Graphs and Combinatorics (1995) 11: 221–231

Graphs and Combinatorics

Hamiltonicity in Balanced k-Partite Graphs

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Abstract. One of the earliest results about hamiltonian graphs was given by Dirac. He showed that if a graph G has order p and minimum degree at least $\frac{p}{2}$ then G is hamiltonian. Moon and Moser showed that a balanced bipartite graph (the two partite sets have the same order) G has order p and minimum degree more than $\frac{p}{4}$ then G is hamiltonian. In this paper, their idea is generalized to k-partite graphs and the following result is obtained: Let G be a balanced k-partite graph with order $p = kn$. If the minimum degree

$$\delta(G) > \begin{cases} \left(\frac{k}{2} - \frac{1}{k+1}\right)n & \text{if } k \text{ is odd} \\ \left(\frac{k}{2} - \frac{2}{k+2}\right)n & \text{if } k \text{ is even} \end{cases},$$

then G is hamiltonian. The result is best possible.

1. Introduction

One of the earliest results in the theory of hamiltonian graphs is due to Dirac [1].

Theorem 1 (Dirac). *If G is a graph with $p \geq 3$ vertices having minimum degree $\delta(G) \geq \frac{p}{2}$, then G is hamiltonian.*

* Research Supported by N.S.A. grant # MDA 904-94-H-2060 and NSF ND-EPSCoR # 4963.
† Research Supported by O.N.R. grant # N00014-J-91-1085 and NSA grant # MDA 904-90-H-4034.
‡ Research Supported by O.N.R. grant # N00014-J-91-1085.
§ Research Supported by O.N.R. grant # N00014-J-91-1098.
¶ Research Supported by O.N.R. grant # N00014-J-93-1-0050.

In 1963, this result was generalized for the special case of bipartite graphs in which each partite set has the same number of vertices, that is, balanced bipartite graphs. Moon and Moser [2] proved:

Theorem 2 (Moon and Moser). *If G is a balanced bipartite graph with $2n$ vertices having minimum degree $\delta(G) > \frac{n}{2}$, then G is hamiltonian.*

Note that the requirement that G be balanced is necessary, and that the minimum degree was lowered to one fourth of the order of the graph. It is the purpose of this paper to generalize this result to balanced k-partite graphs. Although requiring that the graphs be balanced is not necessary for $k \geq 3$, it is crucial to our present proof techniques.

Theorem 3. *Let G be a balanced k-partite graph of order kn. If the minimum degree*

$$\delta(G) > \begin{cases} \left(\frac{k}{2} - \frac{1}{k+1}\right)n & \text{if } k \text{ is odd} \\ \left(\frac{k}{2} - \frac{2}{k+2}\right)n & \text{if } k \text{ is even} \end{cases},$$

then G is hamiltonian.

The proof of this theorem is placed in Section 3. Theorem 3 includes Dirac's Theorem when $n = 1$ (that is, $kn = k$) and gives an improvement of Dirac's result when $n > 1$. This comes at the expense of assuming that G is a balanced k-partite graph. The following two examples show that Theorem 3 is in some sense the best possible.

Example 1. Suppose k is odd. Let $k = 2t + 1$, and $G(n, 2t + 1)$ have partite sets $X_1, X_2, \ldots, X_{2t+1}$ with $|X_i| = n$ for each $i = 1, 2, \ldots, 2t + 1$. The vertices of $X_1, \ldots, X_t$ are adjacent to each vertex in each of the other partite sets. For each $i = t + 1, t + 2, \ldots, 2t + 1$, distinguish a subset of $\left\lfloor \frac{n-1}{2t+2} \right\rfloor$ vertices from each X_i and call it Y_i. Join all the vertices in Y_i to each vertex in each of the other partite sets. There are only two different degrees; there are vertices of degree $2tn$, and vertices of degree $nt + \left\lfloor \frac{n-1}{2t+2} \right\rfloor t$.

Thus $G(n, 2t + 1)$ has minimum degree $\left(n + \left\lfloor \frac{n-1}{2t+2} \right\rfloor\right)t$. Let

$$X = X_1 \cup X_2 \cup \cdots X_t \cup Y_{t+1} \cup \cdots \cup Y_{2t+1}.$$

Note that the vertex set of $G(n, 2t + 1) - X$ is an independent set since $|X| < \frac{(2t+1)n}{2}$ it follows that $G(n, 2t + 1)$ is not hamiltonian. This example shows that we need minimum degree greater than $\left(1 + \frac{1}{2t+2}\right)tn$ in Theorem 3 when $k = 2t + 1$.

Example 2. Suppose that k is even. Let $k = 2t$ and $G(n, 2t)$ have partite sets X_1, $X_2, \ldots, X_{2t}$, with $|X_i| = n$ for each $i = 1, 2, \ldots, 2t$. The vertices of $X_1, \ldots, X_{t-1}$ are adjacent to each vertex in each of the other partite sets. For each $i = t, t+1, \ldots,$ $2t$ distinguish a subset of $\left\lfloor \frac{n-1}{t+1} \right\rfloor$ vertices of X_i, and call it Y_i. Join all the vertices in Y_i to each vertex in each of the other partite sets. As in Example 1, there are only two different degrees, $(2t-1)n$ and $n(t-1) + \left\lfloor \frac{n-1}{t+1} \right\rfloor t$. Hence, $G(n, 2t)$ has minimum degree $n(t-1) + \left\lfloor \frac{n-1}{t+1} \right\rfloor t$ and is not hamiltonian since the vertex set of the graph $G(n, 2t) - X$, where

$$X = X_1 \cup X_2 \cup \cdots \cup X_{t-1} \cup Y_t \cup \cdots \cup Y_{2t},$$

is independent and $|X| < tn$. This example shows that we need minimum degree greater than $\left(t - \frac{1}{t+1}\right)n$ in Theorem 3 for $k = 2t$.

Following Moon and Moser's proof of Theorem 2, we obtain the following result which generalizes Theorem 2 from bipartite graphs to k-partite graphs.

Theorem 4. *Let G be a balanced k-partite graph of order kn. If for all $xy \notin E(G)$, $x \in X_i$, $y \notin X_i$*

$$d(x) + |N(y) \cap X_i| > (k-1)n,$$

then G is hamiltonian.

The proof of the above theorem is similar to the proof of Theorem 2 given in [2]. For this reason we leave it to the reader.

2. A Basic Lemma

We assume that all cycles and paths are given with a fixed orientation. Let C be a cycle of G and $W \subseteq V(C)$. We use W^+ (W^-) to denote the set of successors (predecessors) of vertices of W in C. When $W = \{w\}$, we use w^+ (w^-) for $\{w\}^+$ ($\{w\}^-$). For each pair of distinct vertices u and $v \in C$, let $C[u, v]$ denote the subpath of C from u to v along the orientation of C and $C^-[u, v]$ be the subpath of C from u to v along the reverse direction of C. We make a similar notation when C is replaced by a path P. Let A be a vertex subset of a graph G. We define $N(A) = \bigcup_{a \in A} N(a)$ and $\bar{A} = V(G) - A$, where $N(a)$ is the usual neighborhood of the vertex a.

Let G be a graph of order p and $\chi(G) = k$, with color classes $X_1, X_2, \ldots, X_k$, and let G be "maximally" non-hamiltonian (the addition of any edge between color classes would result in a hamiltonian cycle). Furthermore, assume for all $xy \notin E(G)$, $x \in X_i$ that

$$d(x) + d(y) + |X_i| > \begin{cases} p & \text{if } y \notin X_i \\ p+1 & \text{if } y \in X_i \end{cases}. \qquad (*)$$

Lemma 1 (Basic Lemma). *If G is as described above and $x_i \in X_i$ $(i = 1, 2, \ldots, k)$ then either*

1. $x_i x_j \in E(G)$ *for all* $x_j \in X_j$, $j \neq i$, *or*
2. *there is a* $(p-1)$ *cycle* C *and a vertex* $z \in X_i$ *not on* C, *so that* x_i *is a successor along* C *of a neighbor of* z.

Proof. Let $x_i \in X_i$ and assume condition 1 is not true. Let $x_j \in X_j$, $j \neq i$, so that $x_i x_j \notin E(G)$. By the maximality of G there is a hamiltonian path P from x_i to x_j. Since G is non-hamiltonian, for every adjacency of x_j on P, x_i is not adjacent to the successor of that neighbor of x_j. If all successors of adjacencies of x_j are from color classes other than X_i, then it would follow that x_i would have at least $d(x_j) + |X_i|$ non-adjacencies, which contradicts (*). Thus, for some neighbor y of x_j, the successor $y_i \in X_i$. Subsequently, by following P from x_i to y, the edge to x_j, down P to y_i, there is a hamiltonian path,

$$Q[x_i, y_i] = P[x_i, y]yx_jP^-[x_j, y_i],$$

from x_i to y_i with x_i, $y_i \in X_i$. Again, for every adjacency z of y_i, x_i is not adjacent to the successor of z along path Q. By (*), some of these successors must be from X_i. For each such successor $z_i \neq y_i$ along the path Q, x_i is also not adjacent to its successor z_i^+; so, since $z_i^+ \notin X_i$, (*) would be contradicted since x_i would have at least $d(y_i) + |X_i| - 1$ non-adjacencies. Thus x_i is adjacent to some z_i^+ and

$$x_i \cdots z y_i \cdots z_i^+ x_i$$

is a $(p-1)$ cycle with $z_i \in X_i$ not on the cycle, and x_i is the successor of a neighbor of z_i, namely z_i^+. Thus condition 2 is satisfied when condition 1 does not hold and the Lemma follows. □

3. Proof of Theorem 3

When $k = 2$, it is Moon and Moser's theorem. We assume that $k \geq 3$.

Suppose the theorem is not true; choose an edge maximal counterexample G (the addition of any edge between partite sets of G would result in a hamiltonian cycle) such that $\delta(G) > \left(\frac{k}{2} - \beta\right)n$, where

$$\beta = \begin{cases} 1/(k+1) & k \text{ is odd} \\ 2/(k+2) & k \text{ is even} \end{cases}.$$

Let X_1, X_2, …, X_k denote the partite sets with $|X_i| = n$ for $i = 1, 2, \ldots, k$. It is readily seen that $(1 - 2\beta) \geq \beta$ if $k \geq 3$. If $(1 - 2\beta)n < 1$, the minimum degree $\delta(G) \geq \frac{kn}{2}$; hence, G is hamiltonian by Theorem 1, a contradiction. Thus, $(1 - 2\beta)n \geq 1$, and so the condition (*) is satisfied. Since G is edge maximal and the minimum degree conditions satisfy (*) from Lemma 1, either G is a complete k-partite graph in which case it is hamiltonian, or, without loss of generality, assume that there is a vertex $x \in X_1$ such that $N(x) \neq X_2 \cup X_3 \cdots \cup X_k$. By Lemma 1, there is a cycle C of length $p-1$ and a vertex $z \in X_1$ not on C such that x is a successor along C of some vertex in $N(z)$. Let S be the successor vertex set of $N(z)$ and R be the predecessor set of $N(z)$ on C. Clearly, both S and R are independent vertex sets since C is a longest cycle of G. For each $i = 1, 2, \ldots, k$, set

$$S_i = S \cap X_i \quad \text{and} \quad R_i = R \cap X_i.$$

Without loss of generality, assume that

$$S_i \neq \varnothing \quad \text{for each } i = 1, 2, \ldots, l, \text{ and,}$$
$$|S_{l+1}| = |S_{l+2}| = \cdots = |S_k| = 0.$$

Claim 1.

$$\frac{k}{2} \leq l \leq \begin{cases} \dfrac{k+1}{2} & k \text{ is odd} \\[2mm] \dfrac{k+4}{2} & k \text{ is even} \end{cases}.$$

Proof. Since in any case we have $\beta \leq \frac{1}{2}$, then $l \geq \frac{k}{2}$.

Without loss of generality, assume that $|S_l| = \min\{|S_i|: 1 \leq i \leq l\}$. Then $d(y) < kn - n - \left(\frac{l-1}{l}\right)|S|$ for each $y \in S_l$. Since $\delta(G) \geq \left(\frac{k}{2} - \beta\right)n$, we have

$$\left(\frac{k}{2} - \beta\right)n < kn - n - \left(\frac{l-1}{l}\right)\left(\frac{k}{2} - \beta\right).$$

Solving the above inequality for l, we obtain

$$l < \frac{k - 2\beta}{2(1-2\beta)}.$$

Replacing β in terms of k, we obtain the upper bound on l. □

Claim 2. *For each $y \in S$,*

$$|\overline{N(y)} - S| < \begin{cases} \dfrac{2}{k+1}n & k \text{ is odd} \\[2mm] \dfrac{4}{k+2}n & k \text{ is even} \end{cases}.$$

The above implies that

$$\left|\overline{N(y)} \cap \left(\bigcup_{i=l+1}^{k} X_i\right)\right| < \begin{cases} \dfrac{2}{k+1}n & k \text{ is odd} \\[2mm] \dfrac{4}{k+2}n & k \text{ is even} \end{cases},$$

and for each $i = 1, 2, \ldots, l$,

$$|S_i| > \begin{cases} \dfrac{k-1}{k+1}n & k \text{ is odd} \\ \dfrac{k-2}{k+2}n & k \text{ is even} \end{cases}.$$

Proof. Since C is a longest cycle of G, the vertex subsets $N(y)$, S, and $\overline{N(y)} - S$ are pairwise disjoint. Thus $kn \geq |N(y)| + |S| + |\overline{N(y)} - S|$. So the first inequality holds. The last two inequalities follow directly from the first one. □

Claim 3. *For each* $i = 1, 2, \ldots, l$,

$$|N(z) \cap X_i| < 2\beta n = \begin{cases} \dfrac{2}{k+1}n & k \text{ is odd} \\ \dfrac{4}{k+2}n & k \text{ is even} \end{cases}.$$

Further, if $S_i \cap R \neq \varnothing$, *then*

$$|N(z) \cap X_i| < \beta n = \begin{cases} \dfrac{1}{k+1}n & k \text{ is odd} \\ \dfrac{2}{k+2}n & k \text{ is even} \end{cases}.$$

Proof. Let $y \in S_i$ with $1 \leq i \leq l$. Since C is a longest cycle of G, S is an independent set and

$$\begin{aligned} |V(C)| \geq |S| + |N(z) \cup N(y)| &= d(z) + |N(y)| + |N(z) - N(y)| \\ &\geq d(z) + d(y) + |N(z) \cap X_i|, \end{aligned}$$

so the first result holds.

To prove the second inequality, let $y \in S_i \cap R$; then y is a successor along C of some vertex in $N(z)$, and a predecessor along C of some vertex in $N(z)$ as well. Since C is a longest cycle of G, neither $N(z)$ nor $N(y)$ contains two consecutive vertices along C, and $N(y) \cap ((N(z))^+ \cup (N(z))^-) = \varnothing$. Thus,

$$|V(C)| \geq 2|N(z) \cup N(y)| = 2(d(y) + |N(z) - N(y)|) \geq 2(d(y) + |N(z) \cap X_i|).$$

Since $d(y) > \left(\frac{k}{2} - \beta\right)n$, the second result holds. □

We will break the remainder of the proof into three cases according to whether k is odd, or k is even and $k \geq 6$, or $k = 4$.

3.1 k is odd

Clearly, $k \geq 3$. By Claim 1, we have $l = \dfrac{k+1}{2}$. Since $N(z)$ contains no two consecutive vertices along the cycle C,

$$|S-R| \le |V(C)| - 2d(z) < kn - 2\left(\frac{k}{2} - \frac{1}{k+1}\right)n = \frac{2}{k+1}n.$$

For each $1 \le i \le \frac{k+1}{2}$, $|S_i| > \frac{k-1}{k+1}n \ge \frac{2}{k+1}n$ by Claim 2, and so $S_i \cap R \ne \varnothing$. By Claim 3, $|N(z) \cap X_i| < \frac{1}{k+1}n$ for each $1 \le i \le \frac{k+1}{2}$. Since $z \in X_1$, we have that

$$d(z) = \sum_{i=2}^{k} |N(z) \cap X_i| < \left(\frac{k-1}{2}\right)\left(\frac{1}{k+1}n\right) + \left(\frac{k-1}{2}\right)n = \left(\frac{k}{2} - \frac{1}{k+1}\right)n,$$

a contradiction. □

3.2 k is even and $k \ge 6$

Claim 4. $l = \frac{k}{2}$.

Proof. Again, since C is a longest cycle of G, $|S-R| \le |V(C)| - 2d(z) < kn - 2\left(\frac{k}{2} - \frac{2}{k+2}\right)n = \frac{4}{k+2}n$. For each $1 \le i \le l$, $|S_i| > \frac{k-2}{k+2}n \ge \frac{4}{k+2}n$ by Claim 2; hence $S_i \cap R \ne \varnothing$. By Claim 3, $|N(z) \cap X_i| < \frac{2}{k+2}n$. Since $z \in X_1$,

$$\left(\frac{k}{2} - \frac{2}{k+2}\right)n < d(z) < (l-1)\left(\frac{2}{k+2}n\right) + (k-l)n.$$

Solving the above inequality, we have $l < 1 + \frac{k}{2}$. Since l is an integer, then $l = \frac{k}{2}$ by Claim 1. □

Let $A = \bigcup_{i=1}^{k/2} X_i$ and $B = \bigcup_{i=(k/2)+1}^{k} X_i$.

Claim 5. *For every $x^* \in A$,*

$$|N(x^*) \cap B| > \left(\frac{k}{2} - \frac{4}{k+2}\right)n \ge \frac{kn}{4}.$$

Proof. By Claim 2, the result is true for each $x^* \in S$. We only need to show that $|\overline{N(x^*)} \cap B| < \frac{4}{k+2}n$ for each $x^* \in A - S$. Suppose that, to the contrary, there is a vertex $x^* \in X_{i_0}$ such that $|\overline{N(x^*)} \cap B| \ge \frac{4}{k+2}n$. By Lemma 1, there is a cycle C^* of length $kn - 1$ and a vertex $z^* \in X_{i_0}$ not on C^* such that x^* is a successor along C^* of some vertex of $N(z^*)$. Let S^* be the set of successor vertices of $N(z^*)$ on C^* and $S_i^* = S^* \cap X_i$ for each $i = 1, 2, \ldots, k$. Assume that

$$S^* = S_{i_1}^* \cup S_{i_2}^* \cup \cdots \cup S_{i_{l^*}}^*,$$

with $i_1 = i_0$ abd $S^*_{i_j} \neq \varnothing$ for $j = 1, 2, \ldots, l^*$. By Claim 4, $l^* = \frac{k}{2}$ and, by Claims 2, $|S^*_{i_j}| \geq \frac{(k-2)n}{k+2}$ for $j = 1, 2, \ldots, l^*$. Suppose that there is some $i_j > \frac{k}{2}$. Since $|S_{i_0}| > \frac{k-2}{k+2}n$ and $|S^*_{i_0}| > \frac{k-2}{k+2}n$, by the pigeonhole principle there is a vertex $y \in S_{i_0} \cap S^*_{i_0}$. Then

$$|\overline{N(y)} \cap B| \geq |S^*_{i_j}| > \frac{k-2}{k+2}n.$$

On the other hand, by Claim 2,

$$|\overline{N(y)} \cap B| < \frac{4}{k+2}n,$$

a contradiction. Thus, $\{i_1, i_2, \ldots, i_{k/2}\} = \left\{1, 2, \ldots, \frac{k}{2}\right\}$. By Claim 2, $|\overline{N(x^*)} \cap B| < \frac{4}{k+2}n$, a contradiction. □

Arguing symmetrically, for each vertex $x \in B$, $|N(x) \cap A| > \frac{kn}{4}$. Now consider the spanning bipartite subgraph of G with partite sets A and B having minimum degree more than $\frac{kn}{4}$. By Moon and Moser's theorem it is hamiltonian, hence so is G, completing the proof of this case. □

3.3 $k = 4$

In this case we may have $l = 2$, 3, or 4, and $\delta(G) > \frac{5n}{3}$.

Claim 6. *We have $|S \cap R| > n + 1$, which implies that there exist two distinct integers $1 \leq i_1 \neq i_2 \leq l$ such that $S_{i_1} \cap R \neq \varnothing$ and $S_{i_2} \cap R \neq \varnothing$.*

Proof. Since C is a longest cycle of G,

$$|S \cap R| > d(z) - (|V(C)| - 2d(z)) = 3d(z) - |V(C)| > n + 1.$$

The remaining part of the claim follows directly from the pigeonhole principle. □

Claim 7. $l \leq 3$.

Proof. Assume to the contrary that $l = 4$. By Claim 6, without loss of generality, assume that $S_2 \cap R \neq \varnothing$. Then, by Claim 3, $|N(z) \cap X_2| < \frac{n}{3}$ and $|N(x) \cap X_i| \leq \frac{2n}{3}$,

$i = 3, 4$. Therefore,

$$\frac{5n}{3} < |N(z)| = \sum_{i=2}^{4} |N(z) \cap X_i| < \left(\frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right)n = \frac{5n}{3}.$$

a contradiction. □

Claim 8. *If $l = 2$, that is, $S = S_1 \cup S_2$, we have $|S_1| > \frac{2n}{3}$ and $|S_2| > \frac{2n}{3}$. Further, $|\overline{N(y)} \cap (X_3 \cup X_4)| < \frac{2n}{3}$ for each $y \in S$.*

Proof. The first two inequalities follow directly from the fact that $|S_1| + |S_2| = |S| > \frac{5n}{3}$, $|S_1| \le n$, and $|S_2| \le n$. The third follows directly from Claim 2. □

Claim 9. *If $l = 3$, either $S_2 \cap R = \varnothing$ or $S_3 \cap R = \varnothing$.*

Proof. Assume that, to the contrary, both $S_2 \cap R \ne \varnothing$ and $S_3 \cap R \ne \varnothing$. By Claim 3, $|N(z) \cap X_2| < \frac{n}{3}$ and $|N(z) \cap X_3| < \frac{n}{3}$. Thus

$$\frac{5n}{3} < d(z) = \sum_{i=2}^{4} |N(z) \cap X_i| < \left(\frac{1}{3} + \frac{1}{3} + 1\right)n = \frac{5n}{3},$$

a contradiction. □

Without loss of generality, we will now assume that if $l = 3$ then $S_2 \cap R = \varnothing$ and $S_3 \cap R = \varnothing$. Note that S_3 may be empty.

Claim 10. *If $l = 3$, then $R = R_1 \cup R_2$, $|R_1| > \frac{2n}{3}$, $|R_2| > \frac{2n}{3}$, and $|\overline{N(y)} \cap B| < \frac{2n}{3}$ for each $y \in R$.*

Proof. Clearly, there is a symmetrical froms of Claims 3 and 8 for R_1 and R_2. If $R_3 \ne \varnothing$ and $R_4 \ne \varnothing$, then $|N(z) \cap X_i| < \frac{2n}{3}$ for $i = 3, 4$ by Claim 3. Thus,

$$\frac{5n}{3} < |N(z)| = \sum_{i=2}^{4} |N(z) \cap X_i| < \left(\frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right)n = \frac{5n}{3},$$

a contradiction. □

Suppose that $R_3 \ne \varnothing$ and $R_4 = \varnothing$. Note that $S_3^+ \cap R_3 = \varnothing$. Thus, we have that

$$4n - 1 = |V(C)| \ge |N(z)| + |S| + |S_3^+| + |R_3| > \left(\frac{5}{3} + \frac{5}{3} + \frac{1}{3} + \frac{1}{3}\right)n = 4n,$$

a contradiction. □

Note that, by Claims 8 and 10, respectively when $l = 2$ or $l = 3$, we have that $|S_1| > \frac{2n}{3}$ or $|R_1| > \frac{2n}{3}$.

Claim 11. *For each* $y \in X_1 \cup X_2$, $|N(y) \cap (X_3 \cup X_4)| > n$.

Proof. If $y \in S_1$, then $N(y) \cap S_2 = \varnothing$. Hence $|N(y) \cap X_2| \leq |X_2 - S_2| < \frac{2n}{3}$. Thus,

$$|N(y) \cap (X_3 \cup X_4)| > \frac{5n}{3} - \frac{2n}{3} = n.$$

Similarly, the result holds for $y \in S_2$. Without loss of generality, we assume to the contrary that there is an $x^* \in X_1 - S$ such that $|N(x^*) \cap (X_3 \cup X_4)| \leq n$. By Lemma 1, there is a cycle C^* of length $4n - 1$ and a vertex $z^* \in X_1$ not on C^* such that x^* is a successor along C^* of some vertex in $N(z^*)$. Let S^* be the set of successor vertices of $N(z^*)$ on C^*. Let $S_i^* = S^* \cap X_i$. Clearly $S_1^* \neq \varnothing$, and so $|S_1^*| > \frac{n}{3}$.

If $S_2^* = \varnothing$, then $|S_3^* \cup S_4^*| > \frac{2n}{3}$. Note that $|S_1| > \frac{2n}{3}$ $\left(\text{or } |R_1| > \frac{2n}{3}\right)$ and $|S_1^*| > \frac{2n}{3}$. By the pigeonhole principle, there is a vertex $s_1 \in S_1 \cap S_1^*$ (or $s_1 \in R_1 \cap S_1^*$). Thus,

$$\frac{2n}{3} > |\overline{N(s_1)} \cap (X_3 \cup X_4)| \geq |S_3^* \cup S_4^*| > \frac{2n}{3},$$

a contradiction.

Thus, $S_2^* \neq \varnothing$. By Claim 2, $|S_2^*| > \frac{n}{3}$; hence $|N(x^*) \cap X_2| < \frac{2n}{3}$, which implies that

$$|N(x^*) \cap (X_3 \cup X_4)| > \frac{5n}{3} - \frac{2n}{3} = n,$$

a contradiction. □

Claim 12. *For each* $x^* \in X_3 \cup X_4$, $|N(x^*) \cap (X_1 \cup X_2)| > n$.

Proof. To the contrary, without loss of generality, assume that there is a vertex $x^* \in X_3$ such that $|N(x^*) \cap (X_1 \cup X_2)| \leq n$. By Lemma 1, there is a cycle C^* of length $4n - 1$ and a vertex $z^* \in X_3$ not on C such that x^* is a successor along C^* of some vertex of $N(z^*)$. Let S^* be the set of successor vertices of $N(z^*)$ on C^*. Let $S_i^* = S^* \cap X_i$ for $i = 1, 2, 3, 4$. Let R^* be the set of predecessors of $N(z^*)$ on the cycle C^*. Let $R_i^* = R^* \cap X_i$ for each $i = 1, 2, 3, 4$. Clearly $S_3^* \neq \varnothing$. In the same manner as before, we have either $|S_3^*| > \frac{2n}{3}$ and $|R_3^*| > \frac{2n}{3}$.

If $|S_3^*| > \frac{2n}{3}$, we claim that both $S_1^* = \varnothing$ and $S_2^* = \varnothing$, and so $|N(x^*) \cap (X_1 \cup X_2)| > n$ by Claim 11, a contradiction. Otherwise, without loss of generality, assume that $S_1^* \neq \varnothing$. Then $|S_1^*| > \frac{n}{3}$. If $l = 2$, then $|S_1| > \frac{2n}{3}$ by Claim 8. Using the pigeonhole principle, there is a vertex $s_1 \in S_1 \cap S_1^*$. Again by Claim 2, we have

$$\frac{2n}{3} > |\overline{N(s_1)} \cap (X_3 \cup X_4)| \geq |S_3^*| > \frac{2n}{3},$$

a contradiction. Thus $l = 3$ and by Claim 10, $R = R_1 \cup R_2$ and $|R_1| > \frac{2n}{3}$. By the pigeonhole principle, there is a vertex $s_1 \in S_1^* \cap R_1$. Thus

$$\frac{2n}{3} > |\overline{N(s_1)} \cap (X_3 \cup X_4)| \geq |S_3^*| > \frac{2n}{3},$$

a contradiction.

Here $|S_3^*| \leq \frac{2n}{3}$, and so $|R_3^*| > \frac{2n}{3}$. In particular, we have $S^* = S_3^* \cup S_{i_1}^* \cup S_{i_2}^*$ such that none of the sets are empty. By Claim 10, we assume that $R^* = R_3^* \cup R_{i_1}^*$. If $i_1 \leq 2$, there is a vertex $s_{i_1} \in R_{i_1}^* \cap R_{i_1}$ (or $s_{i_1} \in R_{i_1}^* \cap S_{i_1}$) by the pigeonhole principle. Thus

$$\frac{2n}{3} > |\overline{N(s_{i_1})} \cap (X_3 \cup X_4)| \geq |R_3^*| > \frac{2n}{3},$$

a contradiction. Hence $i_1 = 4$. In the same manner as Claim 11, we have $|N(x^*) \cap (X_1 \cup X_2)| > n$, a contradiction. □

Let $A = X_1 \cup X_2$ and $B = X_3 \cup X_4$. Now consider the spanning bipartite subgraph of G with partite sets $A = X_1 \cup X_2$ and $B = X_3 \cup X_4$. This is a balanced bipartite graph of order $4n$ with minimum degree more than n, and thus by Moon and Moser's theorem it is hamiltonian, and hence so is G, competing the proof of Theorem 3. □

Acknowledgement. The authors would like to thank Hunter S. Snevily for bringing the problem to our attention. The first author would like to thank R.J. Gould for introducing him to the problem and for his hospitality during a visit to Emory University during the preparation of this paper.

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Received: August 26, 1993
Revised: January 1, 1995