On Rotation Numbers for Digraphs

Gary Chartrand
Department of Mathematics and Statistics
Western Michigan University, Kalamazoo, MI 49008, USA

Ronald J. Gould
Department of Mathematics and Computer Science
Emory University, Atlanta, Georgia 30322, USA

Ewa Kubicka
Department of Mathematics
University of Louisville, Louisville, Kentucky 40292, USA

Grzegorz Kubicki
Department of Mathematics
University of Louisville, Louisville, Kentucky 40292, USA

Abstract

The rotation number $h(D)$ of a digraph $D$ of order $p$ is the minimum number of arcs in a digraph $F$ of order $p$ such that for every vertex $x$ of $D$ and every vertex $y$ of $F$, there exists an embedding of $D$ in $F$ with $x$ at $y$.

The rotation number is determined for all asymmetric digraphs whose underlying graph is a star and studied for tournaments as well as for asymmetric digraphs whose underlying graph is a cycle.

1. Introduction

For a graph $G$ of order $p$ rooted at a vertex $x$, the rotation number $h(G, x)$ of this rooted graph is the minimum size of a graph $F$ of order $p$ such that for every vertex $y$ of $F$, there exists an embedding of $G$ in $F$ with $x$ at $y$. The notation $h$ in $h(G, x)$ indicates that $G$ can be homogeneously embedded in $F$.

---

1 Research supported in part by Office of Naval Research Contract N00014-91-J-1060.

2 Research supported in part by Office of Naval Research Contract N00014-88-K-0070 and N00014-91-J-1085.
This concept was introduced in 1980 by Cockayne and Lorimer [5] and emanates from a problem in broadcasting. Rotation numbers have been investigated for complete bipartite graphs [1, 5], unions of cycles [2, 3], unions of stars [7], a class of rooted graphs [8], generalized stars [6] and unicyclic graphs containing exactly one bridge [4].

For a graph $G$ of order $p$ that is not rooted, the rotation number $h(G)$ of $G$ is the minimum size of a graph $F$ of order $p$ such that for every vertex $x$ of $G$ and every vertex $y$ of $F$, there exists an embedding of $G$ in $F$ with $x$ at $y$. For example, $h(G) = 9$ for the graph $G$ of Figure 1. The unique graph $F$ of size 9 in which $G$ can be homogeneously embedded is also shown in Figure 1.

![Figure 1](image1.png)

Rotation numbers for digraphs can be defined similarly. Let $D$ be a digraph of order $p$ rooted at a vertex $x$. A homogeneous superdigraph of $D$ is a digraph $H$ of order $p$ such that for every vertex $y$ of $H$, there exists an embedding of $D$ in $H$ with $x$ at $y$. A homogeneous superdigraph of minimum size is called an optimal digraph for the rooted digraph $D$. The rotation number $h(D, x)$ of this rooted digraph $D$ is then defined as the size of an optimal digraph for $D$. For example, if $D$ is the digraph of Figure 2, then $h(D, x_1) = 4$ while $h(D, x_2) = 6$. Optimal digraphs $F_1$ and $F_2$ for $D$ at $x_1$ and $x_2$, respectively, are also shown in Figure 2.

![Figure 2](image2.png)
In the case of a digraph $D$ of order $p$ that is not rooted, homogeneous superdigraphs and optimal digraphs for $D$ are defined similarly. Then the rotation number $h(D)$ is the size of an optimal digraph for $D$. For the digraph $D$ of Figure 2, it follows that $h(D) = 6$.

If $D$ is a vertex-transitive digraph, then $h(D)$ equals the size $q(D)$ of $D$. We define the rotation ratio $r(D)$ of a digraph $D$ by $r(D) = h(D)/q(D)$. Then $r(D) \geq 1$ for every digraph $D$, and $r(D) = 1$ if and only if $D$ is vertex-transitive. Thus, $r(D)$ provides a measure of the symmetry of $D$, with the more symmetric digraphs having a rotation ratio close to $1$.

2. Rotation Numbers for Stars

When we refer to a star in digraph theory, we mean an asymmetric digraph whose underlying graph is a star. We write $S_n(m)$ to indicate a star with $n$ arcs, $m$ of which are directed outwardly from the central vertex (see Figure 3).

![Figure 3](image)

In what follows, the following two well-known theorems from graph theory will be useful (see [9, pp. 216-217]).

**Theorem A.** If $p \geq 3$ is odd, then the complete graph $K_p$ can be factored into $\frac{p-1}{2}$ hamiltonian cycles.

**Theorem B.** If $p \geq 2$ is even, then the complete graph $K_p$ can be factored into $\frac{p-2}{2}$ hamiltonian cycles and a 1-factor.
Theorem 1. Let $D \equiv S\_n (m)$ where $n \geq 1$. If $x$ is the central vertex of $D$, then

$$h(D, x) = [\max (m, n - m)] (n + 1).$$

Proof. Assume, without loss of generality, that $m \geq n - m$. We show that $h(D, x) = m (n + 1)$. Let $H$ be an optimal digraph for this rooted digraph. Because $D$ has order $n + 1$, so does $H$. Since for every vertex $y$ of $H$, there exists an embedding of $D$ in $H$ with $x$ at $y$, the outdegree of each vertex of $H$ must be at least $m$. Thus, $H$ has at least $m (n + 1)$ arcs; so $h(D, x) \geq m (n + 1)$. To complete the proof, we show that there exists an optimal digraph $F$ for $D$ containing exactly $m (n + 1)$ arcs. This is accomplished by showing that there exists an $m$-regular digraph $D$ (every vertex has outdegree and indegree $m$) of order $n + 1$.

Since $m \geq n - m$, it follows that $m \geq \frac{n}{2}$. Suppose first that $n$ is even. Since $n + 1$ is odd, the complete graph $K\_n + 1$ can be factored into $\frac{n}{2}$ hamiltonian cycles $C_1, C_2, \ldots, C_n$. For each $i = 1, 2, \ldots, \frac{n}{2}$, let $C'_i$ be a directed cycle obtained by cyclically directing the edges of $C_i$ and let $C''_i$ be the directed cycle whose arcs are directed oppositely to that of $C'_i$. Thus the complete symmetric digraph $K\_n + 1$ can be factored into the cycles $C'_1, C'_2, \ldots, C'_\frac{n}{2}, C''_1, C''_2, \ldots, C''_{\frac{n}{2}}$. The digraph $F$ consisting of $m$ of these cycles is $m$-regular and is therefore optimal for $D$.

Next, suppose that $n$ is odd. By Theorem 1, the complete graph $K\_n + 1$ can be factored into $\frac{n - 1}{2}$ hamiltonian cycles $C_1, C_2, \ldots, C_{n-1}$ and a 1-factor $F_1$. Let $C'_i$ and $C''_i$ ($1 \leq i \leq \frac{n - 1}{2}$) be as defined above. Define $F$ to be that digraph consisting of $m$ of the cycles $C'_1, C'_2, \ldots, C'_{\frac{n-1}{2}}, C''_1, C''_2, \ldots, C''_{\frac{n-1}{2}}$ if $m < n$, while define $F$ to be the complete symmetric digraph $K\_n + 1$ if $m = n$. 

G. Chartrand et al., On Rotation Nur

We now consider $h(S\_n (m), x)$, where $n' = n - m$.

Theorem 2. Let $D \equiv S\_n (m)$, where outdegree 0, then

$$h(D, x) =$$

Proof. First we show that $h(D, x)$ is the digraph for $D$. Then $F$ contains a subdigraph $H\_1 \equiv S\_n (m)$ where $m > n - m$. Since $D$ can be an optimal digraph for $D$, we have $m (n + 1)$ arcs. Thus $(c, c_1)$ is an adjacent arc in $F$.

Suppose that $(c, c_1)$ is an arc in $F$ (different from $c$) adjacent to $c$. Thus $(c_1, c)$ is an adjacent arc in $F$.

Since $m > n - m$, we have $m (n + 1)$ arcs. Thus $(c, c_1)$ is an adjacent arc in $F$. If, however, these are embedded in $F$ with $x$ at $c_1$, then the embedding is optimal for $D$.

That $h(D, x) = 2m$ follows homogeneously embedded in the figure.
where \( n \geq 1 \). If \( x \) is the central vertex

\[
 x (m, n-m) \] \( n+1 \).

of generality, that \( m \geq n - m \). We

Let \( H \) be an optimal digraph for

\( D \) has order \( n+1 \), so does \( H \). Since

there exists an embedding of \( D \) in \( H \)

each vertex of \( H \) must be at least \( m \).

1) arcs; so \( h(D, x) \geq m(n+1) \). To

own that there exists an optimal

g exactly \( m(n+1) \) arcs. This is

that there exists an \( m \)-regular

outdegree and indegree \( m \) of order

that \( m \geq \frac{n}{2} \). Suppose first that \( n \)

complete graph \( K_{n+1} \) can be factored

\( C_2, \ldots, C_n \). For each \( i = 1, 2, \ldots, \frac{n}{2} \), let

d by cyclically directing the edges
ced cycle whose arcs are directed

the complete symmetric digraph

cycles \( C_1', C_2', \ldots, C_m, C_1'', C_2'', \ldots, \)
g of \( m \) of these cycles is \( m \)-regular

odd. By Theorem B, the complete

ed into \( \frac{n-1}{2} \) hamiltonian cycles

or \( F_1 \). Let \( C_i' \) and \( C_i'' \) (\( 1 \leq i \leq \frac{n-1}{2} \))

be that digraph consisting of \( m \)

\( C_1', C_2', \ldots, C_{n-1}' \) if \( m \leq n \),

complete symmetric digraph \( K_{n+1} \) if

We now consider \( h(S_n(m), x) \), where \( x \) is not a central vertex.

**Theorem 2.** Let \( D \equiv S_n(m) \), where \( m > n - m \). If \( x \) is a vertex of

outdegree 0, then

\[
 h(D, x) = 2n.
\]

**Proof.** First we show that \( h(D, x) \geq 2n \). Let \( F \) be an optimal

digraph for \( D \). Then \( F \) contains a subdigraph \( H \) isomorphic to

\( S_n(m) \). Let \( c \) be the centre of \( H \). Thus \( od_C c = m \) and \( id_C c = n - m \),

where \( m > n - m \). Since \( D \) can be embedded in \( F \) with \( x \) at \( c \), there

exists a subdigraph \( H_1 \equiv S_n(m) \) of \( F \) with \( x \) at \( c \). Denote the

centre of \( H_1 \) by \( c_1 \). Thus \( (c_1, c) \) is an arc of \( F \).

Suppose that \( (c, c_1) \) is an arc of \( H \). There must be \( n - m \)

vertices of \( F \) (different from \( c \)) adjacent to \( c_1 \), and there must be \( m \)

vertices (including \( c \)) adjacent from \( c_1 \). Thus, \( F \) must contain at

least \( n \) arcs that do not belong to \( H \); so \( h(D, x) \geq 2n \).

Suppose now that \( (c, c_1) \) is not an arc of \( H \). Then \( F \) must

contain at least \( n - 1 \) arcs that do not belong to \( H \); namely \( m - 1 \)

additional arcs directed away from \( c_1 \) and \( n - m \) arcs directed
toward \( c_1 \). If, however, these were the only arcs of \( F \) and \( D \) was

embedded in \( F \) with \( x \) at \( c_1 \), then there is no in-neighbour of \( c_1 \)
available for the centre in this embedding. Therefore, at least

one more arc must be added to produce \( F \), so \( h(D, x) \geq 2n \).

That \( h(D, x) = 2n \) follows by observing that \( D \) can be

homogeneously embedded in the digraph \( F \) of size \( 2n \) shown in

Figure 4.  

\[ \text{Figure 4} \]
Corollary. Let $D \equiv S_n(m)$, where $n-m > m$. If $x$ is a vertex of indegree 0, then

$$h(D, x) = 2n.$$ 

Theorem 3. Let $D \equiv S_n(m)$, where $n-m \geq m$. If $x$ is a vertex of outdegree 0, then

$$h(D, x) = 3n - 2m + 1.$$ 

Proof. First we show that $h(D, x) \geq 3n - 2m + 1$. Let $F$ be an optimal digraph for $D$. Then $F$ contains a subdigraph $H$ isomorphic to $S_n(m)$. Let $c$ be the centre of $H$ and let $S$ be the set of vertices of $H$ adjacent to $c$. Thus $|S| = n-m$. By hypothesis, $D$ can be embedded in $F$ with $x$ at $c$. Let $H_1 \equiv S_n(m)$ be a subdigraph of $F$ with $x$ at $c$. Denote the centre of $H_1$ by $c_1$.

Suppose, first, that $(c, c_1)$ is an arc of $H$. Thus, $c_1 \not\in S$. In $F$ there must be $m$ arcs directed away from $c_1$, one of which is directed toward $c$, and $n-m$ arcs directed toward $c_1$.

Let $k$ be the number of vertices of $S$ that are adjacent to $c_1$ in $H$. Thus, $n-2m+1 \leq k \leq n-m$. If $D$ is embedded in $F$ with $x$ placed at any of these $k$ vertices of $S$, then an additional arc directed toward it is required. Thus, the size of $F$ is at least $2n + k$. Since $k \geq n-2m+1$, it follows that the size of $F$ is at least $2n + (n-2m+1) = 3n-2m+1$, that is, $h(D, x) \geq 3n - 2m + 1$.

Next, suppose that $(c_1, c)$ is an arc of $H$, that is, $c_1 \in S$. In $F$ there must be $m$ arcs directed away from $c_1$, including $(c_1, c)$, and $n-m$ arcs directed toward $c_1$.

Again, let $k$ be the number of vertices of $S$ that are adjacent to $c_1$ in $H$. Thus, $n-2m+1 \leq k \leq n-m$. If $D$ is embedded in $F$ with $x$ placed at any of these $k$ vertices of $S$, then an additional arc directed toward it is needed. Thus, the size of $F$ is at least $2n - 1 + k$. Suppose $D$ is embedded in $F$ with $x$ at $c_1$, and let $H_2 \equiv D$ with $x$ at $c_1$. At least one additional arc is required for this embedding, so that the size of $F$ is at least $2n + k$. Here too then, $h(D, x) \geq 3n - 2m + 1$.

That $h(D, x) = 3n - 2m + 1$ follows from the fact that there exists a homogeneous superdigraph $H$ for $D$ of size $3n - 2m + 1$ (see Figure 5).

Corollary. Let $D \equiv S_n(m)$, where $n - m \geq m$. If $x$ is a vertex of indegree 0, then

$$h(D, x) = 3n.$$ 

Combining all the foregoing results, we have

Theorem 4. If $D \equiv S_n(m)$, where $n$ is even, then

$$h(D) = \left\lfloor \frac{n+1}{2} \right\rfloor$$

According to Theorem 4, rotation ratio $r(D)$ satisfies the inequality $\frac{n+1}{2} \leq r(D)$.

3. Rotation Numbers for $C_n$

We call an asymmetric digraph a cycle. A directed cycle $C_n$ is a cycle. A directed cycle $C_n$.

Although normally we consider rooted digraphs, the following results only hold for the original digraph.

Theorem 5. Let a digraph $D$ be a cycle if $D \equiv C_n$ and $h(D) = 2n$, otherwise...
where $n - m > m$. If $x$ is a vertex of
$$(D, x) = 2n.$$ where $n - m \geq m$. If $x$ is a vertex of
$$(n - 2m + 1).$$ Let $F$ be an
$$\text{the centre of } H \text{ and let } S \text{ be the set}
$$Thus $|S| = n - m$. By hypothesis, $D$
$$\text{ith } x \text{ at } c$. Let $H_3 \subseteq S_n (m)$ be a
$$\text{denote the centre of } H_1 \text{ by } c_1.$$
$$\text{is an arc of } H$. Thus, $c_1 \in S$. In $F$
ed away from $c_1$, one of which is
$$\text{directed toward } c_1.$$
$$\text{ices of } S \text{ that are adjacent to } c_1 \text{ in}$$. If $D$ is embedded in $F$ with
$$\text{ices of } S, \text{ then an additional arc}$$. Thus, the size of $F$ is at least
$$\text{of the size of } F \text{ is at } n + 1, \text{ that is, } h(D, x) \geq 3n - 2m + 1.$$
is an arc of $H$, that is, $c_1 \in S$. In $F$
away from $c_1$, including $(c_1, c)$, and
$$\text{of vertices of } S \text{ that are adjacent}$$. If $D$ is embedded in $F$ with
$$\text{ices of } S, \text{ then an additional arc}$$. Thus, the size of $F$ is at least
$$2n - 1$, in $F$ with $x$ at $c_1$, and let $H_2 \equiv D$
addional arc is required for this
$F$ is at least $2n + k$. Here too then,
$$1 \text{ follows from the fact that there}$$. Let a digraph $D$ be a cycle of order $n$. Then $h(D) = n$
if $D \subseteq \mathcal{C}_n$, and $h(D) = 2n$, otherwise.

**Figure 5**

**Corollary.** Let $D \equiv S_n (m)$, where $n - m \leq m$. If $x$ is a vertex of
$$\text{indegree } 0$, then
$$h(D, x) = 3n - 2m + 1.$$

Combining all the foregoing results, we have the following.

**Theorem 4.** If $D \equiv S_n (m)$, where $n \geq 3$, then
$$h(D) = \lfloor \max (m, n - m) \rfloor (n + 1).$$

According to Theorem 4 then, if $D$ is a star of size $n$, then the
rotation ratio $r(D)$ satisfies the inequalities
$$\frac{n + 1}{2} \leq r(D) \leq n + 1.$$

**3. Rotation Numbers for Cycles**

We call an asymmetric digraph a **cycle** if its underlying
graph is a cycle. A directed cycle of order $n$ will be denoted by
$\mathcal{C}_n$.

Although normally we consider rotation numbers for rooted
and unrooted digraphs, the following observation shows that
only the rooted version is interesting.

**Theorem 5.** Let a digraph $D$ be a cycle of order $n$. Then $h(D) = n$
if $D \subseteq \mathcal{C}_n$, and $h(D) = 2n$, otherwise.
Proof. If \( D \cong \overrightarrow{C}_n \), then \( D \) itself serves as an optimal digraph.

If \( D \) is not isomorphic to \( \overrightarrow{C}_n \), then \( D \) has a vertex of outdegree 2. Therefore, an optimal digraph has at least \( 2n \) arcs, and \( h(D) \geq 2n \). On the other hand, the symmetric cycle \( C_n^d \) is an optimal digraph for \( D \) of size \( 2n \), so \( h(D) \leq 2n \).

By Theorem 5, if \( D \) is a cycle then either \( r(D) = 1 \) or \( r(D) = 2 \).

In the same manner, one can show that for the rooted version, \( h(D, x) = 2n \) whenever \( x \) has indegree or outdegree equal to 2. Consequently, we henceforth consider cycles rooted at \( x \) with \( od x = id x = 1 \). For such cycles, the following notation will be useful. Starting from the selected vertex \( x \), while proceeding around the cycle in some direction, we denote each arc by "\( w \)" (with) if it is consistent with the direction of the travel, or by "\( a \)" (against), otherwise. For example, the pattern (\( wawawaw \)) denotes the cycle \( D \) in Figure 6.

![Graph D: (wwawaw)](image)

In general, if \( D \) is a cycle not isomorphic to \( \overrightarrow{C}_n \), then \( n + 1 \leq h(D, x) \leq 2n \). However, for some special patterns, a better lower bound is possible.

**Theorem 6.** If \( D \) is a rooted cycle of order \( n \) with the pattern \( (w_1 ... w) \) or \( (w ... aw) \) at the vertex \( x \), then \( h(D, x) \geq \frac{3}{2} n \).

**Proof.** Let \( F \) be an optimal digraph for \( D \). For every vertex \( v \) of \( F \), we have \( od v \geq 1 \) and \( id v \geq 1 \). Suppose that there is a vertex \( v \) in \( F \) such that \( od v = 1 \) and \( id v = 1 \). Assume, without loss of generality, that the pattern for \( D \) at \( x \) is \( (w_1 ... w) \). Therefore, if \( x = v_0, v_1, v_2, ..., v_{n-1} \) are the vertices of the cycle \( D \), then

\[(v_{n-1}, x), (x, v_1), (v_2, v_1)\] are arcs by \( y \) and \( z \) the unique vertices of \( F \) (see Figure 7).

![Figure](image)

Suppose that there exists an edge \( (v, y) \in E(D) \) and \( (v, y) \in E(F) \). Then, however, \( v_2 \) must correspond to \((v_2, v_1) \in E(D) \) and \((x, v) \in E(F) \).

From the above observation, \( v \) of \( F \), \( od v + id v \geq 3 \) and, consequently, the lower bound given by Theorem 6 is not possible.

The lower bound given by Theorem 6, if we consider a 4-cycle \( D \) given by Figure 8. Its optimal digraph \( F \) has size \( 6k \).

![Graph D: (waaw)](image)

In general, for a \( 4k \)-cycle \( D \) of \( (waaw ... waaw) \), \( k \) can be any number, and \( D \) is an optimal digraph if and only if \( D \) has exactly 6\( k \) arcs for every second arc of the cycle.
If serves as an optimal digraph.

\[ D \] then \( D \) has a vertex of outdegree

\[ 2n \] graph has at least \( 2n \) arcs, and \( h(D) \)

\[ C_n^* \] is an optimal

\( h(D) \leq 2n. \)

circle then either \( r(D) = 1 \) or \( r(D) = 2. \)

\( s \) indegree or outdegree equal to \( 2 \).

consider cycles rooted at \( x \) with

\( v_0 = x \)

\( v_1 \)

\( v_2 \)

\( \ldots \)

\( v_{n-1} \)

\( z \)

\( y \)

\( v \)

\( (v_{n-1}, x), (x, v_1) \) and \( (v_2, v_1) \) are arcs of \( D \) (see Figure 7). Denote

by \( y \) and \( z \) the unique vertices of \( F \) such that \( (y, v), (v, z) \in E(F) \)

(see Figure 7).

![Figure 6](image)

![Figure 7](image)

Suppose that there exists an embedding of \( D \) in \( F \) with \( x \) at \( y \). The vertex \( v_{n-1} \) cannot correspond to the vertex \( v \), since \( (v_{n-1}, x) \in E(D) \) and \( (v, y) \notin E(F) \). Therefore, \( v_1 \) must correspond to \( v \). Then, however, \( v_2 \) must correspond to \( z \), which is impossible since \( (v_2, v_1) \in E(D) \) and \( (z, v) \in E(F) \).

From the above observation, it follows that for every vertex \( v \) of \( F \), \( od(v) + id(v) \geq 3 \) and, consequently, \( q(F) \geq \frac{3n}{2}. \)

The lower bound given by Theorem 6 is sharp. To see that, let us consider a 4-cycle \( D \) given by the pattern \((waaw)\) (see Figure 8). Its optimal digraph \( F \) has size 6.

![Figure 8](image)

In general, for a 4k-cycle \( D \) \((k = 1, 2, 3, \ldots)\) given by a pattern \((waawwaaw \ldots waaw)\), \( k \) repetitions of \( waaw \), an optimal digraph has size \( 6k \) and may be obtained by adding the reverse arc for every second arc of the digraph \( D \).
Finding the exact value of a rotation number for a cycle seems to be quite difficult, even for cycles with only one arc reversed. For small values of \( n \), a computer search has been performed and the results are summarized in the following table.

<table>
<thead>
<tr>
<th>size of ( D )</th>
<th>pattern of ( D )</th>
<th>( h(D, x) )</th>
<th>number of optimal digraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>((wawww))</td>
<td>10</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>((wwaww))</td>
<td>9</td>
<td>5 (all isomorphic)</td>
</tr>
<tr>
<td>6</td>
<td>((wawww))</td>
<td>12</td>
<td>37</td>
</tr>
<tr>
<td>6</td>
<td>((wwaww))</td>
<td>12</td>
<td>34</td>
</tr>
<tr>
<td>7</td>
<td>((wawww))</td>
<td>14</td>
<td>?</td>
</tr>
<tr>
<td>7</td>
<td>((wwaww))</td>
<td>14</td>
<td>?</td>
</tr>
<tr>
<td>7</td>
<td>((wwaww))</td>
<td>13</td>
<td>7 (all isomorphic)</td>
</tr>
</tbody>
</table>

Table. Rotation number for cycles with one arc reversed

We believe that the following conjecture is true.

**Conjecture.** Let \( D \) be a cycle of length \( n \) obtained from a directed cycle by reversing one arc. Let \( D \) be rooted at a vertex \( v \) with \( \text{od} v = \text{id} v = 1 \).

(a) If \( n \) is even, then \( h(D, x) = 2n \).

(b) If \( n \) is odd, then \( h(D, x) = 2n - 1 \) whenever the reversed arc lies opposite to \( x \), and \( h(D, x) = 2n \) otherwise.

Moreover, the only optimal digraphs for the case with \( h(D, x) = 2n - 1 \) are digraphs obtained by removing one arc from the symmetric cycle \( C_n^r \).

4. **Rotation Number for Tournaments**

A tournament is a digraph obtained by orienting the edges of a complete graph. If \( T \) is a tournament of order \( p \), then its rotation number \( h(T) \) must lie between \( \frac{1}{2} p (p - 1) \) and \( p(p - 1) \). A natural question arises: What are the possible values for rotation numbers of tournaments?

**Theorem 7.** If \( k \) and \( p \) are positive integers such that \( \left\lfloor \frac{p - 1}{2} \right\rfloor \leq k \leq p - 1 \), then there exists a tournament \( T \) of order \( p \) with \( h(T) = kp \).

G. Chartrand et al, On Rotation Num

**Proof.** Assume first that \( p \) is odd. Suppose that \( T \) is a tournament with \( V(T) = \{v_0, v_1, \ldots, v_p\} \) and \( E(T) \) if and only if \( 0 < (j - i) \) (mod \( p \)) and \( v_j \) is not adjacent to \( v_i \) for any \( i, j \) in the vertex-transitive and each vertex is adjacent to \( p \) vertices (see Figure 9, where only vertex \( v_0 \) are shown). Since \( T_0 \) is an optimal digraph for itself. Thus \( h(T_0) \) is the rotation number of \( T_0 \). Theorem 7 applies to \( T_0 \). Suppose next that \( k = \frac{p - 1}{2} + r \), where \( 0 \leq r \leq p - 1 \). For each \( v \) in \( T_0 \), the \( v \) is adjacent to \( v_0 \).

Figu
Proof. Assume first that $p$ is odd. Suppose $k = \frac{p-1}{2}$. Let $T_0$ be a tournament with $V(T_0) = \{v_0, v_1, \ldots, v_{p-1}\}$, and where $(v_i, v_j) \in E(T_0)$ if and only if $0 < (j - i) \pmod{p} \leq \frac{p-1}{2}$. Therefore, $T_0$ is vertex-transitive and each vertex is adjacent to and from $\frac{p-1}{2}$ vertices (see Figure 9, where only the arcs incident with one vertex $v_0$ are shown). Since $T_0$ is vertex transitive, it is the optimal digraph for itself. Thus $h(T_0) = \frac{1}{2} p (p - 1) = pk$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Figure 9}
\end{figure}

Suppose next that $k = \frac{p-1}{2} + 1$. Define $T_1 = T_0 - v_{p-1} v_0 + v_0 v_{p-1}$, that is, $T_1$ is obtained from $T_0$ by reversing the direction of the arc $(v_{p-1}, v_0)$. Since $v_0 = \frac{p+1}{2}$, the rotation number $h(T_1) \geq \frac{1}{2} p (p + 1)$. On the other hand, a digraph $F$ obtained from $T_0$ by adding the reverse arc for each arc on the cycle $v_0, v_1, \ldots, v_{p-1}, v_0$ is a homogeneous superdigraph for $T_1$, and has size $\frac{1}{2} p (p - 1) + p = \frac{1}{2} p (p + 1) = pk$.

For higher values of $k$ we use a similar method, namely for $k = \frac{p-1}{2} + r$, we reverse the last $r$ arcs incident to the vertex $v_0$. Then the outdegree of $v_0$ forces the rotation number of the
obtained digraph $T_r$ to be at least $p \left( \frac{p-1}{2} + r \right)$. An optimal
digraph for $T_r$ can be obtained from $T_0$ by adding to each arc of
the form $(v_i, v_j)$, when $1 \leq (i - j) \mod p \leq r$, its reverse arc. Such
a digraph has also size $\frac{1}{2} p (p - 1) + rp = pk$.

For $p$ even we use a slight modification of the previous
construction. If $k = \left\lfloor \frac{p-1}{2} \right\rfloor = \frac{p}{2}$ then the tournament $T_0$ has $\frac{p}{2}$
vertices of outdegree $\frac{p}{2}$ (namely $v_0, v_1, v_2, \ldots, v_{\frac{p}{2}-1}$) and $\frac{p}{2}$
vertices of outdegree $\frac{p}{2} - 1$ (namely $v_{\frac{p}{2}}, \ldots, v_{p-1}$). On the “main
diagonals” we have arcs $(v_0, v_{\frac{p}{2}}), (v_1, v_{\frac{p}{2}+1}), \ldots, (v_{\frac{p}{2}-1}, v_{p-1})$.
Other arcs are directed in the same manner as for the case with
$p$ odd. An optimal digraph for $T_0$ is obtained by adding $\frac{p}{2}$ “main diagonal” arcs, that is, the arcs $(v_{\frac{p}{2}}, v_0), (v_{\frac{p}{2}+1}, v_1), \ldots, (v_{p-1}, v_{\frac{p}{2}-1})$. Therefore,

$$h(T_0) = \frac{1}{2} p (p - 1) + \frac{p}{2} = p \cdot \frac{p}{2} = pk.$$ 

For larger values of $k$ we construct the corresponding
digraphs from $T_0$ as in the case with $p$ odd.

By Theorem 7, the rotation ratio of every tournament lies
between 1 and 2. In fact, every rational number in the interval
$[1, 2]$ is the rotation ratio of some tournament. In order to see
this, let $r$ be a rational number such that $1 \leq r \leq 2$. Then $r = 1 + \frac{a}{b}$,
where $b$ is a positive integer and $a$ is an integer such that
$0 \leq a \leq b$. Define $k = a + b$ and $p = 2b + 1$, and observe that
$\left\lfloor \frac{p-1}{2} \right\rfloor \leq k \leq p - 1$. By Theorem 7, there exists a tournament $T$
of order $p$ such that $h(T) = kp$. Thus,

$$r(T) = \frac{h(T)}{\binom{p}{2}} = \frac{2k p}{p (p - 1)} = \frac{2k}{p - 1} = \frac{2a + 2b}{2b} = 1 + \frac{a}{b} = r.$$ 

It is natural to ask whether s mentioned in Theorem 7 are attain
tournaments. The next example possible.

Let $T$ be a tournament of order of orders 4 and 3, where every $v$
order 4 is adjacent to each vert order 3. Figure 10 illustrates th
guarantees that the numbers $\mu$ numbers of some tournaments
tournament $T$ of Figure 10, $h(T)$

Recall that the vertex set partitioned into subsets such
induced by each subset is strong property of being strong. These
can be ordered in such a way t vertex in $S_i$ is adjacent to each v

We now examine when a 1 maximum outdegree (indegree $h(T) = p (p - 2)$. The characterizations of such tourna
at least \( p \left( \frac{p-1}{2} + r \right) \). An optimal \( \alpha \) from \( T_0 \) by adding to each arc of \( j \) (mod \( p \)) \( \leq r \), its reverse arc. Such that \( 1 + \frac{p}{2} = pk \).

We shall prove the previous claim. If \( \frac{P}{2} \) then the tournament \( T_0 \) has \( \frac{P}{2} \) vertices \( v_0, v_1, v_2, \ldots, v_{\frac{P}{2}-1} \) and \( \frac{P}{2} \) vertices \( v_{\frac{P}{2}}, \ldots, v_{p-1} \). On the “main” vertices \( v_{\frac{p}{2}}, v_{1}, v_{\frac{p}{2}+1}, \ldots, v_{\frac{p}{2}-1}, v_{p-1} \). The same manner as for the case with \( T_0 \) is obtained by adding \( \frac{P}{2} \) main arcs \( v_{\frac{p}{2}}, v_{0}, v_{1}, \ldots, v_{p-1} \). Let \( T \) be a tournament of order 7 having two strong components of orders 4 and 3, where every vertex in the strong component of order 4 is adjacent to each vertex in the strong component of order 3. Figure 10 illustrates the construction of \( T \). Theorem 7 guarantees that the numbers 21, 28, 35, and 42 are rotation numbers of some tournaments of order 7. However, for the tournament \( T \) of Figure 10, \( h(T) = 36 \).

Recall that the vertex set of every tournament can be partitioned into subsets such that the digraph (tournament) induced by each subset is strong and maximal with respect to the property of being strong. These strong components \( S_1, S_2, \ldots, S_k \) can be ordered in such a way that whenever \( 1 \leq i < j \leq k \), every vertex in \( S_i \) is adjacent to each vertex from \( S_j \).

We now examine when a tournament \( T \) of order \( p \) having maximum outdegree (indegree) \( p - 2 \) has the rotation number \( h(T) = p (p - 2) \). The following theorem gives a characterization of such tournaments.

**Figure 10**
Theorem 8. Let $T$ be a tournament of order $p$, maximum outdegree (or indegree) $p - 2$, and strong components $S_1, S_2, \ldots, S_k$ of orders $p_1, p_2, \ldots, p_k$, respectively. The rotation number $h(T) = p (p - 2)$ if and only if the following two conditions are satisfied:

1. There exist integers $r_1, r_2, \ldots, r_n$ ($r_j \geq 3$) and positive integers $b_j^i$ ($j = 1, 2, \ldots, n; i = 1, 2, \ldots, k$) such that for every $i = 1, 2, \ldots, k$,

$$p_i = \sum_{j=1}^{n} b_j^i r_j.$$

2. If $p_i = b_1^i r_1 + b_2^i r_2 + \ldots + b_n^i r_n$, then for every vertex $u$ of $S_i$ and for every number $r_j$ the disjoint union

$$b_1^i \overrightarrow{C}_{r_1} + b_2^i \overrightarrow{C}_{r_2} + \ldots + b_n^i \overrightarrow{C}_{r_n}$$

of cycles is a subdigraph of $S_i$, where the vertex $u$ lies on some $r_j$-cycle.

Proof. Suppose that $h(T) = p (p - 2)$ and $F$ is an optimal digraph for $T$. Then $F$ is $(p - 2)$-regular and its complement $\overline{F}$ is $1$-regular, which means that $\overline{F}$ is a disjoint union of directed cycles. The fact that for every $x \in V(T)$ and every $y \in V(\overline{F})$ the tournament $T$ is embeddable in $\overline{F}$ with $x$ at $y$ is equivalent to $\overline{F}$ being embeddable in $T$ with $y$ at $x$. Since $\overline{T}$ is obtained from $T$ by reversing all arcs, $\overline{F}$ has the strong components of the same order as $T$ has, namely the components $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k$. Moreover, two corresponding strong components, $S_i$ from $T$ and $S_i$ from $\overline{T}$, have the same cycle structure.

The above facts force the following sequence of statements:

1. All vertices from any cycle in $\overline{F}$ correspond to vertices from only one strong component in $T$.

2. Vertices of each strong component of $\overline{T}$ (or $T$) can be decomposed into subsets forming directed cycles. The union of all these cycles for all strong components is isomorphic to $\overline{F}$. Thus $p_i = b_1^i r_1 + b_2^i r_2 + \ldots + b_n^i r_n$ for some integers $b_j^i$.

Corollary 1. If $T$ has a strong directed cycle $C$, then $h(T) = p (p - 2)$.

Corollary 2. If $T$ is a strong tournament, then $h(T) = p (p - 2)$.

The application of Theorem 8 when the condition (1) is not satisfied allows us to investigate the components. Consider, for example, the presentation in Figure 11. Both the tournaments $T_1$ and $T_2$ have order 6, but only $T_1$ has an odd number of directed cycles, namely $3 = 1 \cdot 1$. However, the vertex $v_1$ in $T_1$ can be partitioned into $n = 3, v_2$ which is not possible in $T_2$ with $h(T_2) = 63$ (in fact $h(T_2) = 6$).
(3) Since any vertex $y$ of $\overline{F}$ can be identified with any vertex $x$ of $\overline{T}$, $b^y_j \geq 1$ and in the aforementioned decomposition, $x$ can be placed on a directed cycle of any of the lengths $r_1, r_2, \ldots, r_n$.

On the other hand, if the conditions (1) and (2) are satisfied we define

$$\overline{F} = \bigcup_{j=1}^{n} (b^1_j + b^2_j + \ldots + b^n_j) \bigcup_{r_j}$$

Then $F$ is a homogeneous superdigraph for $T$ of size $p - 2$; so $F$ is an optimal digraph for $T$.

For tournaments of order $p$ and maximum outdegree (or indegree) $p - 2$, the following corollaries can be easily derived from Theorem 8.

**Corollary 1.** If $T$ has a strong component of order 1, then $h(T) > p - 2$.

**Corollary 2.** If $T$ is a strong tournament, then $h(T) = p - 2$.

**Corollary 3.** If $T$ has strong components of the same order ($\geq 3$), then $h(T) = p - 2$.

The application of Theorem 8 is especially straightforward when the condition (1) is not satisfied. For example, it gives immediate answers (namely $h(T) = p - 2$) for tournaments with the following orders of strong components:

(a) 3 and 5
(b) 5 and 9
(c) 6 and 10
(d) 8, 11 and 15, and so on.

In the case when the condition (1) of Theorem 8 is satisfied, it is necessary to investigate the cycle structure of strong components. Consider, for example, two tournaments $T_1$ and $T_2$ presented in Figure 11. Both $T_1$ and $T_2$ have strong components of order 6 and 3 and therefore we have decomposition $6 = 2 \cdot 3$ and $3 = 1 \cdot 3$. However, the vertices of the larger strong component of $T_1$ can be partitioned into two 3-cycles: $v_1, v_3, v_2, v_7$ and $v_4, v_6, v_5, v_4$ which is not possible for $T_2$. Thus $h(T_1) = 9.7 = 63$ while $h(T_2) > 63$ (in fact $h(T_2) = 65$).
Of course, whenever a tournament $T$ of order $p$ has a maximum outdegree (or indegree) $p - 1$, then $h(T) = p(p - 1)$ and $K_p^*$ is the only optimal digraph. For a tournament $T$ of order $p$ and maximum outdegree (or indegree) $p - 2$, the largest possible value that can be achieved by the rotation number is $h(T) = p(p - 2) + \left\lceil \frac{p}{3} \right\rceil$. To see this, let $p(T) = 3l + r$, where $r = 0, 1, 2$.

Then a digraph $F$ such that $F \equiv I \cdot \overrightarrow{P_3} \cup \overrightarrow{P_2}$ for $r = 2$ and $F \equiv I \cdot \overrightarrow{P_3}$ for $r = 0, 1$, is a homogeneous superdigraph for $T$. The value $h(T) = p(p - 2) + \left\lceil \frac{p}{3} \right\rceil$ is attained by a tournament $T_p, p \geq 7$, that has strong components of order $3, 1, 1, \ldots, 1, 3$ (see Figure 12).
ever a tournament $T$ of order $p$ has a (or indegree) $p - 1$, then $h(T) = p (p - 1)$ and al digraph. For a tournament $T$ of order $p$

gree (or indegree) $p - 2$, the largest possible hied by the rotation number is $h(T) =$
see this, let $p(T) = 3I + r$, where $r = 0, 1, 2$

$h = I \cdot \overrightarrow{P}_3 \cup \overrightarrow{P}_2$ for $r = 2$ and $h = I \cdot \overrightarrow{P}_3$
eneous superdigraph for $T$. The value $h(T)$
tained by a tournament $T_p$, $p \geq 7$, that has
order $3, 1, 1, \ldots, 1, 3$ (see Figure 12).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure11.png}
\caption{Figure 11}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure12.png}
\caption{Figure 12}
\end{figure}

\section*{References}