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Problems Involving Paths or Cycles in Graphs

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October 12, 1994

Abstract

We examine several problems involving paths or cycles in graphs. In particular, we consider the following:

Problem: Given a particular hamiltonian-type property P such as being hamiltonian, traceable, pancyclic (and others), and given a connected (or 2-connected) graph G; when is it possible to determine all pairs of graphs (R, S) such that if G is (R, S)-free, then G has property P?

Problem: A hamiltonian cycle is in some sense the simplest type of 2-factor possible in a graph. Under what conditions can we determine other types of 2-factors? For example, when can we determine that a graph will contain a 2-factor consisting of exactly k disjoint cycles? When can we specify the lengths of these k cycles?

Problem: Given vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of G, then G is called k-linked if there exist vertex disjoint paths joining x_i to y_i for each $i = 1, 2, \ldots, k$. Conditions implying a graph is k-linked will be discussed.

1 Introduction

All graphs considered will be finite and simple. For terms not defined here see [15].

Given a family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ of graphs we say that a graph G is $\mathcal{F}$-free if G contains no induced subgraph isomorphic to any H_i, $i = 1, 2, \ldots, k$. In particular, if $\mathcal{F} = \{H\}$, we simply say G is H-free. We call the graphs in $\mathcal{F}$ *forbidden subgraphs.*

We denote the cycle on n vertices as C_n, path on n vertices as P_n, complete graph on n vertices as K_n and the complete bipartite graph with r vertices in one set and m vertices in the other set as $K_{r,m}$. We define the graphs Z_i $i = 1, 2, \ldots$ to be a triangle with a path P_i attached to one of its vertices (see Figure 1 for Z_3). Recall, a graph is *hamiltonian* if it contains a spanning cycle; *traceable* if it contains a spanning path. Further, let $\sigma_2(G) = min\{deg u + deg v\}$, where the minimum is taken over all pairs of nonadjacent vertices u, v in G.

*Supported by O.N.R. Grant N00014-91-J-1085.

Problems involving paths and cycles in graphs abound. In this paper we will concentrate on three such problems, each very different in nature. A background of related results for each problem is discussed and key remaining questions are highlighted.

2 Characterizing Forbidden Families for Hamiltonian Properties

The use of forbidden subgraphs to obtain classes of graphs possessing special properties has long been a common graphical technique. It has been pointed out that, the star $K_{1,3}$, sometimes called the *claw*, has often been a part of these forbidden families. The reason for this observation shall become clear, at least for hamiltonian properties, as we proceed.

One of the earliest forbidden subgraph results dealing with hamiltonian properties is the following result from [9]. The graphs $K_{1,3}$ and N are shown in Figure 1.

Theorem 1 *Let G be a $\{K_{1,3}, N\}$-free graph. Then*
(1) if G is connected, then G is traceable and
(2) if G is 2-connected, then G is hamiltonian.

Theorem 1 is typical of those in the literature to date. It imposes minor, but necessary, connectivity conditions on the class of graphs defined by a forbidden pair of graphs in order to obtain hamiltonian results. The connectivity conditions used in Theorem 1 are the minimal ones necessary in graphs with the corresponding hamiltonian properties.

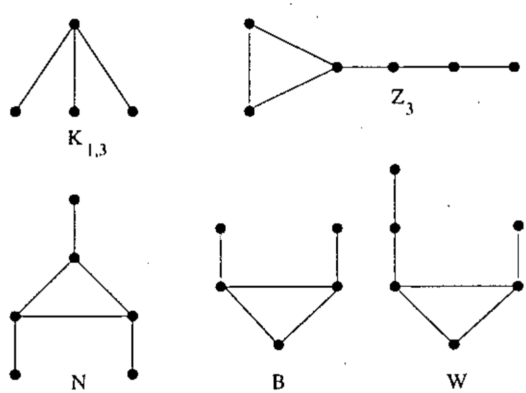


Figure 1: Some common forbidden graphs.

If P is a hamiltonian property (like traceable, hamiltonian, pancyclic, etc.), let $k(P)$ denote the least connectivity possible in a graph with property P. Thus, for example if P is traceability, then $k(P) = 1$; while if P is hamiltonicity, then $k(P) = 2$.

In this section we wish to consider the following problems:

Problem 1 *For fixed $k \geq 1$ and hamiltonian property P, determine all families F consisting of k distinct connected graphs with the property that any $k(P)$-connected F-free graph will possess property P.*

This idea was introduced by Bedrossian [5] who considered it for pairs of connected graphs and the properties of being hamiltonian and pancyclic. However, in proving which graphs must be forbidden, he considered small order graphs in his proofs. In [11] his results were reexamined with the added restriction that only infinite families of graphs would suffice to rule out a particular subgraph. In doing so, Bedrossian's results were extended for graphs of sufficiently large orders.

We concentrate here on one and two graph forbidden families. The single forbidden subgraph problem turns out to be a very restrictive situation and easy to solve. The case of forbidden triples has also been studied for several different hamiltonian properties. However, we shall not consider that problem here. The question for triples, as you might expect, is considerably more involved.

The single forbidden graph problem is trivial. Suppose G is connected, has order $n \geq 3$ and is P_3-free, then G is easily seen to be a complete graph K_n. But if G is complete, then G has every hamiltonian property. Thus, forbidding P_3 alone implies each hamiltonian property P and thus any other graph could be paired with P_3 to obtain the same result. In fact, in [11] it is shown that P_3 is the only single graph that solves our problem (except for induced subgraphs of P_3) and thus we will remove it (and its induced subgraphs) from consideration in forbidden pairs.

We now consider the problem of all forbidden pairs that imply a 2-connected graph is hamiltonian. In order to solve this problem we will need several results from the literature as well as the example graphs of Figure 2, each of which is 2-connected and nonhamiltonian. The positive results were provided for the families $\{K_{1,3}, P_6\}$ in [6], $\{K_{1,3}, Z_2\}$ in [16] and $\{K_{1,3}, W\}$ in [5].

A characterization of all pairs that imply a 2-connected graph is hamiltonian was accomplished in [5]. However, as mentioned earlier, graphs of small order were used in the proof to eliminate certain graphs, namely Z_3. However, recently $\{K_{1,3}, Z_3\}$ (see [13]) was shown to be a viable forbidden family for graphs of order $n \geq 10$. The following result is from [11].

Theorem 2 *Let R and S be connected graphs ($R, S \neq P_3$) and G a 2-connected graph of order $n \geq 10$. Then G is (R, S)-free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is one of the graphs C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N or W.*

Note that the condition that $n \geq 10$ in the last result is only needed for the pair $\{K_{1,3}, Z_3\}$. We now summarize the results of [11] in the following table. Here, Y (Yes) indicates that graph can be paired with the claw to give the corresponding property P. Note that a graph G of order n is *pancyclic* if it contains cycles of all possible lengths from 3 to n, *panconnected* if any two vertices u amd v are joined by paths of all possible lengths from the distance $d(u,v)$ up to n, and finally, G is *cycle extendable* (see [17]) if any nonhamiltonian cycle of G can be extended to a cycle containing exactly one more vertex, that is, C is extended to C' where $V(C') = V(C) \cup \{x\}$ for some x in V. Note, the 2 cases below when $n \geq 10$ represent extensions of Bedrossian's work.

P/Graph	C_3	P_4	P_5	P_6	Z_1	Z_2	Z_3	B	N	W
Traceable	Y	Y			Y			Y	Y	
Hamilton	Y	Y	Y	Y	Y	Y	Y $n \geq 10$	Y	Y	Y
Pancyclic		Y	Y	Y $n \geq 10$ [14]	Y	Y				
Panconnected					Y					
Cycle Extendable	Y	Y			Y	Y				

A very big hole remains, namely for the property of being hamiltonian connected, that is, having the property that each pair of vertices can be joined by a spanning path.

Problem 2 *What pairs of connected graphs (R,S) $(R,S \neq P_3)$ imply that a 3-connected (R,S)-free graph G is hamiltonian connected?*

Unfortunately we have few answers to problem 2. We do know that one of the graphs must again be $K_{1,3}$ (see [11]). However, the only pairs for which we have positive results are $\{K_{1,3}, N\}$ [22] and very recently, $\{K_{1,3}, Z_2\}$ [11]. It would be helpful to determine which of P_6, Z_3, and W, if any, can also be paired with $K_{1,3}$.

3 The Structure of 2-Factors

Recall that a 2-factor of G is a 2-regular spanning subgraph of G. Thus, a 2-factor is clearly the union of cycles. Hence, in finding hamiltonian cycles, we are actually finding 2-factors. In one sense these are the simplest 2-factors as they consist of only one cycle. In another sense, this may make them the most difficult 2-factors to actually find. Can we find 2-factors consisting of two disjoint cycles that together span $V(G)$? Or for that matter, for some specified integer k, can we find k disjoint cycles that form a 2-factor of G? Can we ask for even more?

Problem 3 *Under what conditions can we completely control the structure of a 2-factor? More specifically, when can we specify the number and lengths of the cycles composing the 2-factor?*

Erdös (see [2]) asked a similar, but more particular question.

Problem 4 *Let H be a graph of order $4k$ with $\delta(H) \geq 2k$, then does H contains k vertex disjoint 4-cycles?*

The following result (see [3]) has proven useful in our study of 2-factors, as it supplies a condition for a graph to contain k disjoint cycles (not necessarily spanning $V(G)$).

Theorem 3 *If* $|V(G)| \geq 3k$ *and* $\delta(G) \geq 2k$, *then* G *contains* k *disjoint cycles.*

This result can be deduced from a result of Corradi and Hajnal [7] which provides the special case when $|V(G)| = 3k$.

Aigner and Brandt [2] were able to find a very powerful condition answering our original question on 2-factors.

Theorem 4 *Let* H *be a graph of order* n *with* $\delta(H) \geq (2n-1)/3$, *then* H *contains any graph* G *of order at most* n *with* $\Delta(G) = 2$.

In particular then, given $n_1, n_2, \ldots, n_k$ where $n_i \geq 3$ and $\sum_{i=1}^{k} n_i = n$, the graph H will contain a 2-factor $F = C_1 \cup C_2 \cup \ldots \cup C_k$, where $|V(C_i)| = n_i$, $i = 1, 2, \ldots, k$. That is, we can find a 2-factor of every possible form in G.

The Aigner and Brandt result imposes a very strong condition on the graph, however, that is to be expected when we are asking for so much in return. Now suppose we ask for a little less control on the structure of the 2-factor.

Problem 5 *Under what conditions can we control the number of cycles in a 2-factor?*

In terms of degree conditions, this problem has again been answered.

Theorem 5 *[4] If* G *is a graph of order* n *with*
(1) $\delta(G) \geq n/2$ *and* $n \geq 4k$ *or*
(2) $\sigma_2(G) \geq n$ *and* $n \geq 5k$
then G *has a 2-factor with exactly* k *cycles.*

It is now natural to modify our last problem as follows.

Problem 6 *What conditions allow us to reduce or eliminate the degree conditions of Theorem 5 ?*

One direction that has been explored is the use of forbidden subgraphs. In his Ph.D. thesis, Acree [1] showed that the Corradi-Hajnal condition in 2-connected $K_{1,3}$-free graphs implies the graph contains a 2-factor with k disjoint cycles, in fact, he showed that any collection of k disjoint cycles could be extended to a 2-factor of the graph.

Finally, one other direction should be mentioned. El-Zahar [10] gave the following conjecture.

Conjecture 1 *Let* H *be a graph of order* $n = n_1 + \ldots + n_k$ *with* $\delta(H) \geq \sum_{i=1}^{k} \lceil \frac{n_i}{2} \rceil$, *then* $\cup_{i=1}^{k} C_{n_i} \subset H$.

Note that for $k = 1$, this is the well-known Theorem of Dirac [8] on hamiltonian cycles; while for $n_i = 3$ $(i = 1, 2, \ldots, k)$ this is again a result of Corradi-Hajnal [7]. El-Zahar [10] proved the case $k = 2$. The graph $G = K_{s-1} + K_{\lfloor n-s+1/2 \rfloor, \lceil n-s+1/2 \rceil}$ shows that the conjecture is best possible.

4 k-Linked Graphs

We say that a graph G is *k-linked* if it has at least $2k$ vertices and, for any ordered set

$$\{v_1, v_2, \ldots, v_k, w_1, \ldots, w_k\}$$

of $2k$ distinct vertices, G has k disjoint paths $P_1, \ldots, P_k$ such that P_i connects v_i to w_i, $i = 1, \ldots, k$.

Clearly, a necessary condition for G to be k-linked is that G be $(2k-1)$-connected. Jung [18] and Larman and Mani [19] independently showed the following sufficient condition.

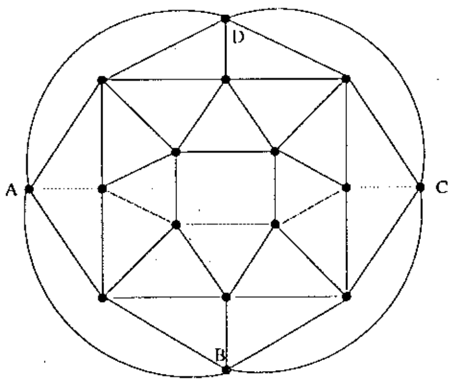


Figure 2: A 5-connected planar graph not 2-linked.

Theorem 6 *If G is a $2k$-connected graph which contains a subdivision of K_{3k}, then G is k-linked.*

The proof used a result of Mader [20] that says if $\delta(G) \geq 2^{\binom{k}{2}}$, then G contains a subdivision of K_k. Szemeredi (see [24]) has dramatically lowered this lower bound. However, the problem that has drawn the most attention to date is the following:

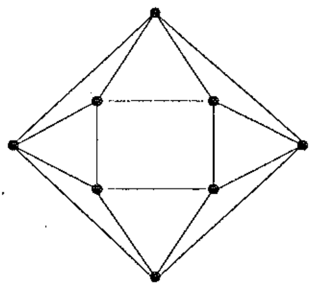
Problem 7 *What is the smallest connectivity necessary to ensure that a graph is k-linked, $k \geq 1$?*

Let $f(k)$ denote the smallest connectivity necessary to ensure that a graph is k-linked. Clearly, $f(1) = 1$ (just the definition of connected). Jung [18] proved that $f(2) = 6$. To see that 5-connectivity is not enough, consider the graph of Figure 2 with $\{A, B, C, D\}$ as the ordered set of vertices. Clearly, any path from A to B in this graph will block all B to D paths. For $k \geq 3$, $f(k)$ is still unknown.

Seymour [21] and Thomassen [23] independently characterized graphs that are not 2-linked. Their characterization is somewhat technical and will not be explored here. Many other questions still remain.

Problem 8 *Determine conditions, other than connectivity, that imply a graph is k-linked.*

Problem 9 *What conditions will reduce the connectivity needed to show a graph is k-linked?*



X

Figure 3: The graph X.

At least some work (see [12]) has been done in this direction.

Theorem 7

1. *If G is a 5-connected $K_{1,3}$-free graph of order $n \geq 4$, then G is 2-linked.*
2. *If G is a 4-connected $\{K_{1,3}, B\}$-free graph, then G is 2-linked or G contains an induced X (see Figure 3).*

Generalizations of this to higher values of k are also possible. The following is also from [12].

Theorem 8 *If G is $K_{1,t}$-free and has connectivity $(2t-2)(k-1)+1$, then G is k-linked.*

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