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Extremal Graphs for Intersecting Triangles

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It is known that for a graph on n vertices $\lfloor n^2/4 \rfloor + 1$ edges is sufficient for the existence of many triangles. In this paper, we determine the minimum number of edges sufficient for the existence of k triangles intersecting in exactly one common vertex.

1. NOTATION

With integers $n \geqslant p \geqslant 1$, we let $T_{n,p}$ denote the *Turán graph*, i.e., the complete p-partite graph on n vertices where each partite set has either $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$ vertices and the edge set consists of all pairs joining distinct parts. K_r represents the complete graph on r vertices.

For a graph G and a vertex $x \in V(G)$, the *neighborhood* of x in G is denoted by $N_G(x) = \{ y \in V(G) : (x, y) \in E(G) \}$, or when clear, simply $N(x)$. The *degree* of x in G, denoted by $\deg_G(x)$, or $\deg(x)$, is size of $N_G(x)$. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees, respectively, in G. For a subset $X \subset V(G)$, let $G[X]$ denote the subgraph of G induced by X. A *matching* in G is a set of edges from $E(G)$, no two of which share a common vertex, and the *matching number* of G, denoted by $\nu(G)$, is the maximum number of edges in a matching in G.

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2. The Main Theorem

Suppose that we are given some fixed graph H. What is the maximum number, $\operatorname{ex}(n, H)$, of edges in a graph G on n vertices that does not contain a copy of H as a subgraph (often said to *forbid* H)? A graph G on n vertices with $\operatorname{ex}(n, H)$ edges and without a copy of H is called an *extremal* graph for H. For $n \geqslant |V(H)|$ adding one more edge to any one of the extremal graphs will produce a graph with a copy of H.

A graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex is called a *k-fan* and denoted by F_k. For each k, the chromatic number of F_k is three, and so by the Erdős–Stone theorem [12], $\operatorname{ex}(n, F_k) = (1 + o(1))\, n^2/4$. Our main result is to determine $\operatorname{ex}(n, F_k)$ for every fixed k whenever n is large.

THEOREM 2.1. *For every $k \geqslant 1$, and for every $n \geqslant 50k^2$, if a graph G on n vertices has more than*

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \dfrac{3}{2}k & \text{if } k \text{ is even} \end{cases} \tag{1}$$

edges, then G contains a copy of a k-fan. Furthermore, the number of edges is best possible.

To prove the lower bound for $\operatorname{ex}(n, F_k)$ we present the following graph, $G_{n,k}$. For odd k (where $n \geqslant 4k-1$) $G_{n,k}$ is constructed by taking a complete equi-bipartite graph and embedding two vertex disjoint copies of K_k in one side. For even k (where now $n \geqslant 4k-3$) $G_{n,k}$ is constructed by taking a complete equi-bipartite graph and embedding a graph with $2k-1$ vertices, $k^2 - (3/2)\,k$ edges with maximum degree $k-1$ in one side. With a little more effort we can prove that the $G_{n,k}$'s are the only F_k-extremal graphs (for $n \geqslant 50k^2$).

Obviously, $\operatorname{ex}(n, F_k) = \binom{n}{k}$ for $1 \leqslant n \leqslant 2k$, and it is easy to check that $\operatorname{ex}(2k+1, F_k) = 2k^2 - 1$ (if $k \geqslant 2$), which is smaller than (1) for odd k and larger than (1) for even k ($k \geqslant 4$). However, we conjecture that (1) gives $\operatorname{ex}(n, F_k)$ for all $n \geqslant 4k$ (rather than $n \geqslant 50k^2$).

In Section 3 a survey of some known related results is given. Section 4 contains theorems and lemmas used in the proof of the main theorem. In Section 5 we prove that an extremal graph with large minimum degree is almost bipartite, and in Section 6 we give a lemma which gives the upper bound (1) for almost bipartite graphs. The proof of the main theorem follows in the last section.

3. Some Extremal Results Concerning Traingles

We briefly examine some results in extremal graph theory. For further results see [3, 21].

Theorem 3.1 (Turán [22]). *If G is a graph on n vertices that does not contain a copy of K_r then $|E(G)| \leqslant |E(T_{n,r-1})|$, with equality only if $G \cong T_{n,r-1}$.*

The following is a corollary Mantel [18] discovered long before Turán's theorem:

$$\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{2}$$

With $\lfloor n^2/4 \rfloor$ edges, we can have a graph containing no triangles, but with the addition of just one more edge, not only one triangle is produced, but as Rademacher proved in 1941 (unpublished) in fact at least $\lfloor n/2 \rfloor$ appear. Erdős gave a simplification of the proof in [7]. The complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ with an additional edge adjoined shows that $\lfloor n/2 \rfloor$ is best possible here.

Moon and Moser [19] proved that if G is any graph on n vertices, then it contains at least $|E(G)|(4|E(G)| - n^2)/(3n)$ triangles. Lovász [15] showed that this theorem can be derived from the sieve formula (see [3] for discussion). Concerning the number of triangles, Bollobás [2] proved the following conjecture by Nordhaus and Stewart [20]: If G is a graph on n vertices and $n^2/4 \leqslant |E(G)| \leqslant n^2/3$, then G contains at least $(n/9)(4|E(G)| - n^2)$ triangles. The best lower bound for the number of triangles was proved by Fisher [13]. That bound is off from the optimal one only by a lower order term (in most cases). Observe that with $|E(G)| = n^2/4 + 1$, Moon–Moser's result gives at least $n/3$ triangles, while Bollobás' theorem yields $(4/9)n$ triangles, and Rademacher's theorem guarantees $n/2$ triangles, the best possible.

Lovász and Simonovits [17] proved that any graph on n vertices with $\lfloor n^2/4 \rfloor + t$ edges contains at least $t\lfloor n/2 \rfloor$ triangles if $t < n/2$. The cases $t = 1, 2, 3$ were done by Erdős [7] in 1955. A few years later he [8] extended it to $t < cn$ for some small positive constant c.

A key to the above result was the following lemma from Erdős [8, 9]. There exists a constant $\rho > 0$ so that for n sufficiently large, $\lfloor n^2/4 \rfloor + 1$ edges in a graph on n vertices yields at least ρn triangles having a common edge. In an unpublished manuscript, Edwards [6] showed that we can take $\rho = 1/6$ and via an appropriate $\lfloor n/2 \rfloor$-regular graph, that this is best possible.

Many of these results have generalizations to larger complete graphs; we mention just one interesting example: a graph on n vertices and $|E(T_{n,p-1})|+1$ edges has an edge contained in at least $n^{p-2}/(10p)^{6p}$ copies of K_p (Erdős [10, 11]).

Now we return to our main object, forbidding F_k. The case $k=1$ is just (2) so the upper bound in (1) gives $\mathrm{ex}(n, F_1)$ for all n. Now we ask how many edges are required to guarantee the existence of F_2. Let us call F_2 a *bowtie*.

THEOREM 3.2. *For $n \geqslant 5$, every graph on n vertices with at least $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles intersecting in a single vertex. Hence,* $\mathrm{ex}(n, F_2) = \lfloor n^2/4 \rfloor + 1$.

For sufficiently large n, the above result is a special case of the following theorem of Bollobás ([3, Problem 36, p. 365]) where he settled a conjecture posed by Busolini [4] in 1976. For any given $p \geqslant 3$ and $k \geqslant 1$ if n is sufficiently large $(n > n_{p,k})$, then a graph on n vertices and $|E(T_{n,p-1})| + k$ edges contains k edge-disjoint copies of K_p which form a connected subgraph. Here we are able to settle the case for F_2 for all possible values of n.

Proof of Theorem 3.2. We use induction on n. There are only two graphs with 5 vertices and 8 edges, both of which contain a bowtie. So assume that for some $n > 5$, every graph with $n-1$ vertices and $\lfloor (n-1)^2/4 \rfloor + 2$ edges contains a bowtie. Let G be a graph on n vertices with $\lfloor n^2/4 \rfloor + 2$ edges. As $\lfloor n^2/4 \rfloor - \lfloor (n-1)^2/4 \rfloor = \lfloor n/2 \rfloor$, we can use the induction hypothesis for $G \backslash x$ if there is any vertex x with $\deg_G(x) \leqslant \lfloor n/2 \rfloor$.

If $\delta(G) \geqslant \lfloor n/2 \rfloor + 1$, then the number of edges of G is at least $(\lfloor n/2 \rfloor + 1)\, n/2$. This is larger than $\lfloor n^2/4 \rfloor + 2$ for $n = 6$, and for all $n \geqslant 8$, a contradiction.

When $n = 7$, $|E(G)| = 14$ and $\delta(G) \geqslant 4$, the graph G is 4-regular. Any such G contains an F_2. One way to see this is to consider any vertex x of G; the graph $G \backslash x$, has degree sequence 4, 4, 3, 3, 3, 3. There are only three graphs with this degree sequence; it is straightforward to verify that two of these contain a bowtie, and to the third, reaffixing x produces the required bowtie.

To see that $\mathrm{ex}(n, F_2) \geqslant \lfloor n^2/4 \rfloor + 1$, examine the complete bipartite graph on n vertices and adjoin one more edge; this graph is F_2-free. ∎

4. PRELIMINARIES

In this section we give preparatory lemmas for the proof of the main theorem.

Define $f(\nu, \Delta) = \max\{|E(G)| : \nu(G) \leqslant \nu,\ \Delta(G) \leqslant \Delta\}$. Chvátal and Hanson [5] proved that for every $\nu \geqslant 1$ and $\Delta \geqslant 1$,

$$f(\nu, \Delta) = \nu\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor \leqslant \nu\Delta + \nu. \tag{3}$$

We will frequently use the following special case proved by Abbott *et al.* [1]:

$$f(k-1, k-1) = \begin{cases} k^2 - \frac{3}{2}k & \text{if } k \text{ is even,} \\ k^2 - k & \text{if } k \text{ is odd.} \end{cases} \tag{4}$$

The extremal graphs are exactly those we embedded into $T_{n,2}$ in Section 2 to obtain the extremal F_k-free graph $G_{n,k}$.

LEMMA 4.1. *If G is F_k-free, then for any $x \in V(G)$,*

$$|E(G[N(x)])| \leqslant (k-1)\,|N(x)|.$$

Proof. As $G[N(x)]$ contains no k-matching we have that the number of its edges is bounded by $f(k-1, |N(x)|-1)$, so the lemma follows from (3). ∎

LEMMA 4.2. *If G is F_k-free, then for $\Delta = \Delta(G)$,*

$$|E(G)| \leqslant (n-\Delta)\,\Delta + (k-1)\,\Delta \leqslant n^2/4 + (k-1)\,\Delta.$$

Proof. Select $x' \in V(G)$ with $\deg_G(x') = \Delta$. There are $n - \Delta$ vertices in $V(G) \backslash N(x')$, each with degree at most Δ, and, by Lemma 4.1, there are at most $(k-1)\,\Delta$ edges in $G[N(x')]$. In all, $|E(G)| \leqslant (n-\Delta)\,\Delta + (k-1)\,\Delta$. The second inequality is elementary. ∎

5. STRUCTURE OF THE EXTREMAL GRAPHS

In Sections 5 and 6, the main theorem is proved only for graphs with large minimum degree. As the cases $k = 1$ and $k = 2$ are settled by (2) and by Theorem 3.2 respectively, we now assume that $k \geqslant 3$. The aim of this section is to prove the following lemma.

LEMMA 5.1. *Suppose that G is an F_k-free graph on n vertices with $n \geqslant 24k^2$, and with minimum degree $\delta > (n/2) - k$, maximum degree $\Delta > n/2$.*

Then there exists a partition $V(G)=V_0 \dot\cup V_1$, *so that* $V_0 \neq \emptyset$, $V_1 \neq \emptyset$, *and for each* $i=0, 1$ *and for every* $x \in V_i$, *the following hold*:

$$\nu(G[V_i]) \leqslant k-1 \qquad \text{and} \qquad \Delta(G[V_i]) \leqslant k-1; \tag{5}$$

$$\deg_{G[V_i]}(x) + \nu(G_{1-i}[N(x) \cap V_{1-i}]) \leqslant k-1. \tag{6}$$

Proof. We give the proof in a sequence of claims.

CLAIM 1. The inequality $\Delta < (n/2) + 3k$ holds.

Proof of Claim 1. Fix x' with maximum degree. By Lemma 4.1, there exists a vertex $y \in N(x')$ with $|N(y) \cap N(x')| \leqslant 2(k-1)$. Then

$$n - \Delta = |V(G) \backslash N(x')| \geqslant |N(y)| - 2(k-1) \geqslant \delta - 2(k-1). \quad \blacksquare$$

Claim 2. Let x' be a vertex of maximum degree. Define the sets V_0 and V_1 so that

$$|N(y) \cap N(x')| > \frac{2k-1}{2k} \Delta \qquad \text{for} \quad y \in V_0,$$

and

$$|N(y) \backslash N(x')| > \frac{2k-1}{2k} (n - \Delta) \qquad \text{for} \quad y \in V_1.$$

Then $V(G) = V_0 \dot\cup V_1$ is a partition of the vertex set $V(G)$ into nonempty parts.

Proof of Claim 2. Obviously, $V_0 \cap V_1 = \emptyset$, otherwise there was a vertex of degree at least $n(2k-1)/(2k)$ contradicting Claim 1. Both V_0 and V_1 are nonempty. Indeed, x' belongs to V_0 and its neighbor y from the proof of Claim 1 belongs to V_1 since

$$\begin{aligned}
|N(y) \backslash N(x')| &\geqslant |N(y)| - 2(k-1) \\
&\geqslant \delta - 2(k-1) \\
&> \frac{n}{2} - k - 2k + 2 \qquad \left(\text{since } \delta > \frac{n}{2} - k\right) \\
&> \frac{n}{2} + k - \frac{16k^2}{4k} - \frac{1}{2} \\
&\geqslant \frac{n}{2} + k - \frac{n}{4k} - \frac{1}{2} \qquad (\text{since } n \geqslant 16k^2)
\end{aligned}$$

$$= \left(1 - \frac{1}{2k}\right)\left(\frac{n}{2} + k\right)$$

$$> \frac{2k-1}{2k}(n - \delta) \qquad \left(\text{since } \delta > \frac{n}{2} - k\right)$$

$$\geqslant \frac{2k-1}{2k}(n - \Delta).$$

Finally, we have to prove that each vertex $z \in V$ belongs to $V_0 \cup V_1$. If $|N(z) \cap N(x')| \leqslant 4k$, then $z \in V_1$, because for $n \geqslant 20k^2$, we have

$$|N(z) \backslash N(x')| \geqslant |N(z)| - 4k \geqslant \delta - 4k > \frac{2k-1}{4k} n \geqslant \frac{2k-1}{2k}(n - \Delta).$$

If $|N(z) \cap N(x')| > 4k$, then there exists a vertex $w \in N(z) \cap N(x')$ such that both $|N(w) \cap N(x')| \leqslant 2k$ and $|N(w) \cap N(z)| \leqslant 2k$. This follows from the fact that for each x, $\nu(G[N(x)]) < k$, so the subgraph induced by a neighborhood cannot have more than $2k-2$ vertices of degree at least $2k$. We obtain

$$|N(w) \backslash (N(x') \cup N(z))| \geqslant |N(w)| - 4k \geqslant \delta - 4k.$$

This implies that $|N(x') \cup N(z)| \leqslant n - \delta + 4k$, so

$$|N(x') \cap N(z)| = |N(x')| + |N(z)| - |N(x') \cup N(z)|$$

$$\geqslant \Delta + \delta - (n - \delta + 4k) > \frac{2k-1}{2k} \Delta,$$

where the last inequality is implied again by the facts that $(n/2) - k < \delta$, $\Delta > n/2$, and $n \geqslant 24k^2$. Thus we obtain $z \in V_0$. ∎

An obvious consequence of the definition of V_0 and V_1 is the following:

Claim 3. *Every $2k$ vertices from V_0 have a common neighbor (in $N(x')$) and every $2k$ vertices from V_1 have a common neighbor (in $V(G) \backslash N(x')$). Moreover, every two vertices from V_0 have at least $2k-1$ common neighbors (in $N(x')$) and every 2 vertices from V_1 have at least $2k-1$ common neighbors (in $V(G) \backslash N(x')$).* ∎

Proof of (5). If G_i contains a k-matching, say $(y_1, y_2), (y_3, y_4), ..., (y_{2k-1}, y_{2k})$, then since each of these y_j's have a common neighbor, say y, we get an F_k with center y, a contradiction. If for some $y \in V_i$, $|N(y) \cap V_i| > k$, say with neighbors $y_1, ..., y_k$ in V_i, then since (by Claim 3), each pair (y, y_i) has at least k common neighbours outside the set

$Y = \{y, y_1, ..., y_k\}$. This implies that one can find distinct vertices $x_1, ..., x_k$ outside Y, such that the triples of the form (y, y_j, x_j) determine k triangles having the common vertex y. ∎

Proof of (6). Let $x_0 \in V_0$ have neighbors $x_1, x_2, ..., x_a$ in V_0 and neighbors $y_1, z_1, y_2, z_2, ..., y_b, z_b$ in V_1 where, for each $j = 1, ..., b$, $(y_j, z_j) \in E(G_1)$. By (5), both a and b are less than k. Suppose for the moment that $a + b \geqslant k$. Then by Claim 3, for each $1 \leqslant i \leqslant a$ there exist at least a vertices outside the set $\{x_0, x_1, ..., x_a, y_1, ..., y_b, z_1, ..., z_b\}$ connected to both x_0 and x_i. Now the existence of a k-fan with central vertex x_0 is immdediate, contrary to our initial assumptions. This also completes the proof of Lemma 5.1. ∎

6. Proof of the Main Lemma

The aim of this section is to prove Lemma 6.2, which is the key to the proof of the main result in this paper. We begin with another technical lemma.

Lemma 6.1. *Let H be a graph and b a nonnegative integer such that $b \leqslant \Delta(H) - 2$, and let $v = v(H)$, $\Delta = \Delta(H)$. Then*

$$\sum_{x \in V(H)} \min\{\deg_H(x), b\} \leqslant v(b + \Delta). \tag{7}$$

Proof. We proceed by induction on v. The inequality is trivial if $v = 0$.

Suppose first that there is a vertex $s \in V$ such that $v(H \backslash s) = v - 1$. Replacing the vertex s, the left-hand side of (7) increases by at most $\min\{\deg_H(s), b\} + \deg_H(s)$ and the right-hand side increases by $b + \Delta$ and we are done.

Suppose now that $v(H \backslash x) = v$ holds for every $x \in V(H)$. Such a graph is said to be *matching-critical*. Gallai [14] (see also [16, p. 89]) proved that a connected matching-critical graph C with matching number $v(C)$ has exactly $2v(C) + 1$ vertices (so it is also *factor-critical*). So a matching-critical graph is a vertex disjoint union of factor-critical components. Let C be such an odd component of size $2a + 1$. Then $v(C) = a$. The partial sum over vertices in C is

$$\sum_{x \in V(C)} \min\{\deg_H(x), b\} \leqslant (2a + 1) \min\{2a, b\} \leqslant a(b + \Delta),$$

for if $b \geqslant 2a$, the middle part is $(2a + 1)\, 2a$ and since $\Delta \geqslant b + 2 \geqslant 2a + 2$, $(2a + 1)\, 2a \leqslant a(b + \Delta)$ holds; while if $b < 2a$ the middle part is $(2a + 1)\, b$, and because $(a+1)\,b/a = b + b/a < b+2 \leqslant \Delta$, we have $(2a+1)\,b \leqslant ab + a\Delta$. Summing over all components gives (7). ∎

Let G be a graph with a partition of the vertices into two non-empty parts $V(G) = V_0 \dot{\cup} V_1$, $G_0 = G[V_0]$, $G_1 = G[V_1]$, and define

$$G_{\mathrm{cr}} = (V(G), \{(v_0, v_1) \in E(G) : v_0 \in V_0, v_1 \in V_1\}),$$

where "cr" denotes "crossing". For each $i \in \{0, 1, \mathrm{cr}\}$ let $d_i(x) = \deg_{G_i}(x)$ and $v_i = v(G_i)$.

One may note that in this section, even though G is F_k-free, we do not use the restrictions for δ and n.

LEMMA 6.2. *Suppose G is partitioned as above so that* (5) *and* (6) *are satisfied. If G is F_k-free, then*

$$|E(G_0)| + |E(G_1)| - (|V_0| \cdot |V_1| - |E(G_{\mathrm{cr}})|) \leqslant f(k-1, k-1). \tag{8}$$

Proof. Observe that G_{cr} is a bipartite graph, and $|V_0| \cdot |V_1| - |E(G_{\mathrm{cr}})|$ is the number of edges missing from the complete bipartite graph. By (5) and the definition of f, we see that for each $i = 0, 1$, $|E(G_i)| \leqslant f(k-1, k-1)$, and so the left hand side of (8) is bounded above by $2f(k-1, k-1)$. Delete vertices of G so that the left hand side of (8) is maximal, and let G be minimal in this sense.

We now claim that for each $i = 0, 1$ and every $x \in V_i$,

$$d_i(x) - (|V_{1-i}| - d_{\mathrm{cr}}(x)) > 0. \tag{9}$$

In fact, if for some $x \in V_0$, $d_0(x) - (|V_1| - d_{\mathrm{cr}}(x)) \leqslant 0$ holds, then

$$\begin{aligned}
&|E(G_0 \backslash x)| + |E(G_1)| - (|V_0 \backslash x| \cdot |V_1| - |E(G_{\mathrm{cr}} \backslash x)|) \\
&\quad = |E(G_0)| + |E(G_1)| - (|V_0| \cdot |V_1| - |E(G_{\mathrm{cr}})|) \\
&\qquad - (d_0(x) - |V_1| + d_{\mathrm{cr}}(x)) \\
&\quad \geqslant |E(G_0)| + |E(G_1)| - (|V_0| \cdot |V_1| - |E(G_{\mathrm{cr}})|),
\end{aligned}$$

contradicting the minimality of G. Hence (9) holds.

We also claim that for each $i = 0, 1$,

$$d_i(x) - (|V_{1-i}| - d_{\mathrm{cr}}(x)) \leqslant k - 1 - v_{1-i}. \tag{10}$$

To see (10), we need only observe that,

$$\begin{aligned}
d_i(x) - [|V_{1-i}| - d_{\mathrm{cr}}(x)] &\leqslant k - 1 - [v(G_{1-i}[N(x) \cap V_{1-i}]) \\
&\qquad + |V_{1-i}| - d_{\mathrm{cr}}(x)] \qquad \text{(by (6))} \\
&\leqslant k - 1 - v_{1-i},
\end{aligned}$$

where the last inequality holds since any matching in G_{1-i} which extends a matching in $G_{1-i}[N(x)\cap V_{1-i}]$ can have at most $|V_{1-i}|-d_{cr}(x)$ additional edges (even though some endpoints of additional edges may be in $N(x)\cap V_{1-i}$). This proves (10).

We can also assume that for each $i=0, 1$,

$$1\leqslant v_i\leqslant k-2, \tag{11}$$

by the following arguments. If $v_0=0$, then G_0 is empty, and is this case,

$$|E(G_1)|-(|V_0|\cdot|V_1|-|E(G_{cr})|)\leqslant|E(G_1)|\leqslant f(k-1, k-1);$$

thus (8) holds trivially, verifying the lemma. If $v_0=k-1$, then by (9) and (10), we would have

$$0<d_1(x)-(|V_0|-d_{cr}(x))\leqslant 0,$$

a contradiction. Similar arguments hold for $i=1$, proving (11).

We may further suppose that

$$2\leqslant v_i. \tag{12}$$

Indeed, if for some i, $v_i=1$, then (11) implies that $v_1+v_2\leqslant k-1$. As

$$f(v_1, \Delta)+f(v_2, \Delta)\leqslant f(v_1+v_2, \Delta)$$

always holds, we get that $|E(G_0)|+|E(G_1)|\leqslant f(k-1, k-1)$ and (8) follows.

Now apply Lemma 6.1 for the graph $G_i (i=0, 1)$ with $\Delta=k-1$ and $b=k-1-v_{1-i}\leqslant\Delta-2$ (by (12)). Using (7) and (10), we get

$$\begin{aligned}\sum_{x\in V_i}[d_i(x)-(|V_{i-1}|-d_{cr}(x))]&\leqslant\sum_{x\in V_i}\min\{d_i(x), k-1-v_{1-i}\}\\&\leqslant v_i(2(k-1)-v_{1-i}).\end{aligned} \tag{13}$$

The sum in the left hand since equals $2|E(G_i)|+|E(G_{cr})|-|V_0||V_1|$, so adding these two (for $i=0, 1$) gives

$$\begin{aligned}2|E(G)|&=2|E(G_0)|+2|E(G_1)|+2|E(G_{cr})|\\&\leqslant v_0(2(k-1)-v_1)+v_1(2(k-1)-v_0)+2|V_0|\cdot|V_1|\\&=2[k^2-2k+1-(k-1-v_0)(k-1-v_1)]+2|V_0|\cdot|V_1|.\end{aligned}$$

This yields $|E(G)|\leqslant k^2-2k+|V_0|\cdot|V_1|$ (by (11), $k-1-v_i\geqslant 1$), finishing the proof of Lemma 6.2. ∎

7. End of the Proof

We can summarize Lemma 6.2 and Lemma 5.1 as follows.

LEMMA 7.1. *Suppose that G is an F_k-free graph on n vertices with $n > 24k^2$, and with minimum degree $\delta > (n/2) - k$. Then $|E(G)| \leqslant \lfloor n^2/4 \rfloor + f(k-1, k-1)$ holds.*

Proof. If $\Delta(G) \leqslant n/2$, then $|E(G)| \leqslant n^2/4$ and we are done. Otherwise, we can apply Lemma 5.1 to get a decomposition of G into G_0, G_1, G_{cr}. The graph G_{cr} consists of the edges between V_0 and V_1. Lemma 6.2 implies that

$$\begin{aligned}|E(G)| &= |E(G_0)| + |E(G_1)| + |E(G_{cr})| \\ &\leqslant |V_0| \cdot |V_1| + f(k-1, k-1) \\ &\leqslant \frac{n^2}{4} + f(k-1, k-1),\end{aligned}$$

and we are done again. ∎

Proof of Theorem 2.1. Suppose that $n \geqslant 50k^2$, and that G is an F_k-free graph on n vertices. We need to show that G has at most $\lfloor n^2/4 \rfloor + f(k-1, k-1)$ edges. If $\delta(G) > (n/2) - k$, then Lemma 7.1 can be applied and we are done. If $|E(G)| \leqslant n^2/4$, then there is nothing to prove. The last case is when there exists a vertex $x = x_n$ with $\deg_G(x_n) \leqslant (n/2) - k$ and $|E(G)| > n^2/4$.

Denote G by G^n, and let $G^{n-1} = G^n \backslash x_n$. Note that for the new graph

$$|E(G^{n-1})| = |E(G^n)| - \delta(G^n) > \frac{(n-1)^2}{4} + k - \frac{1}{4}.$$

If there exists a vertex $x_{n-1} \in V(G^{n-1})$ with $\deg_{G^{n-1}}(x_{n-1}) \leqslant (n-1)/2 - k$, then delete it to obtain $G^{n-2} = G^{n-1} \backslash x_{n-1}$. Continue this process as long as $\delta(G^v) \leqslant (v/2) - k$, and after $n - l$ steps we get a subgraph G^l with $\delta(G^l) > (l/2) - k$. We claim that $l > n/2$ (so $l \geqslant 24k^2$). Indeed, by induction, we can see that $|E(G^l)| \geqslant l^2/4 + (n-l)(k - (1/4))$. On the other hand, Claim 2 says $|E(G)| \leqslant l^2/4 + (k-1)\, l$.

We can apply Lemma 7.1 to get $|E(G^l)| \leqslant l^2/4 + f(k-1, k-1)$ which gives

$$|E(G)| \leqslant l^2/4 + f(k-1, k-1) + \sum_{i=l+1}^{n} \left(\frac{i}{2} - k\right) < n^2/4 + f(k-1, k-1). \quad ∎$$

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