SPANNING CATERPILLARS WITH BOUNDED DIAMETER

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Abstract

A caterpillar is a tree with the property that the vertices of degree at least 2 induce a path. We show that for every graph \( G \) of order \( n \), either \( G \) or \( \overline{G} \) has a spanning caterpillar of diameter at most \( 2 \log n \). Furthermore, we show that if \( G \) is a graph of diameter 2 (diameter 3), then \( G \) contains a spanning caterpillar of diameter at most \( cn^{3/4} \) (at most \( n \)).

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1. Introduction

It is easy to show that for every graph $G$, either $G$ or the complement $\bar{G}$ is connected. Consequently, if $T_n$ denotes the family of all trees of order $n$, then for every graph $G$ of order $n$, either $G$ or $\bar{G}$ contains a member of $T_n$ (as a spanning subgraph). Such a family is called complete, that is, a family $\mathcal{F}_n$ of graphs of order $n$ is complete if for every graph $G$ of order $n$, either $G$ or $\bar{G}$ contains a member of $\mathcal{F}_n$. Thus, $T_n$ is complete and it is easy to show that the subfamily $T_n(4)$ of trees of order $n$ and diameter at most 4 is also complete. In Section 2, we will discuss other complete families of trees and show, in particular, that $C_n(2\log n)$ is complete, where $C_n(2\log n)$ is the family of caterpillars of order $n$ and diameter at most $2\log n$. In Section 3 we will investigate graphs of order $n$ and diameter at most 3 and show that if $G$ has diameter 2 (diameter 3), then $G$ contains a spanning caterpillar of diameter at most $cn^{3/4}$ (at most $n$).

2. Complete Families of Trees

We begin this section by proving a theorem from graph theory folklore. For vertices $x$ and $y$ of a graph $G$, $d_G(x, y)$ will denote the distance between $x$ and $y$ in $G$, i.e., the number of edges in a shortest path from $x$ to $y$. The diameter of $G$, denoted $diam(G)$, is the largest distance between pairs of vertices of $G$.

Theorem 1. Let $T_n(4)$ denote the family of trees of order $n$ and diameter at most 4. Then $T_n(4)$ is complete.

Proof. Without loss of generality, we may assume $n \geq 5$. Let $G$ be a graph of order $n$. If $diam(G) \leq 2$, then clearly $G$ contains a spanning tree with diameter at most 4. Thus we may assume that either $G$ is disconnected or $G$ has diameter at least 3. In either case, $G$ contains nonadjacent vertices $u$ and $v$ which have no common neighbors. Therefore, in $\bar{G}$, $u$ and $v$ are adjacent and every other vertex is adjacent to at least one of $u$ and $v$. Thus, $\bar{G}$ contains a spanning tree of diameter at most 4.

Let $G$ be the graph of order 5s obtained by replacing each vertex of a 5-cycle with a copy of the complete graph $K_5$ and adding edges between two vertices in different copies of $K_5$ if the corresponding vertices of the 5-cycle were adjacent. Then neither $G$ nor $\bar{G}$ contains a spanning tree of diameter at most 3. Thus, with respect to diameter, Theorem 1 cannot be improved.
Recently, Bialostocki, Dierker and Voxman [1] investigated other complete families of trees. Moreover, they conjectured that the family $B_n$ of brooms of order $n$ is complete, where a broom (of order $n$) is a tree consisting of a star and a path, with one end of the path identified with the central vertex of the star. The brooms of order 6 are shown in Figure 1.

In [2], Burr settled their conjecture in the affirmative and suggested that, in fact, only about half of $B_n$ is needed for a complete family. We note that any complete subfamily of $B_n$ necessarily contains the broom of diameter $n-1$, i.e. the path of order $n$.

![Figure 1](image_url)

One property of brooms is that all non-endvertices lie along a single path. In the remainder of this paper we will focus primarily on complete families of trees with this property having small diameter.

A *caterpillar* is a tree with the property that the vertices of degree at least 2 induce a path. These vertices form the spine of the caterpillar. Note that if $S$ is the spine of a caterpillar $C$ of order at least 3, then $\text{diam}(C) = |S| + 1$. In Theorem 2, we will show that $C_n(2 \log n)$ is complete, where $C_n(2 \log n)$ is the family of caterpillars of order $n$ and diameter at most $2 \log n$. (Here, $\log n$ is $\log_2 n$.) The following lemma will be useful.

**Lemma 1.** Let $G$ be a graph of order $n$ and diameter 2. If $G$ contains a caterpillar $C$ of diameter $d$, then $G$ contains a spanning caterpillar with diameter at most $d + (|V(G)| - |V(C)|)$.
Proof. Let \( v_1, v_2, \ldots, v_{d-1} \) be the vertices in the spine of \( C \), where \( v_i v_{i+1} \in E(C), 1 \leq i \leq d - 2 \). We first construct a caterpillar \( C' \) such that (i) \( |V(C')| = |V(C)| + 1 \) and (ii) \( \text{diam}(C') \leq \text{diam}(C) + 1 \).

Without loss of generality we may assume that if \( x \) is an endvertex of \( C \) and \( x \) is adjacent to \( v_i \), then \( x \) is not adjacent to \( v_j \) for \( j < i \). For convenience, we will say that the end vertices have been "shifted left". Furthermore, we may assume that no vertex in the spine is adjacent to a vertex of \( V(G) - V(C) \) since in that case we immediately obtain \( C' \) with \( \text{diam}(C') = \text{diam}(C) \). Let \( y \in V(G) - V(C) \). Then, since \( d_G(y, v_1) = 2 \) it follows that there is a vertex \( x \) of \( C \) such that \( xv_1 \in E(C) \) and \( yx \in E(G) \). Thus we obtain \( C' \) with spine \( \{ x, v_1, v_2, \ldots, v_{d-1} \} \) and \( \text{diam}(C') = \text{diam}(C) + 1 \).

Clearly, by repeating this procedure we obtain the desired spanning caterpillar.

A set \( X \) of vertices in a graph \( G \) is a dominating set if every vertex of \( V(G) - X \) is adjacent to at least one vertex of \( X \). In [3] it was shown that for every graph \( G \) of order \( n \), either \( G \) or \( \bar{G} \) has a dominating set \( X \) with \( |X| \leq \log n \). This result will be used in the proof of Theorem 2.

Theorem 2. Let \( C_n(2\log n) \) denote the family of caterpillars of order \( n \) and diameter at most \( 2\log n \). Then \( C_n(2\log n) \) is complete.

Proof. It is straightforward to verify the result for \( n \leq 4 \). Thus we assume \( n \geq 5 \). If \( G \) or \( \bar{G} \) is complete, then \( G \) or \( \bar{G} \) contains a spanning caterpillar of diameter \( 2 \) (i.e., a spanning star), where \( 2 \leq 2\log n \). Furthermore, if \( G \) or \( \bar{G} \) is disconnected or has diameter at least \( 3 \) then, as in the proof of Theorem 1, either \( G \) or \( \bar{G} \) contains a spanning caterpillar of diameter at most \( 3 \) and \( 3 \leq 2\log n \). Thus we may assume that \( \text{diam}(G) = \text{diam}(\bar{G}) = 2 \).

Let \( uv \in E(G) \) and let \( A \) denote those vertices adjacent to neither \( u \) nor \( v \) in \( G \). Suppose \( |A| \leq 2\log n - 3 \). Then, in \( \bar{G} - A \), \( u \) and \( v \) are either in different components or at distance at least \( 3 \). Consequently, as in the proof of Theorem 1, \( \bar{G} - A \) contains a spanning caterpillar of diameter at most \( 3 \). Thus \( \bar{G} \) contains a caterpillar of diameter at most \( 3 \) and it follows from Lemma 1 that \( \bar{G} \) contains a spanning caterpillar of diameter at most \( 3 + |A| \leq 2\log n \). Thus we may assume that if \( uv \in E(G) \) then \( u \) and \( v \) have at least \( 2\log n - 3 \) common neighbors in \( \bar{G} \). Similarly, if \( uv \notin E(G) \), then \( u \) and \( v \) have at least \( 2\log n - 3 \) common neighbors in \( G \).

Let \( X \subseteq V(G) \) with \( |X| \leq \log n \) such that \( X \) is a dominating set in \( G \) or \( \bar{G} \). (The existence of such a set is guaranteed by the aforementioned
result in [3]). Assume, without loss of generality, that $X$ dominates $G$ and $X = \{v_1, v_2, \ldots, v_t\}$. We claim that there is a $v_1 - v_t$ path $P$ containing the vertices of $X$ in the order $v_1, v_2, \ldots, v_t$ and such that between $v_i$ and $v_{i+1}$ there is at most one vertex. Suppose such a $v_1 - v_t$ path $P$ has been constructed for $l < t$. If $v_l v_{l+1} \in E(G)$ then we may extend $P$ to include $v_{l+1}$. If $v_l v_{l+1} \not\in E(G)$ then $v_l$ and $v_{l+1}$ have at least $2 \log n - 3 \geq 2l - 1$ common neighbors in $G$. Consequently there is a common neighbor $v \in V(G) - V(P) - X$ and $P$ can be extended to include $v_{l+1}$. Thus $G$ contains a $v_1 - v_t$ path of order at most $2t - 1$ containing $X$ and this path forms the spine of a spanning caterpillar of diameter at most $2 \log n$.

In [3] it was shown that for fixed $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for each $n \geq n_0$ there is a graph $G$ of order $n$ such that no set of at most $(1 - \varepsilon) \log n$ vertices dominates either $G$ or $\overline{G}$. Thus the bound in Theorem 2 on the diameter of the spanning caterpillars is, in fact, the correct order of magnitude.

In the proof of Theorem 2, we began with either a caterpillar of diameter at most 3 or a dominating set of cardinality at most $\log n$ and built a spanning caterpillar of diameter at most $2 \log n$. The same proof technique can be used to establish Theorem 3.

**Theorem 3.** If $D_n$ denotes the family of trees of order $n$ with diameter at most 6 and domination number at most $\log n$, then $D_n$ is complete.

### 3. Spanning Trees of Small Diameter Graphs

If $G$ is the graph of Figure 2, then $G$ has diameter 4 and no spanning caterpillar. In this section we will show that every graph of diameter at most 3 has a spanning caterpillar.

![Figure 2](image)
Theorem 4. If $G$ is a graph with diameter at most 3, then $G$ contains a spanning caterpillar.

Proof. If $\text{diam}(G) = 1$ then $G$ is complete and contains a spanning star. If $\text{diam}(G) = 2$ then Lemma 1 guarantees the existence of a spanning caterpillar. Thus we need only show that if $G$ is a graph of diameter 3 then $G$ has a spanning caterpillar. Assume, to the contrary, that $G$ is an edge-maximal counterexample. Thus, by edge maximality, $G$ contains two vertex disjoint caterpillars that together span $G$. Among all such pairs $C_1, C_2$ of disjoint caterpillars that together span $G$ select a pair such that $|V(C_1)|$ is as large as possible. Let $v_1, v_2, \ldots, v_l$ be the vertices (in order) of the spine of $C_1$ and $v_{l+1}, v_{l+2}, \ldots, v_m$ be the vertices of the spine of $C_2$. As in the proof of Lemma 1, assume that the endvertices of $C_1$ have been "shifted left". Let $w$ be an endvertex of $C_1$ adjacent to $v_l$ and let $u$ be an endvertex of $C_2$ adjacent to $v_{l+1}$. If $C_2$ is trivial, let $u = v_{l+1}$. Clearly, $d_G(u, w) \neq 1$ since, by assumption, $G$ has no spanning caterpillar. Thus, $2 \leq d_G(u, w) \leq 3$. Furthermore, by the choice of $C_1$ and $C_2$ we know that:

1. $w$ is adjacent to no vertex of $C_2$,
2. $w$ is adjacent to no $v_i$, $i < l$,
3. $v_l$ is adjacent to no vertex of $C_2$,
4. $u$ is adjacent to no $v_i$, $i \leq l$, and
5. there is no $u - w$ path whose interior vertices are all endvertices of $C_1$ and $C_2$.

By (1) and (2), every adjacency of $w$ other than $v_l$ in $G$ is an endvertex of $C_1$. Thus, by (4) and (5) there is no $u - w$ path of length 2. Therefore, $d_G(u, w) = 3$. Let $u, x_1, x_2, w$ be a $u - w$ path of length 3. Then by (1) and (2), either $x_2 = v_l$ or $x_2$ is an endvertex of $C_1$. If $x_2 = v_l$ then by (3) and (4) it follows that $x_1$ is an endvertex of $C_1$. Subsequently $C_1$ can be extended by including $x_1$ in the spine and $u$ as an endvertex, contradicting the maximality of $C_1$. Therefore $x_2$ is an endvertex of $C_1$. However, then by (4) and (5), $x_1$ must be a spine vertex of $C_2$ and again the maximality of $C_1$ is contradicted, and the proof is complete.

For even $n$, let $G$ be the graph of order $n$ obtained from the graph $K_{n/2} \cup \overline{K_{n/2}}$ by adding a matching between the set of $n/2$ isolated vertices and the remaining $n/2$ vertices. Then every spanning caterpillar has diameter $n/2 + 1$. Thus the (implied) bound in Theorem 4 of $n - 1$ on the smallest diameter of a spanning caterpillar is the correct order of magnitude for graphs of diameter 3. For graphs of diameter 2, some improvement can be
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The following notation will be useful. Let $G$ be a graph, $u$ a vertex of $G$, and $H$ a subgraph of $G$. Then

$$N_H[u] = \{w \in V(H) | uw \in E(G)\} \cup \{u\}.$$ 

**Theorem 5.** There is a constant $c$ such that if $G$ is a graph with $\text{diam}(G) = 2$, then $G$ contains a spanning caterpillar of diameter at most $cn^{3/4}$.

**Proof.** We first show that $G$ contains a dominating set with at most $2n^{3/4}$ vertices. Let $u_1$ be a vertex of $G$ with $\text{deg}_G u_1 \geq n^{1/4}$ and set $U_1 = N_G[u_1]$. Let $u_2 \in V(G)$ with $\text{deg}_{G-U_1} u_2 \geq n^{1/4}$ and set $U_2 = N_{G-U_1}[u_2]$. Continue in this fashion to obtain a maximal length sequence of vertices $u_1, u_2, \ldots, u_t$, $t \geq 1$, where $\text{deg}_{G-U_1-U_2-\ldots-U_{t-1}} u_t \geq n^{1/4}$ and $U_l = N_{G-U_1-U_2-\ldots-U_{l-1}}[u_l]$ for $l = 1, 2, \ldots, t$, and let $A = V(G) - \bigcup_{i=1}^{t-1} U_i$. Then $t \leq n^{3/4}$ and $\Delta(\langle A \rangle) < n^{1/4}$. If $|A| < n^{3/4}$, then $A \cup \{u_1, u_2, \ldots, u_t\}$ is the desired dominating set. We show that this must be the case. Assume, to the contrary, that $|A| = kn^{3/4}$, where $k > 1$. Each of the $\binom{k}{2}$ pairs of vertices of $A$ are at distance 1 or 2 in $G$. Since $\Delta(\langle A \rangle) < n^{1/4}$, $\langle A \rangle$ has fewer than $(|A| \cdot n^{3/4})/2$ edges. Furthermore, the number of pairs of vertices of $A$ with a common neighbor in $A$ is less than $|A| \cdot \binom{n^{3/4}}{2}$. Thus, more than

$$\binom{kn^{3/4}}{2} - \frac{kn}{2} - kn^{3/4} \cdot \binom{n^{1/4}}{2}$$

pairs of vertices of $A$ have a common neighbor in $V(G) - A$, implying that more than

$$\frac{k^2n^{3/2}}{2} - \frac{kn}{2} - \frac{kn^{5/4}}{2}$$

pairs of vertices in $A$ have a common neighbor in $V(G) - A$. However, each vertex in $V(G) - A$ is adjacent to fewer than $n^{1/4}$ vertices of $A$. Therefore, the number of pairs of vertices in $A$ with a common neighbor in $V(G) - A$ is less than

$$n \cdot \binom{n^{1/4}}{2}.$$ 

We conclude that

$$\frac{k^2n^{3/2}}{2} - \frac{kn}{2} - \frac{kn^{5/4}}{2} < \frac{n^{3/2}}{2} - \frac{n^{5/4}}{2},$$

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which is a contradiction for $k > 1$ and $n$ sufficiently large. Thus $G$ has a dominating set $X$ with $t \leq 2n^{3/4}$ vertices.

We complete the proof by showing that the vertices of $X$ are contained in the spine $S'$ of a caterpillar of $G$ in which

1. consecutive vertices of $X$ in $< S >$ are at distance at most 3 in $< S >$ and
2. $< S >$ begins and ends with a vertex of $X$.

Suppose $l < t$ vertices of $X$ are contained in such a caterpillar $C$ with spine $S'$. We assume that no vertex of $X$ is an endvertex of $C$ and that the endvertices of $C$ have been “shifted left.” Furthermore, we assume that if $u \in V(G) - X - S'$ and $u$ is adjacent to a vertex in $S'$, then $u$ is an endvertex of $C$. Let $x_1 \in X$ be the rightmost spine vertex of $C$ and let $x_2 \in X - V(C)$. Furthermore, let $w$ be an endvertex of $C$ adjacent to $x_1$. If no such $w$ exists, then we may replace $x_1$ in $X$ by its predecessor on the spine $S'$ and continue. Then $d_G(w, x_2) \leq 2$. If $wx_2 \in E(G)$ we can easily extend $C$ to include $w$ and $x_2$ as spine vertices. If $d_G(w, x_2) = 2$, then, as in the proofs of previous results, $w$ and $x_2$ must have a common neighbor $y$ that is not on the spine of $C$ (where $y$ may or may not be in $X$.) In either case, we can extend the spine of $C$ to include $w, y, x_2$, and the proof is complete.

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