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Note

On isomorphic subgraphs

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Abstract

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We prove that every 3-uniform hypergraph with q edges contain two edge disjoint isomorphic subgraphs with at least $\lfloor \frac{1}{23}\sqrt{q} \rfloor$ edges. This answers a question of Erdős, Pach and Pyber.

Recently, the following question was independently raised by M.S. Jacobson (personal communication) and J. Schönheim (see [1]).

> In an arbitrary graph or hypergraph G, what is the maximum possible s such that G contains a pair of edge disjoint isomorphic subgraphs of size s?

Erdős et al. [1] provided bounds on the maximum size of such isomorphic subgraphs for graphs and hypergraphs. Let $f_k(q)$ denote the maximum integer such that in every graph ($k=2$) or k-uniform hypergraph ($k \geqslant 3$) of size q, one can find a pair of edge disjoin isomorphic subgraphs of size $f_k(q)$. In [1] it is shown that there exist constants c_1 and c_2 (that depend only on k) such that

$$c_1 q^{\frac{2}{2k-1}} \leqslant f_k(q) \leqslant c_2 q^{\frac{2}{k+1}} \frac{\log q}{\log \log q}. \tag{1}$$

For graphs (i.e., $k=2$), the bounds given by (1) are quite tight. In [1], the authors further asked about the proper behavior when $k=3$? The purpose of this note is to answer their question. For terms not defined here see [2].

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Theorem. *If G is a 3-hypergraphs of size q, then G contains two edge disjoint isomorphic subgraphs of size at least* $\lfloor \frac{1}{23}\sqrt{q} \rfloor$.

Proof. Let $G=(V,E)$ be as above and let $|V|=n$ ($\geqslant 3$). Without loss of generality we assume that

$$q \geqslant (23)^2 \tag{2}$$

for otherwise the statement of the Theorem is vacuous.

We begin by partitioning the vertex set of G into three sets, X, Y and Z so that as many 3-edges as possible have a vertex in each set. A simple averaging argument shows that every 3-hypergraph with q edges contains such a 3-partite subhypergraph with at least $\frac{2}{9}q$ edges. Let G_1 be such a subhypergraph. We will find two large edge disjoint subgraphs of G_1. To do this we use the following.

Claim. *If F is a forest with q edges consisting of disjoint stars, then F contains two edge disjoint isomorphic subgraphs, each of size at least* $(q-1)/3$.

Proof of Claim. Let $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_t$ be the sizes of the stars. Let m be the largest index with $s_m > 1$. We will split all stars with $s_j > 1$ as equally as possible and match the remaining edges. This leaves us with two subgraphs each with at least

$$\sum_{i=1}^{m} \left\lfloor \frac{s_i}{2} \right\rfloor + \left\lfloor \frac{t-m}{2} \right\rfloor \tag{3}$$

edges. Expression (3) is minimized when $s_1 = s_2 = \cdots = s_m = 3$ and $t-m$ equals 0, 1 or 2 depending on the congruence class of q mod 3. This leaves us with two edge disjoint isomorphic subgraphs, each of size at least $(q-1)/3$, as desired, completing the proof of the claim. □

Our next goal is to infer that two of the sets X, Y and Z contain small subsets (with less than $c\sqrt{q}$ vertices, $c \geqslant \frac{1}{23}$) with the property that there are $\frac{1}{27}q$ edges meeting both of these sets.

In order to see this, suppose we select a star forest F in G_1 containing the maximum number of edges. If this forest contains at least $3c\sqrt{q}+1$ edges, then by the claim we would find two edge disjoint isomorphic star forests, each with size at least $c\sqrt{q}$. Thus, the forest F must contain less than $3c\sqrt{q}+1$ edges. But then, since this forest is maximal with respect to size, every other edge must meet $V(F)$, hence one of the sets $V(F) \cap X$, $V(F) \cap Y$, $V(F) \cap Z$ (which contains at most $3c\sqrt{q}$ vertices) must meet at least $\frac{1}{3}|E(G_1)| = \frac{2}{27}q$ edges of G_1. Without loss of generality assume that $\bar{X} = V(F) \cap X$ is that set and let G_2 be the graph induced by the set of all such edges.

Next let I be the largest system of disjoint pairs $\{y,z\}$ such that $y \in Y$, $z \in Z$ and $\{y,z\}$ is a subset of an edge of G_2. For each such pair $\{y,z\} \in I$ select a representative $x \in X$ such that $\{x,y,z\} \in E(G_2)$. The collection of representatives for the pairs of I, along with the pairs, induces a star forest in G_2. Thus, if $|I|\geqslant 3c\sqrt{q}+1$, we can apply the claim to obtain two isomorphic edge disjoint star forests each of size at least $c\sqrt{q}$. Thus, we assume that $|I|\leqslant 3c\sqrt{q}$. This however means that for Y or Z (say Y) there exists a subset $\bar{Y}\subset Y$ with $|\bar{Y}|\leqslant 3c\sqrt{q}$ such that at least $\frac{1}{2}|G_2|=\frac{1}{27}q$ edges of G_2 meet both $\bar{X}$ and $\bar{Y}$. Let G_3 be the subgraph of all such edges. Further, let $\bar{Z}$ be the minimum subset of Z such that there are at least $\frac{1}{2}|G_3|=\frac{1}{54}q$ edges of G_3 which meet each of $\bar{X}$, $\bar{Y}$ and $\bar{Z}$.

We now recognize two cases.

Case 1: *Suppose that* $|\bar{Z}|\geqslant 3c\sqrt{q}+1$.

Consider a maximal integer $m=t_1+t_2+\cdots+t_k$ such that there exists a system S of edges

$$\{x_i, y_i, z_i^j\}\in G_3, \quad 1\leqslant j\leqslant t_i, \quad i=1,2,\ldots,k$$

satisfying:

(i) $x_i\in\bar{X}$, $y_i\in\bar{Y}$ and $z_i^j\in Z$ for $i=1,2,\ldots,k$ and $j=1,2,\ldots,t_i$,

(ii) $t_i>1$ for all $i=1,2,\ldots,k$ and,

(iii) all $t_1+t_2+\cdots+t_k$ vertices z_i^j are distinct.

Due to the maximality of the above system of edges, for every pair $\{x,y\}$, $x\in\bar{X}$, $y\in\bar{Y}$ different from $\{x_i,y_i\}$, $i=1,2,\ldots,k$, there exists at most one $z\notin\{z_i^j|i=1,2,\ldots,k, 1\leqslant j\leqslant t_i\}$ such that $\{x,y,z\}\in G_3$. Hence there are at most

$$|\bar{X}||\bar{Y}|\leqslant 9c^2q<\tfrac{1}{2}|G_3|=\tfrac{1}{54}q \tag{4}$$

such edges $\{x,y,z\}$. (Again note that (4) holds for $c=\frac{1}{23}$.) Thus, all other edges ($>\frac{1}{2}|G_3|$) of G_3 intersect the set

$$Z^*=\{z_i^j|i=1,2,\ldots,k,\ 1\leqslant j\leqslant t_i\}.$$

Hence, (due to the minimality of $\bar{Z}$), $|Z^*|\geqslant|\bar{Z}|\geqslant 3c\sqrt{q}+1$. Consider now the permutation

$$\phi:\bar{X}\cup\bar{Y}\cup Z^*\to\bar{X}\cup\bar{Y}\cup Z^*$$

that fixes each point of $\bar{X}$ and $\bar{Y}$. To define how ϕ acts on Z^*, split the stars $\{x_i,y_i,z_i^j\}$ in a manner similar to the claim. The same argument gives the existence of two edge disjoint subgraphs S_1, S_2 of S with

$$|S_1|=|S_2|=c\sqrt{q}$$

such that $\phi:S_1\to S_2$ is an isomorphism between S_1 and S_2.

Case II: *Suppose that* $|\bar{Z}|\leqslant 3c\sqrt{q}$.

In this case we have three sets $\bar{X}, \bar{Y}, \bar{Z}$ each of cardinality at most $3c\sqrt{q}$ and at least $\frac{1}{54}q=q_1$ edges of the form $\{x,y,z\}$, $x\in\bar{X}$, $y\in\bar{Y}$, $z\in\bar{Z}$. Let G_4 be the subgraph of all such edges. Consider a random permutation

$$\phi:\bar{X}\cup\bar{Y}\cup\bar{Z}\to\bar{X}\cup\bar{Y}\cup\bar{Z}$$

such that $\phi(\bar{X})=\bar{X}$, $\phi(\bar{Y})=\bar{Y}$ and $\phi(\bar{Z})=\bar{Z}$. Then for two distinct edges e_1 and e_2

$$\mathrm{Prob}(\phi(e_1)=e_2)=\frac{1}{|\bar{X}|\,|\bar{Y}|\,|\bar{Z}|}=\frac{1}{27c^3q^{3/2}}.$$

Thus,

$$\mathrm{Ex}(|\{e_1,e_2)|\,\phi(e_1)=e_2 \text{ and } e_1\neq e_2\}|)=\frac{q_1(q_1-1)}{|\bar{X}|\,|\bar{Y}|\,|\bar{Z}|}$$

$$=\frac{\frac{q}{54}\left(\frac{q}{54}-1\right)}{(3c\sqrt{q})^3}>3c\sqrt{q}. \tag{5}$$

(We note that in order to insure that the subgraphs are edge disjoint, we can use at most $\frac{1}{3}$ of the pairs. Hence, the above inequality). Inequality (5) is true for $c\leqslant\frac{1}{23}$ since $q\geqslant(23)^2$ recalling (2). Thus, we can find two edge disjoint subgraphs of G_4, each with at least $\lfloor\frac{1}{23}\sqrt{q}\rfloor$ edges. □

Our result leads us to the following conjecture.

Conjecture. If G is a k-hypergraph of size q, then G contains two isomorphic subgraphs of size $cq^{2/(k+1)}$.

References

[1] P. Erdős, J. Pach and L. Pyber, Isomorphic subgraphs in a graph, preprint.
[2] R.J. Gould, Graph Theory (Benjamin/Cummings, Menlo Park, CA, 1988).