Generalized Degrees and Short Even Cycles

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Abstract

Let $G$ be a graph of order $n$ such that $|E(G)| > c_k n^{\frac{k+1}{2}}$. In 1963 Erdős showed that this implies $G$ contains a $C_{2k}$. He conjectured that this edge density condition implies $G$ contains a $C_{2l}$ for every integer $l \in [k, n^k]$. In 1974 Bondy and Simonivits proved the conjecture with $c_k = 100k$.

The purpose of this paper is to provide a generalized degree analogue to this classic result of Erdős. Here we use the following idea of generalized minimum degree. Let

$$\delta_k(G) = \min |N(u_1) \cup N(u_2) \cup \cdots \cup N(u_k)|$$

where the minimum is taken over all independent sets of vertices $\{u_1, \cdots, u_k\}$ of size $k$ and $N(u_i)$ denotes the neighborhood of the vertex $u_i$. We call $\delta_k(G)$ the minimum generalized $k$-degree for $G$.

1 Introduction

All graphs considered in this paper are finite and simple. For a vertex $x \in V(G)$ we denote the degree of $x$ as $d(x)$. We define the neighborhood of a vertex $x$ to be the set $N(x)$ where

$$N(x) = \{y \in V(G) \mid xy \in E(G)\}.$$ 

For terms not defined here, see [5].

Many results in extremal theory are based on edge density conditions, for example, a minimum degree requirement. Turan's [8] classic result on the existence of complete subgraphs is a prime example of such an edge density result.

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Over the last few years a wide variety of results using various forms of generalized degree conditions have been found (see for example [6] and [7]). In this paper we are concerned with the following minimum generalized degree

$$
\delta_k(G) = \min |N(u_1) \cup N(u_2) \cup \cdots \cup N(u_k)|
$$

where the minimum is taken over all independent sets of size \(k\).

In 1989, Noga Alon, Ralph Faudree and Zoltan Furedi [1] proved the following theorem based on this minimum generalized degree, again a form of edge density condition.

**Theorem 1** Let \(G\) be a graph of order \(n\). Let \(k\) and \(d\) be integers where \(k \geq 1\) and \(d \geq 2\). Let

$$
\delta_k(G) = \min |N(u_1) \cup N(u_2) \cup \cdots \cup N(u_k)|
$$

where the minimum is taken over all independent sets of size \(k\). If \(G\) satisfies the condition

$$
\delta_k(G) \geq \frac{d - 2}{d - 1} n,
$$

then for sufficiently large \(n\), \(G\) contains a \(K_d\).

Their result, as stated, can be viewed as the \(\delta_k(G)\) analogue of Turan’s Theorem. Note that when \(k = 1\) we have \(\delta(G) = \delta_1(G)\).

The purpose of this paper is to provide another generalized degree analogue, this time to an edge density result due to Erdős [3]. In particular, Erdős proved the following theorem.

**Theorem 2** There exists a \(c_k\) and an \(n_0(k)\) such that if

$$
|E(G)| > c_k n^{1+1/k}
$$

and \(n > n_0\), then \(C_{2k}\) is contained in \(G\).

Erdős [3] conjectured that this edge density condition implies \(G\) contains a \(C_{2l}\) for every integer \(l \in [k, n^{1/k}]\). Bondy and Simonivits [2] proved this conjecture with \(c_k = 100k\). Our analogue of Theorem 2 is the following.

**Theorem 3** Let \(G\) be a graph of order \(n\). Let \(k\) and \(p\) be integers greater than or equal to one. If \(G\) satisfies the generalized minimum degree condition

$$
\delta_k \geq cn^{\frac{1}{p}}
$$

for some real number \(c\), then for sufficiently large \(n\), \(G\) contains a \(C_{2p}\).
2 Results

We first consider the case where $G$ is a bipartite graph. Consider $u_0 \in V(G)$ such that $d(u_0) \geq c_0 n^{\frac{1}{p}}$. Let $U_i = N^i(u_0)$ denote the set of all vertices $V(G)$ whose minimum distance from $u_0$ is $i$, that is,

$$U_i = \{x \in V(G) \mid d(x, u_0) = i\}.$$  

Note that $U_0 = \{u_0\}$ and that $U_1 = N(u_0)$.

**Lemma 1** Let $p$ and $k$ be a fixed integers greater than $1$. Let $G$ be a bipartite graph of order $n$ that satisfies the generalized minimum degree condition

$$\delta_k(G) \geq cn^{\frac{1}{p}}$$

for some real number $c$. If $G$ does not contain a $C_{2p}$, then for sufficiently large $n$, $|U_i| \geq c_i n^{\frac{1}{p}}$ for $1 \leq i \leq p$ and some real number $c_i$.

**Proof:** The proof is by induction on $i$. Note that the claim is true for $i = 0$ with $c_0 = 1$ as in this case $U_0 = \{u_0\}$ and $c_0 n^{2/p} = 1$. Now assume for some $j$, $0 < j < p$, that the claim is true for all $0 \leq i \leq j$.

Given any collection of $p$ vertices in $U_j$, there must exist at least two vertices in the collection, say $u$ and $v$, such that

$$|N_{j+1}^*(u) \cap N_{j+1}^*(v)| < p - 1$$

where $N_{j+1}^*(u) = N(u) \cap U_{j+1}$. If this were not the case, given any collection of $p$ vertices in $U_j$, the intersection would be at least $p - 1$, that is,

$$|N_{j+1}^*(u) \cap N_{j+1}^*(v)| \geq p - 1.$$

In order to see that is is true, let $z_m$ be a vertex in $U_m$, $0 \leq m \leq j - 1$, such that there exists vertex-disjoint paths from $z_m$ to $u$ and $v$ and let $m$ be maximal with respect to this property. Note that such a $z_m$ must exist as $u$ and $v$ are both at distance $j$ from $u_0$. Then there exist a path of length $j - m$ from $z_m$ to $u$, a path of length $2(p - j + m)$ from $u$ to $v$ via the vertices in $U_j$ and $U_{j+1}$ (using the fact that any two such vertices have at least $p - 1$ common neighbors), and a path from $v$ to $z_m$ of length $j - m$. Hence, we have a cycle of length $2p$ in $G$ obtained by combining the path from $z_m$ to $u$, the path from $u$ to $v$ and the path from $v$ to $z_m$. Therefore, given any collection of size $p$ in $U_j$, the two vertices $u$ and $v$ that satisfy (1) must exist.

This implies that at most $p - 1$ vertices in $U_j$ can cover the same neighborhood set $S$ in $U_{j+1}$ if $|S| > p - 1$. This implies

$$|U_{j+1}| \geq \frac{(c_j n^{\frac{1}{p}})}{(p - 1)} = c_{j+1} n^{\frac{j+1}{p}}$$

for some real number $c_{j+1}$, hence completing the proof.  

With the use of Lemma 1 we can now prove a version of our main result for bipartite graphs.
Lemma 2 Let $k \geq 1$ and $p \geq 1$ be integers. Let $G$ be a bipartite graph of order $n$ that satisfies the generalized minimum degree condition

$$
\delta_k(G) \geq cn^{\frac{1}{p}}
$$

for some real number $c$. Then for $n$ sufficiently large, $G$ contains a $C_{2p}$.

Proof: Let $G$ be a graph that satisfies the conditions of the lemma and does not contain a $C_{2p}$. We shall show that this leads us to a contradiction. Let $u_0 \in V(G)$ such that $d(u_0) \geq \frac{1}{p}n^{\frac{1}{p}}$. Such a vertex must exist or our condition on $\delta_k(G)$ would be violated. Then, by Lemma 1, $|N^p(u_0)| \geq c_0 n$ for some real number $c_0$. Let $u_1 \in N(u_0)$ such that $d(u_1) \geq \frac{1}{p}n^{\frac{1}{p}}$. Then again by Lemma 1, $|N^p(u_1)| \geq c_1 n$ for some real number $c_1$ and since $G$ is bipartite, $N^p(u_0) \cap N^p(u_1) = \phi$. For each $t \in Z^+$, we may choose $u_t \in N(u_{t-1})$. Note, for example, that if $x \in N^p(u_0) \cap N^p(u_2)$, then the minimum distance from $x$ to $u_0$ would be both $p$ and $p+2$. Thus, in fact, we obtain a sequence of pairwise disjoint subsets of $V(G)$, each of order $c_t n$ for some real number $c_t$. Since $n$ is finite, this process must terminate, which is a contradiction to Lemma 1. ∙

We are now able to prove our main result.

Proof of Main Result: Let $G$ be a graph of order $n$ and $\delta_k \geq cn^{\frac{1}{p}}$ for some real number $c$. There can be at most $k-1$ vertices whose degree is less than $\frac{1}{p}n^{\frac{1}{p}}$. Erdős [4] showed $G$ contains a spanning bipartite subgraph $H$ where each vertex $x \in V(H)$ has degree at least half its degree in $G$, that is,

$$
deg_H x \geq \frac{\deg_G x}{2}.
$$

Such a spanning subgraph satisfies the generalized minimum degree condition $\delta_k \geq c'n^{\frac{1}{p}}$ for some real number $c'$. Hence, by Lemma 2, $H$ contains a $C_{2p}$, and thus, so does $G$. ∙

3 Future Directions

We can now ask the same question posed originally by Erdős: Do all cycles $C_{2t}$ exist for $t \in [p, n^{1/p}]$? If so, for what constants can we verify this result?

References


