C_1, \ldots, C_{k-1}\} \) is minimal, there exist two disjoint cycles in \( \langle C_i \cup T \rangle \) by Lemma 2. Thus \( G \) contains \( k \) disjoint cycles. □

**Lemma 4.** Let \( G \) be a graph satisfying the assumption of Theorem 1, and let \( C_1, \ldots, C_{k-1} \) be \( k - 1 \) minimal disjoint cycles of \( G \). Suppose that \( H = G - \bigcup_{i=1}^{k-1} C_i \) has at least two components at least one of which is a tree \( T \) with at least three leaves. Then there exist two disjoint cycles in \( \langle C_i \cup T \rangle \) for some \( 1 \leq i \leq k - 1 \) or there exists a triangle \( C \) in \( \langle H \cup C \rangle \) such that \( \omega(\langle H \cup C \rangle - C) < \omega(H) \).
Proof. Let \( \mathcal{C} = \bigcup_{i=1}^{k-1} C_i \). Let \( x_1, x_2, x_3 \) be three leaves of the tree \( T \), and let \( x_4 \) be a leaf from another component, and \( X = \{x_1, x_2, x_3, x_4\} \). Since \( X \) is an independent set, \( d_\mathcal{C}(X) \geq (8k - 3) - 4 = 8(k - 1) + 1 \). Then there exists a cycle \( C_j \) for some \( 1 \leq j \leq k - 1 \) such that \( d_\mathcal{C}(X) \geq 9 \). If \( d_\mathcal{C}(x_4) < 2 \), then \( d_\mathcal{C}(x_1, x_2, x_3)) \geq 7 \). By Lemma 5, there exist two disjoint cycles in \( (C \cup T) \) or there exists a cycle \( C \) in \( (C \cup T) \) such that \( |C| < |C| \). Since \( |C_1, \ldots, C_{k-1} \) is minimal, the lemma holds. If \( d_\mathcal{C}(x_4) \geq 3 \), then \( C_i \) is a triangle by Lemma 4. Thus the lemma holds by Lemma 1. □

Lemma 5. Let \( C_1 \) and \( C_2 \) be two disjoint cycles such that \( |C_2| \geq 6 \). Suppose that \( C_2 \) contains vertices with at least one of the following degree sequences from \( C_2 \) to \( C_1 \). Then \((C_1 \cup C_2)\) contains two disjoint cycles \( C'_1 \) and \( C'_2 \) such that \( |C'_1| + |C'_2| < |C_1| + |C_2| \).

(i) \((2, 2, 2, 2, 2)\)
(ii) \((5, 3)\)
(iii) \((3, 1, 1, 1, 1, 1)\)
(iv) \((3, 2, 1, 1)\)
(v) \((3, 3, 1)\)

Lemma 6. Let \( H \) be a graph with two components \( H_1, H_2 \), where \( H_1 = x_1, \ldots, x_s \) (s \( \geq 1 \)) is a path and \( H_2 = y_1, \ldots, y_t \) (t \( \geq 3 \)) is a path. Let \( W = \{x_1, y_1, y_i, y_j\} \) for any \( 2 \leq i \leq t - 1 \), and let \( C \) be a triangle. If there exists a degree sequence \((3, 3, 2, 0)\) or \((3, 3, 1, 1)\) from \( W \) to \( C \), then \((H \cup C)\) contains two disjoint cycles.

3. Proof of Theorem 1

Suppose that the theorem does not hold. Let \( G \) be an edge-maximal counter-example. If \( G \) is a complete graph, then \( G \) contains \( k \) disjoint cycles. Thus we may assume that \( G \) is not a complete graph. Let \( xy \notin E(G) \) for some \( x, y \in V(G) \), and define \( G' = G + xy \). Since \( G' \) is not a counter-example by the maximality of \( G, G' \) contains \( k \) disjoint cycles \( C_1, \ldots, C_k \). Without loss of generality, we may assume that \( xy \notin \bigcup_{i=1}^{k-1} E(C_i) \), that is, \( G \) contains \( k - 1 \) disjoint cycles \( C_1, \ldots, C_{k-1} \). Let \( \mathcal{C} = \bigcup_{i=1}^{k-1} C_i \) and \( H = G - \mathcal{C} \). Choose \( C_1, \ldots, C_{k-1} \) such that

1. \( \sum_{i=1}^{k-1} |C_i| \) is minimal, and
2. subject to (1), \( \omega(H) \) is minimal.

Note that any cycle \( C \) in \( \mathcal{C} \) has no chords by (1). Clearly, \( H \) is a forest, otherwise, since \( H \) contains a cycle, \( G \) contains \( k \) disjoint cycles, a contradiction. If \( H \) contains at least two components at least one of which is a tree with at least three leaves, then by Lemma 4, \( G \) contains \( k \) disjoint cycles, or contradicting (2). Thus if \( H \) contains at least two components, then \( H \) must be a collection of paths. If \( H \) has only one component, then it is a tree. If \( H \) is a tree with at least four leaves, then the theorem holds by Lemma 3. Thus if \( H \) has only one component, then \( H \) is a tree with at most three leaves.

Now, we consider two cases on \( |H| \).

Case 1. \( |H| \leq 7 \)

Let \( C \) be a longest cycle in \( \mathcal{C} \). Suppose that \( |C| \leq 7 \). Then \( |C'| \leq 7 \) for any cycle \( C' \) in \( \mathcal{C} \), and \( |\mathcal{C}| \leq 7(k - 1) \). Since \( |G| \geq 7k + 1 \), \( |H| = |G| - |\mathcal{C}| \geq (7k + 1) - 7(k - 1) = 8 \), contradicting the assumption of this case. Thus \( |C| \geq 8 \). Let \( |C| = 4t + r, t \geq 2 \) and \( 0 \leq r \leq 3 \). Then there exist at least \( t \) disjoint independent sets in \( V(C) \) each of which has four vertices. By (1) and \( |C| \geq 8 \), \( d_\mathcal{C}(v) \leq 1 \) for any \( v \in V(H) \). Thus \( |E(H, C)| \leq 7 \).

Suppose that \( k = 2 \). Then \( \mathcal{C} \) has only one cycle, and \( H = G - C \). Since \( |C| \geq 8 \), \( C \) contains at least two independent sets each of which has four vertices. Let \( C_1 \) and \( C_2 \) be such sets. Since \( d_\mathcal{C}(X_i) = 8 \) for each \( i \in \{1, 2\}, d_\mathcal{C}(X_i) = 8(k - 3) - 8 = 8k - 11 \). Then \( d_\mathcal{H}(X_1 \cup X_2) \geq 16k - 22 \geq 10 \), since \( k \geq 2 \). Thus \( |E(H, C)| \geq 10 \), a contradiction.

Suppose that \( k \geq 3 \). We claim that \( |E(C, C')| \geq 8t \) for some cycle \( C' \) in \( \mathcal{C} \). Note that each of \( t \) disjoint independent sets in \( V(C) \) sends at least \( 8(k - 3) - 8 = 8k - 11 \) edges out of \( C \). Since \( |E(H, C)| \leq 7 \) and \( t \geq 2 \), \( |E(C, \mathcal{C})| \geq |E(H, C)| = 7 \geq 8t(k - 2) \). Thus the claim holds. Since \( |C| = 4t + r \leq 4t + 3 \) and \( |E(C, C')|/|C| \geq 8t/(4t + 3) > 8t(4t + 4) = 2t/(t + 1) > 1 \), \( d_\mathcal{C}(v) \geq 2 \) for some \( v \in V(C) \).

Suppose that \( \max\{d_\mathcal{C}(v)\} \leq 2 \) for some \( C \). Let \( X = \{v \in V(C)|d_\mathcal{C}(v) \leq 1\} \) and \( Y = V(C) - X \). Then noting that \( t \geq 2 \) and \( r \leq 3 \),

\[
8t \leq |E(C, C')| \leq |X| + 2|Y| = (|C| - |Y|) + 2|Y| = |C| + |Y|
\Rightarrow |Y| \geq 8t - |C| = 8t - (4t + r) = 4t - r \geq 8 - 3 = 5
\]

Thus we have the degree sequence \((2, 2, 2, 2, 2)\) from \( C \) to \( C' \). By Lemma 5(i), \( \langle C \cup C' \rangle \) contains two shorter disjoint cycles, contradicting (1).

Suppose that \( h = \max\{d_\mathcal{C}(v)\} \leq 2 \) for any \( v \in V(C) \). Since \( |C'| \leq |C| = 4t + r \) by the choice of \( C \), \( d_\mathcal{C}(v) \leq |C'| \leq 4t + r \). Then since \( t \geq 2 \) and \( r \leq 3 \), \( |E(C - v^*, C')| \geq 8t - (4t + r) = 4t - r \geq 5 \). This implies that \( N_{C}(C - v^*) \neq \emptyset \). Let \( Z = \{v \in V(C)|N_{C}(v) \neq \emptyset\} \). Then \( |Z| \geq 2 \).

Suppose that \( |Z| = 2 \). Then \( d_\mathcal{C}(v) \geq 5 \) for any \( v \in Z \) by the above observations. By Lemma 5(ii), \( \langle C \cup C' \rangle \) contains two shorter disjoint cycles, contradicting (1).

Suppose that \( |Z| \geq 3 \). Since \( |E(C - v^*, C')| \geq 5 \), we may assume that the minimum degree sequence \( S \) from vertices of \( C \) to \( C' \) is at least one of \((h, 4, 1), (h, 3, 2), (h, 3, 1, 1), (h, 2, 2, 1), (h, 2, 1, 1, 1), (h, 1, 1, 1, 1, 1) \), where by the definition of \( h \),
if \( S = (h, 4, 1) \), then \( h \geq 4 \), and if \( S \) is the other degree sequence, then \( h \geq 3 \). If \( S = (h, 4, 1) \) or \( (h, 3, 2) \), then by Lemma 5(v), \( (C \cup C') \) contains two shorter disjoint cycles. If \( S = (h, 3, 1, 1), (h, 2, 2, 1) \) or \( (h, 2, 1, 1, 1) \), then by Lemma 5(iv), \( (C \cup C') \) contains two shorter disjoint cycles. If \( S = (h, 1, 1, 1, 1) \), then by Lemma 5(iii), \( (C \cup C') \) contains two shorter disjoint cycles.

**Case 2.** \( |H| \geq 8 \).

**Claim 1.** \( H \) is connected.

**Proof.** Suppose to the contrary that \( H \) is disconnected. Then note that \( H \) is a collection of paths. Suppose that \( X \) is an independent set that consists of four leaves from at least two components in \( H \) such that \( d_H(X) \leq 4 \). Then \( d_H(X) \geq (8k - 3) - 4 = 8(k - 1) + 1 \), and \( d_C(X) \geq 9 \) for some \( 1 \leq i_0 \leq k - 1 \). Thus \( d_C(x) \geq 3 \) for some \( x \in X \), and \( |C_0| = 3 \) by Lemma A. By Lemma 1 and (2), \( (H \cup C_0) \) contains two disjoint cycles, and \( G \) contains \( k \) disjoint cycles, a contradiction. Thus \( H \) does not contain such an independent set.

Now, we consider three cases on \( \alpha(H) \).

**Case 1.** \( \alpha(H) \geq 4 \).

We take four leaves \( x_1, x_2, x_3, x_4 \), one from each component of \( H \). Then \( X = \{x_1, x_2, x_3, x_4\} \) is an independent set such that \( d_H(X) \leq 4 \), a contradiction.

**Case 2.** \( \alpha(H) = 3 \).

We take three leaves \( x_1, x_2, x_3 \), one from each component of \( H \). Since \( |H| \geq 8 \), some component of \( H \), say \( H_1 \), has the order at least 3. Now, we take the other leaf from \( H_1 \), call it \( x_4 \). Then \( X = \{x_1, x_2, x_3, x_4\} \) is an independent set such that \( d_H(X) \leq 4 \), a contradiction.

**Case 3.** \( \alpha(H) = 2 \).

Let \( H_1, H_2 \) be two distinct components in \( H \). Without loss of generality, we may assume that \( |H_1| \leq |H_2| \). Suppose that \( |H_1| \geq 5 \). Then we take two leaves from each component of \( H \), yielding a set \( X \) of four independent vertices such that \( d_H(X) = 4 \), a contradiction. Suppose that \( |H_1| \leq 4 \). Since \( |H_1| \geq 8, |H_2| \leq 6 \). Let \( H_1 = x_1, x_2, x_3, x_4 \). Suppose that \( |H_2| = 4 \), \( |H_3| = 3 \), \( |H_4| = 2 \), and \( |H_5| = 1 \). Thus \( d_H(X) \leq 4 \), a contradiction.

Suppose that \( |C_0| = 4 \). Let \( C_0 = v_1, v_2, v_3, v_4 \). Suppose that \( d_H(u) \leq 2 \) for any \( u \in W \) by Lemma A. Thus we must have degree sequence \( (2, 2, 2, 2) \). If some \( u \in W \) has degree \( d_H(u) \geq 3 \), then we must have \( d_H(X) \leq 5 \), a contradiction.

**Subcase 1.** \( S = (3, 3, 2, 0) \) or \( S = (3, 3, 1, 1) \).

By Lemma 6, we can find two disjoint cycles in \( (C_0 \cup H) \), a contradiction.

**Subcase 2.** \( S = (3, 3, 2, 1) \).

If \( d_C(y_3) = 1 \), then \( \{x_1, y_3, y_4\} \) satisfies the conditions of Lemma B, we get a contradiction. Thus \( d_C(y_3) \geq 2 \).

First, suppose that \( d_C(y_1) = 1 \). Let \( v_1 \in N_C(x_1) \). Note that \( d_C(y_1) \geq 2 \) for each \( v_i \in \{1, 3, 5\} \). If \( v_1 \notin N_C(y_0) \) for some \( i_0 \in \{1, 3, 5\} \), then \( d_C(y_1) = 2 \), and \( C_0 = y_0, v_3, v_2, y_3 \) is a 3-cycle. Since \( d_C(y_1) = 3 \) for some \( i_1 \in \{1, 3, 5\} \), \( v_1 \in N_C(y_1) \). Then \( (C_0 \cup H) \sim C_0 \) is connected, contradicting (2) (see Fig. 1.1). Thus \( v_1 \in N_C(y_1) \). Since \( d_C(y_3) = 2 \) for some \( i_2 \in \{1, 3, 5\} \), \( C_0 = y_3, v_2, v_3, y_1 \) is a 3-cycle. Then \( (C_0 \cup H) \sim C_0 \) is connected, contradicting (2).

Next, suppose that \( d_C(y_1) = 2 \). Suppose that \( d_C(y_1) = 3 \). If \( v_1 \in N_C(y_1) \) for some \( m_1 \in \{1, 3, 5\} \), then \( H^2 \{y_1, y_2, y_3, y_4\} \) is a 2-cycle. Then \( \{x_1, x_2, x_3, x_4\} \) is two disjoint cycles. Thus \( C_0 \sim N_C(y_i) \) for each \( i \in \{1, 3, 5\} \), \( (C_0 \cup H) \sim C_0 \) is connected, contradicting (2) (see Fig. 2).
Finally, suppose that $d_{C_0}(x_1) = 3$. Since $d_{C_0}(y_0) = d_{C_0}(y_9) = 2$ for some $i_0, j_0 \in \{1, 3, t\}$ with $i_0 < j_0$, we may assume that $v_1 \in N_{C_0}(y_0) \cap N_{C_0}(y_9)$. Then $H_2[y_0, y_9]$, $v_1$, $y_0$ is a cycle. Since $d_{C_0}(x_1) = 3$, $v_3, v_2, x_1$ is the other disjoint cycle.

**Subcase 3.** $S = \{2, 2, 2, 2\}.$

Without loss of generality, we may assume that $N_{C_0}(x_1) = \{v_1, v_2\}$. If $v_3 \in N_{C_0}(y_0) \cap N_{C_0}(y_9)$ for some $i_0, j_0 \in \{1, 3, t\}$ with $i_0 < j_0$, then $H_2[y_0, y_9]$, $v_3, y_0$ and $x_1$, $v_2, v_1, x_1$ are two disjoint cycles. Thus at most one in $\{y_1, y_3, y_t\}$ can be adjacent to $v_3$. Suppose that $v_3 \in N_{C_0}(y_0)$ for some $i_0 \in \{1, 3, t\}$. Since $d_{C_0}(y_0) = 2$, we may assume that $v_2 \in N_{C_0}(y_0)$. Then $C_0' = y_0, v_3, v_2, y_0$ is a 3-cycle. For each $i \in \{1, 3, t\} - \{i_0\}, N_{C_0}(y_i) = \{v_1, v_2\}$. Then $(C_0 \cup H) - C_0'$ is connected, contradicting (2). Thus $v_3 \notin N_{C_0}(y_i)$ for each $i \in \{1, 3, t\}$, that is, $N_{C_0}(y_t) = \{v_1, v_2\}$. Then $C_0'' = H_2[y_1, y_3], v_2, y_1$ is a 3-cycle, and $(C_0 \cup H) - C_0''$ is connected, contradicting (2). This completes the proof of **Claim 1.**

**Claim 2.** $H$ is a path.

**Proof.** Suppose that $H$ is not a path. Then recall that $H$ is a tree with one branch vertex of degree 3 in $H$. Then $H$ has three leaves, say $x_1, x_2, x_3$. Removing the branch vertex in $H$, there exist three disjoint paths each of which has one in $\{x_1, x_2, x_3\}$ as an endpoint. Also, some path has a length at least two, say $P$, since there exist at least seven vertices distributed over three paths. Without loss of generality, we may assume that $x_1$ is one of the endpoints of $P$, and let the other endpoint be $x_4$. Let $X = \{x_1, x_2, x_3, x_4\}$ (see Fig. 3). Then $X$ is an independent set. Since $d_H(X) = 5, d_{\varphi}(X) \geq (8k - 3) - 5 = 8(k - 1)$. Thus there exists a cycle $C_0$ in $\varphi$ such that $d_{C_0}(X) \geq 8$ for some $1 \leq i_0 \leq k - 1$. Then $d_{C_0}(x) \geq 2$ for some $x \in X$. By Lemma A, $d_{C_0}(x) \leq 3$ and $|C_0| \leq 4$.

**Case 1.** $|C_0| = 3.$
Let $C_0 = v_1, v_2, v_3, v_4$. Suppose that $d_{C_0}(x) = 2$ for each $x \in X$. Let $v_1, v_2 \in N_{C_0}(x_1)$. Since $|C_0| = 3, N_{C_0}(x_2) \cap N_{C_0}(x_3) \neq \emptyset$. If $v_3 \in N_{C_0}(x_2) \cap N_{C_0}(x_3)$, then $H[x_2, x_3], v_3, x_2$ and $x_1, v_2, v_1$ are two disjoint cycles. Thus without loss of generality, we may assume that $v_1 \in N_{C_0}(x_2) \cap N_{C_0}(x_3)$. Then $H[x_2, x_3], v_1, x_2$ is a cycle. Since $d_{C_0}(x_2) = 2, N_{C_0}(x_2) \neq \emptyset$. If $v_3 \in N_{C_0}(x_4)$, then $H[x_1, x_4], v_3, x_1$ is the other disjoint cycle, and if $v_3 \in N_{C_0}(x_4)$, then $H[x_1, x_4], v_3, x_1$ is the other disjoint cycle. Thus there exists at least one vertex $x \in X$ such that $d_{C_0}(x) = 3$. Then the minimum possible degree sequences from $X$ to $C_0$ are $(3, 3, 2, 0), (3, 3, 1, 1)$ or $(3, 2, 2, 1)$.

We claim that if there exists a degree sequence $(3, 3, 1, 0)$ from $X$ to $C_0$, then there exist two disjoint cycles in $(H \cup C_0)$.

First, suppose that $d_{C_0}(x_0) = 1$ for some $1 \leq j_0 \leq 3$. Let $v_1 \in N_{C_0}(x_0)$. If $d_{C_0}(x_4) = 0$, then since $d_{C_0}(x_4) = 3$ for each $m \in \{1, 2, 3\} - \{j_0\}, H[x_0, x_m], v_1, x_m$ is a cycle. Since $d_{C_0}(x_m) = 3$ for $m \in \{1, 2, 3\} - \{j_0, m\}, x_m$, $v_3, v_2, x_m$ is the other disjoint cycle. If $d_{C_0}(x_4) = 3$, then $H[x_0, x_4], v_1, x_4$ is a cycle, and since $d_{C_0}(x_m) = 3$ for some $m_0 \in \{1, 2, 3\} - \{j_0, x_m, v_3, v_2, x_m\}$ is the other disjoint cycle. Next, suppose that $d_{C_0}(x_4) = 1$. Let $v_1 \in N_{C_0}(x_4)$. Then $d_{C_0}(x_m) = 3$ and $d_{C_0}(x_m) = 3$ for some $1 \leq m_1 < m_2 \leq 3$, and $H[x_1, x_4], v_1, x_1$ and $x_4, v_3, x_2, x_m$ are two disjoint cycles.

Thus by the claim, we have only to consider the degree sequence $(3, 2, 2, 1)$. If the degree 3 vertex does not lie on the path connecting the degree 2 vertices, then since the two vertices with degree 2 must have a common neighbor by $|C_0| = 3$, we can easily find two disjoint cycles. Thus the degree 3 vertex does lie on the path connecting the two vertices with degree 2. Then $d_{C_0}(x_4) = 3, d_{C_0}(x_1) = 2$, and we may assume that $d_{C_0}(x_2) = 1$ and $d_{C_0}(x_3) = 2$. Let $v_1 \in N_{C_0}(x_2)$. Since $|N_{C_0}(x_1) \cap N_{C_0}(x_4)| = 2$, there exists $v_{h_0} \in N_{C_0}(x_1) \cap N_{C_0}(x_4)$ for some $h_0 \in \{2, 3\}$. Then $H[x_1, x_4], v_{h_0}, x_4$ is a cycle. Since $d_{C_0}(x_2) = 2$, there exists $v_{h_1} \in N_{C_0}(x_3)$ for some $h_1 \in \{1, 2, 3\} - \{h_0\}$. If $h_1 = 1$, then $H[x_2, x_3], v_1, x_2$ is the other disjoint cycle, and if $h_1 = 2, 3$, then $H[x_2, x_3], v_{h_1}, x_1, v_2, x_3$ is the other disjoint cycle.

Case 2. $|C_0| = 4$.

Let $C_0 = v_1, v_2, v_3, v_4, v_1$. By Lemma A, $d_{C_0}(x) = 2$ for each $x \in X$. Since $d_{C_0}(X) \geq 8, d_{C_0}(x) = 2$ for each $x \in X$. Any vertex in $X$ does not have consecutive neighbors in $C_0$, otherwise, we can immediately find a 3-cycle, contradicting (1). Thus for each $x \in X$, either $N_{C_0}(x) = \{v_1, v_3\}$ or $N_{C_0}(x) = \{v_2, v_4\}$.

Subcase 1. All four vertices in $X$ have the same two neighbors in $C_0$.

We may assume that $N_{C_0}(x) = \{v_1, v_3\}$. Then $H[x_1, x_4], v_1, x_1$ and $H[x_2, x_3], v_2, x_2$ are two disjoint cycles.

Subcase 2. Three vertices in $X$ have the same two neighbors in $C_0$.

Suppose that $x_1, x_4$ have the same two neighbors in $C_0$. Then we may assume that $v_1 \in N_{C_0}(x_1) \cap N_{C_0}(x_4)$, and $H[x_1, x_4], v_1, x_1$ is a cycle. Since $d_{C_0}(x_j) = 2$ for each $j \in \{2, 3\}, N_{C_0}(x_1) \neq \emptyset$. Then $(H[x_2, x_3] \cup (C_0 - \{v_1\}))$ contains the other disjoint cycle. Suppose that $x_1, x_4$ do not have the same two neighbors in $C_0$. Since $x_2, x_3$ have the same two neighbors in $C_0$, we repeat the above arguments, replacing $x_1, x_4$ with $x_2, x_3$.

Subcase 3. Two vertices of $X$ have the same two neighbors in $C_0$, and the other two vertices of $X$ have the same two neighbors, different from the neighbors of the first two.

Suppose that $x_1, x_4$ have the same two neighbors. We may assume that $v_1 \in N_{C_0}(x_1) \cap N_{C_0}(x_4)$. Then $H[x_1, x_4], v_1, x_1$ is a cycle. Since $x_2, x_3$ have the same two neighbors, different from the neighbors of $x_1$ and $x_4$, $H[x_2, x_3], v_2, v_3$ is the other disjoint cycle. Suppose that $x_1, x_4$ have different neighbors. We may assume that $v_1 \in N_{C_0}(x_1)$ and $v_2 \in N_{C_0}(x_4)$. Then $H[x_1, x_4], v_2, v_1, x_1$ is a cycle. Since $x_2, x_3$ have the neighbors, different from $v_1, v_2, (H[x_2, x_3] \cup \{v_3, v_4\})$ contains the other disjoint cycle. \(\square\)

Since $H$ is a path by Claim 2, let $H = x_1, x_2, \ldots, x_t$ (t ≥ 8). Let $X = \{x_1, x_3, x_5, x_1\}$. Then $X$ is an independent set with $d_{\overline{H}}(X) = 6$, and $d_{\overline{H}}(X) \geq (8k - 3) - 6 = 8k - 9 \geq 7(k - 1), \text{ since } k \geq 2$. Thus either $d_{C_0}(X) \geq 8$ for some cycle $C_0 \subset \overline{H}$, or $d_{C}(X) = 7$ for every cycle $C$ in $\overline{H}$. If $d_{C}(X) \geq 8$ for some cycle $C \subset \overline{H}$, then we have the minimum possible degree sequences $(3, 3, 2, 0), (3, 3, 1, 1), (3, 2, 2, 1)$ or $(2, 2, 2, 2)$ from $X$ to $C$. If $d_{C}(X) = 7$ for some cycle $C \subset \overline{H}$, then we have the minimum possible degree sequences $(3, 3, 1, 0), (3, 2, 1, 1), (3, 2, 2, 0)$ or $(2, 2, 2, 1)$ from $X$ to $C$.

Subclaim 1. If there exists a degree sequence $(3, 3, 1, 0)$ from $X$ to $C$, then there exist two disjoint cycles in $(H \cup \overline{C})$.

Proof. By Lemma A, $|C| = 3$. Let $C = v_1, v_2, v_3, v_4$. We may assume that $d_{C_0}(x_0) = 1$ for some $i_0 \in \{1, 3\}$, otherwise, $i_0 \in \{5, t\}$, and we may argue in a similar manner from the other end of the path $H$. Let $v_1 \in N_{C_0}(x_0)$. First, suppose that $i_0 = 1$, that is, $d_{C}(x_1) = 1$. Then $d_{C}(x_1) = d_{C}(x_2) = 3$ for some $j_1, j_2 \in \{3, 5, t\}$ with $j_1 < j_2$. Thus $H[x_1, x_2], x_1, x_1$ and...
Proof. Let $v_1, v_2, \ldots, v_q, v_1$, where $q = |C|$. We may assume that $d_C(x_0) = 1$ for some $i_0 \in \{5, t\}$, otherwise, $i_0 \in \{1, 3\}$, and we may argue in a similar manner from the other ends of the path $H$. Let $v_1 \in N_C(x_0)$.

Case 1. $|C| = 4$. Then $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3$, and $d_C(x_0) = 2$ for each $i \in \{1, 3\}, N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 3$, and either $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, or $|C| = 3
Suppose that $N_C(x_4) \neq \emptyset$. Then replacing $x_6$ in the above argument with $x_4$ and $x_5$ with $x_1$, we can prove this case by the same arguments above. Thus $N_C(x_i) = \emptyset$ for each $i \in \{4, 6\}$. This implies that $d_C(x_2) = 3$. Then $x_0, x_2, v_1, x_0$ and $x_5, v_3, v_2, x_5$ are two disjoint cycles. □

4. Proofs of Lemmas

4.1. Proof of Lemma 1

Let $F, C, x_i (1 \leq i \leq 4)$ be as in Lemma 1. Let $F_1, F_2$ be two components of $F$, $C = v_1, v_2, v_3, v_4$, and $X = \{x_1, x_2, x_3, x_4\}$. Now, we consider two cases.

Case 1. At most two vertices of $X$ lie in the same component of $F$.

Since $d_C(X) \geq 5$, $d_C(x_i) \geq 3$ for some $1 \leq i \leq 4$. By $|C| = 3$, $d_C(x_i) \leq 3$ for each $1 \leq i \leq 4$. Thus $d_C(x_6) = 3$. Without loss of generality, we may assume that $i_0 = 1$, that is, $d_C(x_1) = 3$. Then $d_C(x_2, x_3, x_4) \geq 6$. Also, we may assume that $d_C(x_2) \geq d_C(x_3) \geq d_C(x_4)$. Now, we claim that $d_C(x_2, x_3) \geq 4$. Otherwise, if $d_C(x_2, x_3) \leq 3$, then $d_C(x_6) \leq 1$ for some $j_0 \in \{2, 3\}$. That implies that $d_C(x_4) \leq 1$, since $d_C(x_4)$ is the smallest degree in $x_2, x_3, x_4$. Then $d_C(x_2, x_3) \leq 3 + 1 = 4$, a contradiction. Thus the claim holds. Noting our assumption of this case, $\{x_1, x_2, x_3\}$ is a set of leaves from at least two components of $F$. Since $d_C(x_1, x_2, x_3) \geq 3 + 4 = 7$, Lemma B applies, completing this case.

Case 2. Three vertices of $X$ lie in the same component of $F$.

Without loss of generality, we may assume that $x_1, x_2, x_3 \in V(F_1), x_4 \in V(F_2)$, and $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3)$. Recall that $d_C(X) \geq 9$. It follows that the minimum possible degree sequence 5 from $X$ to $C$ is $(3, 3, 3, 0), (3, 3, 2, 1)$ or $(3, 2, 2, 2)$.

Subcase 1. $S = (3, 3, 3, 0)$.

If $d_C(x_6) = 0$ for some $1 \leq i \leq 3$, then $i_0 = 3$, that is, $d_C(x_3) = 0$. Now, we take $\{x_1, x_2, x_4\}$ that is a set of leaves from at least two components of $F$. Since $d_C(x_1, x_2, x_4) = 9$, Lemma B applies. If $d_C(x_4) = 0$, then $d_C(x_1) = 3$ for each $1 \leq i \leq 3$. Since all the $x_i$s are leaves, $x_3$ does not lie on the path in $F_1$ connecting $x_1$ and $x_2$. Then $F_1[x_1, x_2], v_1, x_1$ and $x_3, v_2, x_3$ are two disjoint cycles in $(F \cup C)$.

Subcase 2. $S = (3, 3, 2, 1)$.

Take $\{x_1, x_2, x_4\}$. If $d_C(x_4) \in \{1, 2\}$, then $d_C(x_1, x_2) \geq 6$. If $d_C(x_4) = 3$, then $d_C(x_1, x_2) \geq 5$. Since $d_C(x_1, x_2, x_4) \geq 7$ for all cases, Lemma B applies.

Subcase 3. $S = (3, 2, 2, 2)$.

Take $\{x_1, x_2, x_4\}$. If $d_C(x_4) = 2$, then $d_C(x_1, x_2) \geq 5$. If $d_C(x_4) = 3$, then $d_C(x_1, x_2) \geq 4$. Since $d_C(x_1, x_2, x_4) \geq 7$ for all cases, Lemma B applies. □

4.2. Proof of Lemma 5

Proof of (i). Let $v_1, v_2, v_3, v_4, v_5$ be the vertices such that $d_C(v_i) = 2$ for each $1 \leq i \leq 5$, appearing in this order on $C$. Let $w_1, w_2 \in N_C(v_1)$ appear in this order on $C$. The neighbors of $v_1$ partition $C_1$ into two intervals $C_1[w_1, w_2]$ and $C_1[w_2, w_1]$. We claim that each of $v_2, v_3, v_4, v_5$ has one neighbor in different interval of $C_1$.

First, suppose that $v_1, v_2, v_3, v_4$ for some $2 \leq i_1 < i_2 < i_3 \leq 5$ have both their neighbors in a common interval of $C_1$, say $C_1[w_1, w_2]$. We may assume that at least one of their neighbors is not $w_2$. Let $z_1 \in N_C(w_1,w_2)(v_1)$ and $z_2 \in N_C(w_1,w_2)(v_2)$. Then $C_1[z_1, z_2, C_2, v_3, v_4, v_5, z_0]$, and $C_1[w_2, w_1]$, $v_1, v_2$ are shorter two disjoint cycles, since $v_3$ is not used.

Next, suppose that $v_1, v_2$, for some $2 \leq i_1 < i_2 \leq 5$ have both their neighbors in a common interval of $C_1$, say $C_1[w_1, w_2]$. Then we may assume that $i_1 = 2$ and $i_2 = 5$, otherwise, we can prove the other pairs of $i_1$ and $i_2$ by the same arguments above. Let $z_1 \in N_C(w_1,w_2)(v_2)$ and $z_2 \in N_C(w_1,w_2)(v_5)$. If $N_C(w_1, w_2)(v_1) \neq \emptyset$ for some $i_0 \in \{3, 4\}$, then there exist shorter two disjoint cycles. Thus $N_C(w_1, w_2)(v_j) = \emptyset$ for each $j \in \{3, 4\}$. Since $d_C(v_j) = 2$ for each $j \in \{3, 4\}$, $N_C(w_1, w_2)(v_j) = \emptyset$. Let $z_4 \in N_C(v_1,v_2)(v_3)$ and $z_4 \in N_C(v_2,v_1)(v_4)$. Then $C_1[z_1, z_2, C_2, v_3, v_4, v_5, z_3]$, and $C_1[z_1, z_4, C_2, v_1, v_2, z_0]$, $C_1[v_2, v_3, v_4]$ are shorter two disjoint cycles, since $v_5$ is not used.

Finally, suppose that $v_0$ for some $2 \leq i_0 \leq 5$ has both the neighbors in an interval of $C_1$, say $C_1[w_1, w_2]$. Then we have only to consider $i_0 = 2$ or $i_0 = 3$, otherwise, we take a cycle from $v_1$ in the opposite direction. First, suppose that $i_0 = 2$. Let $x_1, x_2 \in N_C[w_1,w_2](v_2)$, appearing in this order on $C$. If $x_2 \neq w_2$, then $C_1[x_1, x_2, v_1, x_1]$ and $C_1[w_1, w_2, v_1]$, $v_1, w_2$ are shorter two disjoint cycles, since $v_3$ is not used. Thus $x_2 = w_2$. Let $y_1, y_2 \in N_C(v_3)$, appearing in this order on $C$. Suppose that $y_1 \in C_1[w_1, w_2]$. Then $C_1[x_1, y_1, C_2, v_2, v_1, x_1]$ and $C_1[w_1, w_2, v_1]$, $v_1, w_2$ are shorter two disjoint cycles, since $v_4$ is not used. Thus $y_1 \notin C_1[w_1, w_2]$, that is, $y_1 \in C_1[w_2, w_1]$. Note that $y_2 \in C_1[w_2, w_1]$. If $y_1 \neq w_2$, then $C_1[x_1, w_2, v_1, x_1]$ and $C_1[y_1, v_3, y_2]$, $v_2$ are shorter two disjoint cycles, since $v_3$ is not used. Thus $y_1 = w_2$. If $y_2 \neq w_2$, then $C_1[w_2, y_2]$, $v_2, w_2$ and $C_1[w_1, x_1]$, $C_2[v_2, v_1]$, $w_1$ are shorter two disjoint cycles, since $v_4$ is not used. Thus $y_2 = w_1$. Let $z_1, z_2 \in N_C(v_4)$, appearing in this order on $C$. Suppose that $z_1 \in C_1[w_1, w_2]$. Then $C_1[w_1, z_1, C_2[v_4, v_3], v_1, C_2[v_2, v_1, z_0]$, $C_1[w_2, z_1]$, $C_2[v_4, v_3]$, $w_2$ are shorter two disjoint cycles, since $v_3$ is not used. Next, suppose that $i_0 = 3$. Then, by the same arguments as the case where $i_0 = 2$, we have shorter two disjoint cycles, replacing $v_2$ with $v_3$. 


Thus each of $v_2, v_3, v_4, v_5$ has one neighbor in each interval of $C_1$. Let $x \in N_{C_1[w_1,w_2]}(v_2), y \in N_{C_1[w_1,w_2]}(v_5), z \in N_{C_1[w_1,w_2]}(v_4), u \in N_{C_1[w_1,w_2]}(v_3)$. Then $C_1^{+}[x, y], C_1^{+}[v_2, v_3], x$ and $C_1^{-}[z, u], C_1^{-}[v_3, v_4], z$ are shorter two disjoint cycles, since $v_1$ is not used. □

**Proof of (ii).** Let $v_1, v_2 \in V(C_2)$ such that $d_{C_2}(v_1) = 5$ and $d_{C_2}(v_2) = 3$, appearing in this order on $C_2$. Let $w_1, w_2, w_3, w_4, w_5 \in N_{C_1}(v_1)$, appearing in this order on $C_1$, and let $u_1, u_2, u_3 \in N_{C_1}(v_2)$, appearing in this order on $C_1$. The neighbors of $v_1$ partition $C_1$ into five intervals $C_1[w_1, w_{i+1}], 1 \leq i \leq 5$ (mod 5). Suppose that $u_0, u_0 \in C_1[w_{m_0}, w_{m_0+1}]$ (mod 5) for some $1 \leq i_0 < j_0 \leq 3$ and for some $1 \leq m_0 \leq 5$. Without loss of generality, we may assume that $i_0 = 1, j_0 = 2$ and $m_0 = 1$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_4], v_1, w_3$ are shorter two disjoint cycles, since $w_1$ is not used. Thus neighbors of $v_2$ are contained in different intervals. Since $C_1$ is partitioned into five intervals, some two neighbors of $v_2$ lie in neighboring intervals, say $u_1 \in \{w_1, w_2\}$ and $u_2 \in C_1[w_2, w_3]$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_4, w_5], v_1, w_4$ are shorter two disjoint cycles, since $w_1$ is not used. □

**Proof of (iii).** Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the vertices on $C_2$ with the degree sequence $(3, 1, 1, 1, 1, 1)$, appearing in this order on $C_2$. Without loss of generality, we may assume that $d_{C_1}(v_1) = 3$ and $d_{C_1}(v_2) = 2$ and $d_{C_1}(v_i) = 1$ for each $i \in \{3, 4, 5\}$. Suppose that $v_1, v_2$ are in this order on $C_2$. Let $w_1, w_2, w_3 \in N_{C_1}(v_1)$ be in this order on $C_1$, and let $u_1, u_2 \in N_{C_1}(v_2)$ be in this order on $C_1$. Let $v_3, v_4$ be in this order on $C_2$. Let $z_1 \in N_{C_1}(v_3)$, and let $z_2 \in N_{C_1}(v_4)$. The neighbors of $v_1$ partition $C_1$ into three intervals: $C_1[w_1, w_2], C_1[w_2, w_3], C_1[w_3, w_1]$. If $v_2$ has both its neighbors in the same interval in $C_1$, then we can find shorter two disjoint cycles. If the neighbors of $v_2$ are in two different intervals of $C_1$ and neither is in $\{w_1, w_2\}$, then we can also find shorter two disjoint cycles. Thus the neighbors of $v_2$ are in two different intervals of $C_1$ and at least one of them is at an endpoint of these intervals. Without loss of generality, we may assume that $u_1 \in C_1[w_1, w_2]$ and $u_2 \in C_1[w_2, w_3]$. Now, we consider two cases.

**Case 1.** $v_3, v_4 \in C_2(v_1, v_2)$ or $v_3, v_4 \in C_2(v_2, v_1)$.

Without loss of generality, we may assume that $v_3, v_4 \in C_2(v_1, v_2)$. If $z_2 \in C_1[w_1, w_3], C_2[v_4, v_2], v_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_3$ is not used. If $z_2 \in C_1[w_3, w_1], C_1[v_2, v_4], v_2$ and $C_1[w_1, w_2], v_1, w_1, v_3$ are shorter two disjoint cycles, since $v_3$ is not used. Thus $z_2 = w_1$.

If $u_2 \in C_1[w_2, w_3], C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_3$ is not used. Thus $u_2 = w_3$.

If $z_1 \in C_1[w_3, w_1], C_1[v_3, v_1], z_1$ and $C_1[w_1, w_3], v_3$ are shorter two disjoint cycles, since $v_3$ is not used. Thus $z_1 \in C_1[w_1, w_3]$.

Suppose that $u_1 \in C_1[w_1, w_3]$. If $z_1 \in C_1[w_1, w_3], C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_3$ is not used. If $z_1 \in C_1[w_1, w_3], C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_3$ is not used. If $z_1 \in C_1[w_1, w_3], C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_3$ is not used. Thus $u_1 \in C_1[w_1, w_3]$.

Now, we consider two disjoint cycles $C' = w_1, C_2[v_1, v_4], v_1$ and $C'' = C_1[w_1, w_3], v_2$. Note that $|C_2| \geq 6$ if $C_2(v_4, v_2) \neq \emptyset$ or $C_2(v_2, v_2) \neq \emptyset$, then $C'$ and $C''$ are shorter two disjoint cycles. Thus $C_2(v_4, v_2) = \emptyset$ and $C_2(v_2, v_1) = \emptyset$. First, suppose that $z_1 \in C_1[w_2, w_3]$. If $C_2(v_2, v_1) \neq \emptyset$, then $C_1[w_3, w_1], v_1, w_3$ and $C_2(v_2, v_3), C_1[w_2, z_1], v_3$ are shorter two disjoint cycles. If $C_2(v_2, v_3) \neq \emptyset$, then $C_1[w_1, w_3], v_1, w_3$ and $C_2[v_3, v_1], v_1, w_3$ are shorter two disjoint cycles. Next, suppose that $z_1 = w_3$. If $C_2(v_3, v_3) \neq \emptyset$, then $C_1[w_1, w_3], v_1, w_3$ and $C_2[v_3, v_2], v_2, v_3$ are shorter two disjoint cycles. If $C_2(v_3, v_3) \neq \emptyset$, then $C_1[v_1, v_3], v_3, v_1$ and $C_1[w_1, w_3], C_2[v_3, v_2], v_3$ are shorter two disjoint cycles.

**Case 2.** $v_3 \in C_2(v_1, v_2)$ and $v_4 \in C_2(v_2, v_1)$.

If $z_1 \in C_1[w_3, w_1], C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_4$ is not used. If $z_1 \in C_1[w_3, w_1], C_1[w_3, w_1], v_1, w_3$ are shorter two disjoint cycles, since $v_4$ is not used. Thus $z_1 = w_1$. Then $C_1[w_1, w_3], v_1, w_3$ and $C_1[w_1, w_3], v_1, w_3$ are shorter two disjoint cycles, since $v_4$ is not used. □

**Proof of (v).** Let $v_1, v_2, v_3$ be the vertices on $C_3$ with the degree sequence $(3, 3, 1)$. Suppose that $v_1, v_2, v_3$ exist in this order on $C_3$. Without loss of generality, we may assume that $d_{C_3}(v_i) = 3$ each $i \in \{1, 2, 3\}$ and $d_{C_3}(v_1) = 1$. Suppose that $w_1, w_2, w_3 \in N_{C_1}(v_1)$ exist in this order on $C_1$. Let $W = \{w_1, w_2, w_3\}$. These neighbors of $v_1$ partition $C_1$ into three intervals: $C_1[w_1, w_2], C_1[w_2, w_3], C_1[w_3, w_1]$. Let $u_1, u_2, u_3 \in N_{C_1}(v_2)$ and suppose that $u_1, u_2, u_3$ are in this order on $C_1$.

**Case 1.** Some two neighbors of $v_2$ are in the same interval of $C_1$.

Without loss of generality, we may assume that $u_1, u_2 \in C_1[w_1, w_2]$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_1], v_3, w_3$ are shorter two disjoint cycles, since $v_1$ is not used.

**Case 2.** No two neighbors of $v_2$ are in the same interval of $C_1$.
Then $u_1 \in C_1(w_1, w_2), u_2 \in C_1(w_2, w_3)$, and $u_3 \in C_1(w_3, w_1)$. First, suppose that $u_{i_0}, u_{j_0} \notin W$ for some $1 \leq i_0 < j_0 \leq 3$. Without loss of generality, we may assume that $i_0 = 1$ and $j_0 = 2$, that is, $u_1 \in C_1(w_1, w_2)$ and $u_2 \in C_1(w_2, w_3)$. Then $C_1[u_1, u_2], v_2, u_1$ and $C_1[w_3, w_1], v_1, w_3$ are two shorter two disjoint cycles, since $v_3$ is not used.

Next, suppose that $u_{i_0} \notin W$ for only some $1 \leq i_0 \leq 3$. Without loss of generality, we may assume that $i_0 = 1$, that is, $u_1 \in C_1(w_1, w_2)$. Then note that $u_3 = w_1, C_1[w_1, u_1], v_2, w_1$ and $C_1[w_2, w_3], v_1, w_2$ are two shorter two disjoint cycles, since $v_3$ is not used.

Finally, suppose that $u_i = w_{i+1} \pmod{3}$ for each $1 \leq i \leq 3$. Without loss of generality, we may assume that $v_2 z_1 \in E(G)$ for $z_1 \in (w_2, w_3)$. Now, we have two choices for constructing shorter two disjoint cycles. We may construct $C_1[w_1, w_2], v_2, w_1$ and $C_1[z_1, w_3], C_2[v_1, v_3], z_1$, or $C_1[w_1, w_2], v_1, w_1$ and $C_1[z_1, w_3], C_2[v_2, v_3], z_1$. Since $|C_2| \geq 6$, one of these two choices must leave out a vertex of $C_2$, and hence we may form shorter two disjoint cycles. □

4.3. Proof of Lemma 6

Let $C = v_1, v_2, v_3, v_1$.

Case 1. The sequence is $(3, 3, 2, 0)$.

Suppose that $d_C(x_1) = 0$. Then $d_C(y_{i_0}) = 3$ for some $i_0 \in \{1, i, t\}$, and we may assume that $i_0 = 1$, that is, $d_C(y_1) = 3$. Since $d_C(y_i) \geq 2$ for each $r \in \{i, t\}$ and $|C| = 3$, $v_{m_0} \in N_C(y_i) \cap N_C(y_j)$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_1, y_2], v_1, y_1$ and $v_1, v_3, v_2, y_1$ are two disjoint cycles.

Suppose that $d_C(x_1) = 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_1)$. Then $x_1, v_2, v_1, x_1$ is a cycle. Since $d_C(y_{i_0}) = d_C(y_{j_0}) = 3$ for some $i_0, j_0 \in \{1, i, t\}$ with $i_0 < j_0$ and $|C| = 3$, $v_3 \in N_C(y_{i_0}) \cap N_C(y_{j_0})$. Then $H_2[y_0, y_{j_0}], v_3, y_0$ is the other disjoint cycle.

Suppose that $d_C(x_1) = 3$. Since $d_C(y_{i_0}) \geq 2$ and $d_C(y_{j_0}) \geq 2$ for some $i_0, j_0 \in \{1, i, t\}$ with $i_0 < j_0$ and $|C| = 3$, $v_{m_0} \in N_C(y_{i_0}) \cap N_C(y_{j_0})$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_0, y_{j_0}], v_1, y_0$ and $x_1, v_3, v_2, x_1$ are two disjoint cycles.

Case 2. The sequence is $(3, 3, 1, 1)$.

Suppose that $d_C(x_1) = 1$. Then $d_C(y_{i_0}) = 3$ for some $i_0 \in \{1, i, t\}$, and we may assume that $i_0 = 1$, that is, $d_C(y_1) = 3$. Since one of $y_i$ and $y_t$ has degree 3 to $C$ and the other one of them has degree 1 to $C$, noting that $|C| = 3$, $v_{m_0} \in N_C(y_i) \cap N_C(y_t)$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_1, y_t], v_1, y_1$ and $v_1, v_3, v_2, y_1$ are two disjoint cycles.

Suppose that $d_C(x_1) = 3$. Since one of $y_1, y_i, y_t$ has degree 3 to $C$ and the others of them have degree 1 to $C$, $d_C(y_{i_0}) = 3$ and $d_C(y_{j_0}) = 1$ for some distinct $i_0, j_0 \in \{1, i, t\}$. Then note that either $i_0 < j_0$ or $i_0 > j_0$. Since $|C| = 3$, $v_{m_0} \in N_C(y_{i_0}) \cap N_C(y_{j_0})$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_0, y_{j_0}], v_1, y_0$ and $v_1, v_3, v_2, x_1$ are two disjoint cycles. □

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