Generalized Degree Conditions for Graphs with Bounded Independence Number

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ABSTRACT

We consider a generalized degree condition based on the cardinality of the neighborhood union of arbitrary sets of $r$ vertices. We show that a Dirac-type bound on this degree in conjunction with a bound on the independence number of a graph is sufficient to imply certain hamiltonian properties in graphs. For $K_{1,m}$-free graphs we obtain generalizations of known results. In particular we show:

Theorem. Let $r \geq 1$ and $m \geq 3$ be integers. Then for each non-negative function $f(r, m)$ there exists a constant $C = C(r, m, f(r, m))$ such that if $G$ is a graph of order $n$ ($n \geq r, n > m$) with $\delta_i(G) \geq (n/3) + C$ and $\beta(G) \leq f(r, m)$, then

(a) $G$ is traceable if $\delta(G) \geq r$ and $G$ is connected;
(b) $G$ is hamiltonian if $\delta(G) \geq r + 1$ and $G$ is 2-connected;
(c) $G$ is hamiltonian-connected if $\delta(G) \geq r + 2$ and $G$ is 3-connected.

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Dirac [2] proved that if $G$ is a graph of order $n \geq 3$ with $\delta(G) \geq n/2$, then $G$ is hamiltonian. In [5], Matthews and Sumner lowered the minimum degree condition for hamiltonicity by imposing the condition that $G$ be clawfree (i.e., $G$ contains no induced subgraph isomorphic to $K_{1,3}$).

**Theorem A** [5]. If $G$ is a 2-connected $K_{1,3}$-free graph of order $n \geq 3$ with $\delta(G) \geq (n - 2)/3$, then $G$ is hamiltonian.

Recently, Markus [4] obtained similar results for $K_{1,m}$-free graphs, $m \geq 3$.

**Theorem B** [4]. If $G$ is a 2-connected $K_{1,m}$-free graph of order $n \geq 3$ with $\delta(G) \geq (n + m - 2)/3$, then $G$ is hamiltonian.

Both of the previous theorems have analogs for traceable graphs and hamiltonian-connected graphs.

The idea of minimum degree can be generalized as follows. For a graph $G$ of order $n$ and $r \leq n$, define

$$\delta_r(G) = \min_{\{S \subseteq V(G) \mid |S| = r\}} \left| \bigcup_{u \in S} N(u) \right| .$$

Then, of course, $\delta(G) = \delta_1(G)$. In [3], the following results involving $\delta_2(G)$ were established.

**Theorem C** [3]. If $G$ is connected $K_{1,3}$-free graph of order $n$ such that $\delta_2(G) \geq (n + 1)/3$, then for $n$ sufficiently large $G$ is traceable.

**Theorem D** [3]. If $G$ is a 2-connected $K_{1,3}$-free graph of order $n$ such that $\delta_2(G) \geq (n + 1)/3$, then for $n$ sufficiently large $G$ is hamiltonian.

**Theorem E** [3]. If $G$ is a 3-connected $K_{1,3}$-free graph of order $n$ such that $\delta_2(G) \geq (n + 24)/3$, then for $n$ sufficiently large $G$ is hamiltonian-connected.

Here we will prove results that in some sense incorporate and generalize Theorems A–E. Undefined terms and notations can be found in [1]. We begin with Theorem 1, which establishes sufficient conditions for traceability, hamiltonicity, and hamiltonian-connectedness based on $\delta_r(G)$ and the independence number $\beta(G)$ of a graph $G$.

**Theorem 1.** Let $r \geq 1$ and $m \geq 3$ be integers. Then for each non-negative function $f(r,m)$ there exists a constant $C = C(r,m,f(r,m))$ such that if $G$ is a graph of order $n$ ($n \geq r, n > m$) with $\delta_r(G) \geq (n/3) + C$ and $\beta(G) \leq f(r,m)$ then

1. $G$ is traceable if $\delta(G) \geq r$ and $G$ is connected;
(b) $G$ is hamiltonian if $\delta(G) \geq r + 1$ and $G$ is 2-connected;
(c) $G$ is hamiltonian-connected if $\delta(G) \geq r + 2$ and $G$ is 3-connected.

Proof. We proceed by induction on $n$ and assume that (a), (b), and (c) have been established for all graphs of order less than $n$. (The proof is anchored by selecting $C$ large.) Let $G$ be a graph of order $n$ such that $\delta_r(G) \geq (n/3) + C$ and $\beta(G) \leq f(r, m)$. Assume that $G$ satisfies the hypotheses of (a), (b), or (c). We first show that

(i) if $G$ satisfies the hypotheses of (a), then $G$ has a path of order at least $(2n/3) - (2r/3)$;
(ii) if $G$ satisfies the hypotheses of (b), then $G$ has a cycle of order at least $(2n/3) - (2r/3)$;
(iii) if $G$ satisfies the hypotheses of (c), then $G$ has a $u - v$ path of order at least $(2n/3) - (2r/3)$ for each pair $u, v \in V(G)$.

Let $X$ denote a longest path of $G$, longest cycle of $G$, or longest $u - v$ path of $G$ depending on whether we are in (i), (ii), or (iii). We first show that $|V(X)| \geq n/6r$. Since $\delta_r(G) \geq (n/3) + C$, for $C$ sufficiently large every vertex of $G$ with at most $r - 1$ exceptions has degree at least $(n/3) + r - 1$. Let $S$ be the set of vertices of degree less than $(n/3r) + r - 1$ and let $H = \langle V(G) - S \rangle$. Then every vertex of $H$ has degree at least $n/3r$ (in $H$). Let $P$ be a longest path in $H$, with initial vertex $w$. Then every adjacency of $w$ in $H$ is on $P$ so that one of these adjacencies together with a segment of $P$ forms a cycle $C$ in $H$ with at least $n/3r$ vertices. This cycle (or path) is also in $G$. It is straightforward to use this cycle to show that in the hamiltonian-connected case, any two vertices $u$ and $v$ can be joined by a path using at least half the vertices of the cycle.

Thus, in all cases, $|V(X)| \geq n/6r$.

Next, if $L$ denotes the vertices of $G$ not on $X$ of degree less than $C/r$, then $|L| \leq r - 1$. Thus, if $V(G) = V(X) \cup L$, then $|V(X)| \geq n - r + 1 \geq (2n/3) - (2r/3)$. Assume, then, that $V(G) \neq V(X) \cup L$.

We wish to show that the removal of $l$ vertices from $G - V(X) - L$, $0 \leq l \leq 2$, results in at most two components, and each such component $H$ satisfies

$$|V(H)| \geq \frac{n}{3} + C - f(r, m) - r - 2 \quad (1)$$
$$\delta(H) \geq r + 2 \quad (2)$$
$$\delta_r(H) \geq \frac{|V(H)|}{3} + C. \quad (3)$$

To do so, let $H$ be such a component and $w \in V(H)$. Then $deg_G w \geq C/r$. Suppose $G$ satisfies the hypotheses of (a) and let $X: v_1, v_2, \ldots, v_k$. If $w$ is adjacent to $v_i$ and $v_j$, $1 \leq i < j < k$, then $v_{i+1}v_{j+1} \notin E(G)$; otherwise,
the path

\[ X': v_1, v_2, \ldots, v_i, w, v_j, v_{j-1}, \ldots, v_{i+1}, v_{j+1}, \ldots, v_k \]

has order greater than \( X \). Thus, since \( \beta(G) \leq f(r, m) \) we have that \( \deg_X w \leq \beta(G) + 1 \leq f(r, m) + 1 \) (where the extra 1 is only needed in the hamiltonian-connected case). Similarly, if \( G \) satisfies the hypotheses of (b) or (c), then \( \deg_X w \leq f(r, m) + 1 \). Thus,

\[
\deg_H w \geq \frac{C}{r} - f(r, m) - 1 - (r - 1) - l \geq r + 2
\]

for \( C \) sufficiently large. Thus, \( \delta(H) \geq r + 2 \). Let \( S \) be a set of \( r \) vertices of \( H \). Then

\[
|N_G(S)| \geq \frac{n}{3} + C .
\]

However, since \( H \) is connected and \( \beta(G) \leq f(r, m) \) we have that \( |N_X(S)| \leq f(r, m) + 1 \). Thus

\[
|N_H(S)| \geq \frac{n}{3} + C - f(r, m) - 1 - (r - 1) - l
\]

\[
\geq \frac{n}{3} + C - f(r, m) - r - 2 .
\]

Thus, \( |V(H)| \geq (n/3) + C - f(r, m) - r - 2 > n/3 \) for \( C \) sufficiently large, so the removal of \( l \) vertices from \( G - V(X) - L \) results in at most two components. Since \( \delta_r(H) \geq (n/3) + C \), it follows that \( n \geq C \). Thus, by choosing \( C \) at least \( 18rf(r, m) + 18r^2 + 36r \), we have that

\[
\frac{n}{18r} \geq f(r, m) + r + 2
\]

so that

\[
\frac{n}{3} + C - f(r, m) - r - 2 \geq \frac{n}{3} - \frac{n}{18r} + C \geq \frac{|V(H)|}{3} + C .
\]

Since each component \( H \) of \( G - V(X) - L \) satisfies (1), (2), and (3) and has independence number at most \( f(r, m) \), it follows by induction that each such component is traceable. Furthermore, any 2-connected component is hamiltonian and any 3-connected component is hamiltonian-connected. Also, if \( |V(X)| > (n/3) - 2(C - f(r, m) - r - 2) \), then \( G - V(X) - L \) consists of one component, which is necessarily 3-connected.
Assume now that \( G \) satisfies the hypotheses of (a). We wish to show that 
\[ |V(X)| \geq (2n/3) - (2r/3). \]
Since each component \( H \) of \( G - V(X) - L \) is traceable and \( X \) is a longest path, we conclude that
\[
|V(X)| \geq |V(H)| \geq \frac{n}{3} + C - f(r, m) - r - 2
\]
\[
> \frac{n}{3} - 2(C - f(r, m) - r - 2)
\]
for \( C \) sufficiently large. Thus, \( G - V(X) - L \) is hamiltonian-connected.

Let \( X: v_1, v_2, \ldots, v_k \). Since \( G \) is connected there is a path from \( V(G) - V(X) - L \) to some vertex \( v_i \) on \( X \). Let \( P_1 \) be a shortest such path and let \( w \) be the vertex of \( G - V(X) - L \) on \( P_1 \). Let \( z \) be any other vertex of \( G - V(X) - L \) and let \( P_2 \) be any hamiltonian \( z - w \) path in \( G - V(X) - L \). Finally, let \( P_3 \) denote the longer of the subpaths \( v_1, v_2, \ldots, v_i \) and \( v_i, v_{i+1}, \ldots, v_k \) of \( X \). Then

\[
P_2, P_1, P_3
\]
is a path of \( G \) of order at least
\[
n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}
\]
Since, by assumption, \( X \) is a longest path in \( G \), it follows that
\[
|V(X)| \geq n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}
\]
and so \( |V(X)| \geq (2n/3) - (2r/3) \).

Assume next that \( G \) satisfies the hypotheses of (b). If \( G - V(X) - L \) is 2-connected, then \( G - V(X) - L \) is hamiltonian. If \( \kappa(G - V(X) - L) \leq 1 \), then the removal of 0 or 1 vertices results in two 2-connected components, each of order at least \((n/3) + C - f(r, m) - r - 2\). In either case, we obtain a hamiltonian subgraph of \( G \) of order at least \((n/3) + C - f(r, m) - r - 2\). Since \( X \) is a longest cycle of \( G \), we conclude that \( |V(X)| \geq (n/3) + C - f(r, m) - r - 2 \). Thus, \( G - V(X) - L \) is hamiltonian-connected for \( C \) sufficiently large.

Let \( X: v_1, v_2, \ldots, v_k, v_1 \). Since \( G \) is 2-connected, there are two vertex-disjoint paths, the first from \( V(G) - V(X) - L \) to \( V(X) \) and the second from \( V(X) \) to \( V(G) - V(X) - L \). Let \( P_1, P_2 \) be a shortest pair of such paths. Assume, without loss of generality, that \( P_1 \) intersects \( V(X) \) at \( v_i \) and \( P_2 \) intersects \( V(X) \) at \( v_j \), with \( i < j \). Let \( w \) be the initial vertex of \( P_1 \) and let \( z \) be the final vertex of \( P_2 \). Let \( P_3 \) be any hamiltonian \( z - w \) path of \( G - V(X) - L \), and finally, let \( P_4 \) denote the longer of the subpaths \( v_i, v_{i+1}, \ldots, v_j \) and \( v_i, v_{i-1}, \ldots, v_j \) of \( X \). Then

\[
P_1, P_4, P_2, P_3
\]
is a cycle of $G$ of order at least 

$$n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}.$$ 

Since, by assumption, $X$ is a longest cycle in $G$, it follows that 

$$|V(X)| \geq n - |V(X)| - (r - 1) + \frac{|V(X)|}{2}.$$ 

and so $|V(X)| \geq (2n/3) - (2r/3)$.

Next, assume that $G$ satisfies the hypotheses of (c). In this case, $G$ also satisfies the hypotheses of (b). Thus, a longest cycle of $G$ has order at least $(2n/3) - (2r/3)$. This implies that 

$$|V(X)| \geq \frac{n}{3} - \frac{r}{3} > \frac{n}{3} - 2(C - f(r, m) - r - 2)$$

for $C$ sufficiently large, and so $G - V(X) - L$ is hamiltonian-connected. Since $G$ is 3-connected, there are three vertex-disjoint paths from $V(G) - V(X) - L$ to $V(X)$. Using two of these paths, a hamiltonian path in $G - V(X) - L$, and all but an appropriate segment of $X$ we conclude that $|V(X)| \geq (2n/3) - (2r/3)$.

Thus, we have established that if $G$ satisfies the hypotheses of (a), (b), or (c), then $G$ has a path, cycle or $u - v$ path, respectively, of order at least $(2n/3) - (2r/3)$. If $G$ satisfies the hypotheses of (a), let $\alpha$ denote the maximum number of vertices of degree less than $C/r$ on a path of order at least $(2n/3) - (2r/3)$, and let $Y$ be a longest path containing $\alpha$ vertices of degree less than $C/r$. Define $\alpha$ similarly if $G$ satisfies the hypotheses of (b) or (c) and obtain either a longest cycle $Y$ or a longest $u - v$ path $Y$ containing $\alpha$ vertices of degree less than $C/r$.

If $G - V(Y)$ has a vertex $w$ such that $\deg_G w \geq C/r$, then in a manner analogous to earlier arguments, we can show that $G - V(Y)$ has a component $H$ with $|V(H)| \geq (n/3) + C - f(r, m) - 1$ that, for $C$ sufficiently large, contradicts the fact that $|V(Y)| \geq (2n/3) - (2r/3)$. Thus, every vertex of $G$ of degree at least $C/r$ lies on $Y$. We complete the proof by showing that every vertex of $G$ of degree less than $C/r$ also lies on $Y$. Assume, to the contrary, that there are $\gamma > 0$ vertices of degree less than $C/r$ that do not lie on $Y$. Since the number of vertices of $G$ of degree less than $C/r$ is at most $r - 1$, we have $\alpha + \gamma \leq r - 1$ and $r \geq 2$.

Assume first that $G$ satisfies the hypotheses of (a). Let $Y: v_1, v_2, \ldots, v_k$ and let $w \in V(G) - V(Y)$. Since $\delta(G) \geq r$, we have $\deg_G w \geq r$. Thus, $\deg_Y w \geq r - (\gamma - 1) = (r - 1) - \gamma + 2 \geq \alpha + 2$. Furthermore, by the definition of $\alpha$, neither $v_1$ nor $v_k$ is adjacent to $w$. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_{\alpha+2}}$ be $\alpha + 2$ adjacencies of $w$ on $Y$, $i_1 \leq i_2 \leq \ldots \leq i_{\alpha+2}$.
Let $I_0 = \{v_1, v_2, \ldots, v_{i-1}\}$, let $I_{\alpha+2} = \{v_{i\alpha+2+1}, v_{i\alpha+2+2}, \ldots, v_k\}$ and for $j = 1, 2, \ldots, \alpha + 1$ let

$$I_j = \{v_{i_j+1}, v_{i_j+2}, \ldots, v_{i_j+1}\}.$$

Since $Y$ contains exactly $\alpha$ vertices of degree less than $C/r$, it follows that three of the sets $I_0, I_1, \ldots, I_{\alpha+2}$ contain no vertices of degree less than $C/r$. Let $I_s$ be the smallest such set. If $1 \leq s \leq \alpha + 1$, let

$$P: v_1, v_2, \ldots, v_s, w, v_{i_s+1}, v_{i_s+1+1}, \ldots, v_k.$$

If $s = 0$, let

$$P: v_{i_1-1}, v_{i_2-2}, \ldots, v_1, w, v_{i_2}, v_{i_2+1}, \ldots, v_k.$$

If $s = \alpha + 2$, let

$$P: v_1, v_2, \ldots, v_{i_{\alpha+1}}, w, v_{i_{\alpha+2}}, v_{i_{\alpha+2}-1}, \ldots, v_{i_{\alpha+1}+1}.$$

Then $P$ contains $\alpha + 1$ vertices of degree less than $C/r$. By the choice of $\alpha$, then, this means that $P$ has order less than $(2n/3) - (2r/3)$. However,

$$|V(P)| \geq n - \left[ \left( \gamma - 1 \right) + \frac{n - (\gamma - 1) - (\alpha + 2)}{3} \right]$$

$$= n - \left( \frac{n + 2\gamma - \alpha - 4}{3} \right)$$

$$\geq n - \left( \frac{n + 2\gamma - 4}{3} \right)$$

$$\geq n - \left( \frac{n + 2(r - 1) - 4}{3} \right)$$

$$= n - \left( \frac{n + 2r - 6}{3} \right) = \frac{2n}{3} - \frac{2r}{3} + 2,$$

which gives a contradiction. Thus, $Y$ contains every vertex of degree less than $C/r$, which completes the proof in the case that $G$ satisfies the hypotheses of (a).

If $G$ satisfies the hypotheses of (b) or (c), the proof is completed in an analogous manner. In these cases, we have $\delta(G) \geq r + 1$ or $\delta(G) \geq r + 2$, respectively, so that every vertex $w \in V(G) - V(Y)$ has $\deg_x w \geq \alpha + 3$ or $\deg_x w \geq \alpha + 4$. In either case, we are able to contradict the choice of $\alpha$. This completes the proof of Theorem 1. \[\square\]

An immediate corollary of Theorem 1 provides the result that in some sense generalizes Theorems A–E.
Corollary. Let \( r \geq 1 \) and \( m \geq 3 \) be integers. Then there exists a constant \( C = C(r, m) \) such that if \( G \) is a \( K_{1,m} \)-free graph of order \( n (n \geq r, n > m) \) with \( \delta_r(G) \geq (n/3) + C \), then

(a) \( G \) is traceable if \( \delta(G) \geq r \) and \( G \) is connected;
(b) \( G \) is hamiltonian if \( \delta(G) \geq r + 1 \) and \( G \) is 2-connected;
(c) \( G \) is hamiltonian-connected if \( \delta(G) \geq r + 2 \) and \( G \) is 3-connected.

Proof. It suffices to show that if \( G \) is a \( K_{1,m} \)-free graph of order \( n \) and \( \delta_r(G) > n/3 \), then \( \beta(G) \leq 3(m - 1)r \). Let \( t = \beta(G) \). If \( t < r \) then we are done. Otherwise, let \( T \) be a set of \( t \) independent vertices of \( G \) and let \( S = V(G) - T \). Since \( G \) is \( K_{1,m} \)-free, each vertex of \( S \) is adjacent to at most \( m - 1 \) vertices of \( T \). Thus, the number of edges from \( S \) to \( T \) is at most \( (m - 1)(n - i) \). However, if \( T' \) is a set of \( r \) vertices of \( T \), then \( |N_G(T')| > n/3 \). Thus, the number of edges from \( T' \) to \( S \) is greater than \( n/3 \). It follows that the number of edges from \( T \) to \( S \) is greater than

\[
\frac{\binom{t}{r}\left(\frac{n}{3}\right)}{\binom{t-1}{r-1}}.
\]

Thus, \( (m - 1)(n - t) > \binom{t}{r}(n/3)/(\binom{t-1}{r-1}) \). This however, implies that \( t \leq 3(m - 1)r \), which completes the proof of the corollary.

Since Theorems A–D are best possible with respect to the bounds on \( \delta_1(G) \) and \( \delta_2(G) \), the bound given on \( \delta_r(G) \) in the corollary is of the correct order of magnitude. The graph \( G \) of Figure 1 indicates that a minimum degree condition of at least \( r - 1 \) is required in (a). The connected \( K_{1,m} \)-free graph \( G \) satisfies \( \delta_r(G) \geq (n - r + 1)/2 \) and \( \delta(G) = r - 2 \). However, \( G \) is not traceable.

The graph \( G \) of Figure 2 indicates that a minimum degree condition of at least \( r - 1 \) is also required in (b) for \( r \geq 4 \). The 2-connected \( K_{1,m} \)-free
graph $G$ satisfies $\delta_r(G) \geq (n - r + 1)/2$ and $\delta(G) = r - 2$. However, $G$ is not hamiltonian.

In our next result we restrict ourselves to lower bounds on $\delta_3(G)$ in $K_{1,3}$-free graphs. Here we can lower the minimum degree conditions for traceable, hamiltonian and hamiltonian-connected from $r = 3$, $r + 1 = 4$ and $r + 2 = 5$ to $2$, $3$, and $4$ respectively. We observe that in this case, the property of being $K_{1,3}$-free is used heavily throughout the proof. Furthermore, the constant $C$ in the statement of Theorem 2 must be chosen so that $n$ is sufficiently large for Theorem E to be applicable.

**Theorem 2.** There exists a constant $C$ such that if $G$ is a $K_{1,3}$-free graph of order $n$ with $\delta_3(G) \geq (n/3) + C$, then

(a) $G$ is traceable if $\delta(G) \geq 2$ and $G$ is connected;
(b) $G$ is hamiltonian if $\delta(G) \geq 3$ and $G$ is 2-connected;
(c) $G$ is hamiltonian-connected if $\delta(G) \geq 4$ and $G$ is 3-connected.

**Proof.** We proceed by induction on $n$ and assume that (a), (b), and (c) have been established for all graphs of order less than $n$. Let $G$ be a $K_{1,3}$-free graph of order $n$ such that $\delta_3(G) \geq (n/3) + C$, and assume that $G$ satisfies the hypotheses of (a), (b), or (c). Since $G$ is $K_{1,3}$-free, $\beta(G) \leq 18$.

In a manner analogous to the proof of Theorem 1, we can show that $G$ has a path, cycle, or $u - v$ path of order at least $(2n/3) - 2$, depending on whether $G$ satisfies the hypotheses of (a), (b), or (c). This, however, implies that $G$ has a path, cycle, or $u - v$ path $X$ that contains all vertices of $G$ of degree at least $C/3$. Thus, $|V(X)| \geq n - 2$. To complete the proof, we show that $G$ has a path, cycle or $u - v$ path $Y$ of order at least $(2n/3) - 2$ that contains all vertices of $G$ of degree less than $C/3$.

Suppose, first, that $G$ has exactly one vertex $y$ of degree less than $C/3$. If $y$ is on $X$, then let $Y = X$. If $y$ is not on $X$, then since $\text{deg}_G y$ is at least $2$, $3$, or $4$ depending on whether $G$ satisfies the hypotheses of (a), (b), or (c), we can delete an appropriate segment of $X$ and add $y$ together with two adjacent edges to obtain the required $Y$. Thus, we assume that $G$ has two vertices $x$ and $y$ of degree less than $C/3$. If both $x$ and $y$ are on $X$, let $Y = X$. Suppose, then, that at least one of $x$ and $y$ are not on $X$. 
I. Assume that $G$ satisfies the hypotheses of (c).

Case 1. Suppose $xy \not\in E(G)$ and that exactly one of $x$ and $y$, say $x$, is on $X$. If $\deg_G y > 4$, then we can delete an appropriate segment of $X$ and add $y$ to obtain the required $u - v$ path $Y$. Thus, we may assume that $\deg_G y = 4$. Let $X: u = x_1, x_2, \ldots, x_{n-1} = v$ and suppose $N_G(y) = \{x_i, x_j, x_k, x_l\}$, where $i < j < k < l$. Let $x = x_i$. We may assume $i < t < l$; otherwise, we can easily obtain the desired $Y$. Then (by symmetry) either $j < t < k$ or $k < t < 1$.

Subcase (i). Suppose $j < t < k$. Then $j \geq i + \lceil n/3 \rceil + 4$; otherwise, let

$$Y: u = x_1, x_2, \ldots, x_i, y, x_j, x_{j+1}, \ldots, x_{n-1} = v.$$  

Similarly, $l \geq k + \lceil n/3 \rceil + 4$. Furthermore, since $G$ is $K_{1,3}$-free and $\deg_G y = 4$, it follows that $x_{j-1}x_{j+1} \in E(G)$. Consider the vertex $x_j$. Since $\delta_3(G) \geq (n/3) + C$, we have that $|N_G(x, y, x_j)| \geq (n/3) + C$. Since $\deg_G x$ and $\deg_G y$ are less than $C/3$, it follows that $\deg_G x_j > n/3$. Thus $x_jx_p \in E(G)$ for some $p$ with $i + 1 \leq p \leq i + \lceil n/3 \rceil$ or $k - \lceil n/3 \rceil \leq p \leq k - 1$. Thus we have either

$$Y: u = x_1, x_2, \ldots, x_i, y, x_j, x_p, x_{p+1}, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{n-1} = v$$  

or

$$Y: u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p, x_j, y, x_lx_{l+1}, \ldots, x_{n-1} = v.$$  

Subcase (ii). Suppose $k < t < l$. Then, necessarily, $j \geq i + \lceil n/3 \rceil + 4$ and $k \geq j + \lceil n/3 \rceil + 4$. Furthermore, since $G$ is $K_{1,3}$-free, $x_{j-1}x_{j+1} \in E(G)$. As in the previous case, $\deg_G x_j > n/3$. Thus, $x_jx_p \in E(G)$ for some $p$ with $i + 1 \leq p \leq i + \lceil n/3 \rceil$ or $k - \lceil n/3 \rceil \leq p \leq k - 1$. Thus we have either

$$Y: u = x_1, x_2, \ldots, x_i, y, x_j, x_p, x_{p+1}, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_{n-1} = v$$  

or

$$Y: u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_p, x_j, y, x_k, x_{k+1} \ldots, x_{n-1} = v.$$  

Case 2. Suppose $xy \not\in E(G)$ and that neither $x$ nor $y$ is on $X$. Since $\deg_G x \geq 4$, we obtain a $u - v$ path $X'$ of order at least $(2n/3) - 2$ that contains $x$. If we choose a longest such path $X'$ then $X'$ contains all vertices of degree at least $C/3$. If $x$ and $y$ are on $X'$, let $Y = X'$ and if only $x$ is on $X'$ we may proceed as in Case 1.
Case 3. Suppose \( xy \in E(G) \) and one of \( x \) and \( y \), say \( x \) is on \( X \). Since \( xy \in E(G) \) and \( \deg_G y \geq 4 \), we can clearly add \( y \) and delete an appropriate segment of \( X \) to obtain the required \( u - v \) path \( Y \).

Case 4. Suppose \( xy \in E(G) \), neither \( x \) nor \( y \) is on \( X \) and \( |N_X(\{x, y\})| \geq 4 \). We then obtain a \( u - v \) path \( X' \) of order at least \((2n/3) - 2\) that contains one or both of \( x \) and \( y \). A longest such path \( X' \) contains all vertices of degree at least \( C/3 \). If \( x \) and \( y \) are on \( X' \), let \( Y = X' \); otherwise, we may proceed as in Case 3.

Case 5. Suppose \( xy \in E(G) \), neither \( x \) nor \( y \) is on \( X \) and \( |N_X(\{x, y\})| = 3 \). Let

\[
X: \ u = x_1, x_2, \ldots, x_{n-2} = v
\]

and suppose \( N_X(\{x, y\}) = \{x_i, x_j, x_k\} \) where \( i < j < k \). We may assume that \( j \geq i + \lceil n/3 \rceil + 4 \) and \( k \geq j + \lceil n/3 \rceil + 4 \) since otherwise we can easily obtain the desired \( u - v \) path \( Y \). As in Case 1, \( x_{j-1}x_{j+1} \in E(G) \) and \( \deg_G x_j > n/3 \). Thus, \( x_jx_p \in E(G) \) for some \( p \) with \( i + 1 \leq p \leq i + \lceil n/3 \rceil \) or \( k - \lceil n/3 \rceil \leq p \leq k - 1 \). Thus, we have either

\[
Y: \ u = x_1, x_2, \ldots, x_j, x_{j+1}, x_{j+2}, \ldots, x_{n-2} = v
\]
or

\[
Y: \ u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_p, x_j, x, y, x_kx_{k+1}, \ldots, x_{n-2} = v
\]

II. Assume that \( G \) satisfies the hypotheses of (b).

Case 1. Suppose \( xy \notin E(G) \) and that exactly one of \( x \) and \( y \), say \( x \), is on \( X \). If \( \deg_G y > 3 \), then we can delete an appropriate segment of \( X \) and add \( y \) to obtain the required cycle \( X \). Thus, we may assume that \( \deg_G y = 3 \).

Let \( X = x_1, x_2, \ldots, x_{n-1}, x_t \) and suppose \( N_G(y) = \{x_i, x_j, x_k\} \), where \( i < j < k \). Without loss of generality, we may assume that \( x = x_t \), where \( k < t \leq n - 1 \). As in previous cases, we may assume that \( j \geq i + \lceil n/3 \rceil + 4 \), \( k \geq j + \lceil n/3 \rceil + 4 \), \( x_{j-1}x_{j+1} \in E(G) \), and \( \deg_G x_j > n/3 \). Thus \( x_jx_p \in E(G) \) for some \( p \) with \( i + 1 \leq p \leq i + \lceil n/3 \rceil \) or \( k - \lceil n/3 \rceil \leq p \leq k - 1 \). In either case, we obtain the desired cycle \( Y \).

Case 2. Suppose \( xy \notin E(G) \) and that neither \( x \) nor \( y \) is on \( X \). Since \( \deg_G x \geq 3 \), we may proceed as in I, Case 2.

Case 3. Suppose \( xy \in E(G) \) and that exactly one of \( x \) and \( y \) is on \( X \), say \( x \). Since \( xy \in E(G) \) and \( \deg_G x \geq 3 \), we may proceed as in I, Case 3.
Case 4. Suppose \( xy \in E(G) \), neither \( x \) nor \( y \) is on \( X \) and \( |N_X(\{x, y\})| \geq 3 \). Here we may proceed as in I, Case 4.

Case 5. Suppose \( xy \in E(G) \), neither \( x \) nor \( y \) is on \( X \) and \( |N_X(\{x, y\})| = 2 \). Let

\[
X; \ x_1, x_2, \ldots, x_{n-2}, x_1
\]

and assume, without loss of generality, that \( N_X(\{x, y\}) = \{x_1, x_j\} \), where \( j < n - 2 \). We may also assume \( j \geq \lfloor n/3 \rfloor + 5 \) and \( n - 2 \geq j + \lfloor n/3 \rfloor + 3 \); otherwise we easily obtain the desired cycle \( Y \). As in previous cases, \( x_{j-1}, x_{j+1} \in E(G) \) and \( \deg_G x_j > n/3 \). Thus, \( x_jx_p \in E(G) \) for some \( p \) with \( 2 \leq p \leq \lfloor n/3 \rfloor + 1 \) or \( n - \lfloor n/3 \rfloor - 1 \leq p \leq n - 2 \). In either case, we obtain the desired cycle \( Y \).

III. Assume that \( G \) satisfies the hypotheses of (a).

Cases 1–4 follow exactly as they did in I and II. We list them without proof.

Case 1. Suppose \( xy \notin E(G) \) and exactly one of \( x \) and \( y \), say \( x \), is on \( X \).

Case 2. Suppose \( xy \notin E(G) \) and that neither \( x \) nor \( y \) is on \( X \).

Case 3. Suppose \( xy \in E(G) \) and exactly one of \( x \) and \( y \) is on \( X \).

Case 4. Suppose \( xy \in E(G) \), neither \( x \) nor \( y \) is on \( X \), and \( |N_X(\{x, y\})| \geq 2 \).

Case 5. Suppose \( xy \in E(G) \), neither \( x \) nor \( y \) is on \( X \), and \( |N_X(\{x, y\})| = 1 \). Consider the connected graph \( G' = G - \{x, y\} \). Since each of \( x \) and \( y \) has degree 2 in \( G \) it follows that \( \delta(G') > n/3 \). If \( G' \) is 2-connected then, by the Matthews-Sumner result \( G' \) is hamiltonian and we obtain a hamiltonian path \( Y \) in \( G \). If \( G' \) has a cutvertex \( w \), consider \( G' - w \). Then, since \( \delta(G' - w) > n/3 \) and, consequently, \( \delta_2(G' - w) > n/3 \), it follows that \( G' - w \) has exactly two components, both of which are 3-connected and hence hamiltonian-connected by Theorem E. But then since \( G \) is \( K_{1,3} \)-free, \( G \) has a hamiltonian path \( Y \) and the proof is complete.

The graph \( G \) of Figure 1, with \( r = 3 \), indicates that for the traceable case, \( \delta(G) \geq 2 \) is a necessary condition.
ACKNOWLEDGMENT

RF and RJG were supported by O.N.R. grant N00014-91-J-1085. LL was supported by O.N.R. grant N00014-93-1-0050. TL was supported by Pew Midstates Science and Mathematics Consortium Faculty Development Program.

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Received September 3, 1992