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Forbidden Triples of Subgraphs and Traceability

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Abstract

Given a family $\mathcal{F}$ of connected graphs, a graph G is said to be $\mathcal{F}$-free if G contains no induced subgraph that is isomorphic to a graph in $\mathcal{F}$, and the graphs in such a family are called forbidden subgraphs. A topic considered recently has been the investigation of which sets of subgraphs can be forbidden in order to imply a particular hamiltonian property in a graph. Faudree and Gould have characterized all pairs of subgraphs that imply traceablity in connected graphs. In this paper we identify two families of triples of subgraphs that imply traceability when forbidden.

1 Introduction and Definitions

All graphs considered here are simple—no loops or multiple edges. For terms not defined here, see [3].

Let G and S be connected graphs. If G does not contain an induced subgraph that is isomorphic to S, then G is said to be *S-free*. Furthermore, if $\mathcal{F}$ is a family of connected graphs, and if G does not contain an induced subgraph that is isomorphic to any graph in $\mathcal{F}$, then G is said to be *$\mathcal{F}$-free*. In these cases, the graph S and the graphs in $\mathcal{F}$ are called *forbidden subgraphs*.

A topic that has long been of interest is the determination of various families of subgraphs that imply certain hamiltonian properties. In particular, this paper investigates the relationship between forbidden subgraphs and traceability, the existence of a spanning path. The reader will notice that each result presented here assumes connectivity. This is a necessary assumption since a graph must first be connected in order to have a chance at being traceable.

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A reasonable first question to ask is whether or not there exists a connected graph S such that if a connected graph G is S-free, then G is traceable. The graph P_3, the path on three vertices, provides an affirmative answer to this question. It is relatively easy to see that if a connected graph G is P_3-free, it is necessarily complete and, of course, traceable. In [2] Faudree and Gould show that P_3 is in fact the only single graph that implies traceability when forbidden.

Having taken care of the single graphs, let us now shift our attention to forbidden pairs of subgraphs. First of all, it is trivial to see that if S is *any* graph, the pair $\{P_3, S\}$ will imply traceability. This being the case, it is more interesting to consider pairs that do not involve P_3. An early result of this type comes from Duffus, Gould, and Jacobson [1]. The graphs $K_{1,3}$ and N can be seen in Figure 1. The graph $K_{1,3}$ is also called the "claw," and it appears frequently in forbidden subgraph results.

Theorem 1.1 *If G is a connected $\{K_{1,3}, N\}$-free graph, then G is traceable.*

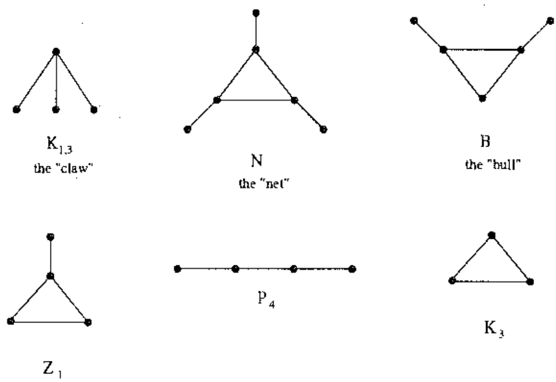

Figure 1: Graphs involved in forbidden pairs.

For a graph G, let $CI(G)$ denote the set of all connected induced subgraphs of G. If $S \in CI(N)$, then it is relatively easy to see that the pair $\{K_{1,3}, S\}$ is another pair that implies traceability when forbidden. For if G is $\{K_{1,3}, S\}$-free, it is certainly $\{K_{1,3}, N\}$-free as well, and is therefore traceable. There are five nonisomorphic connected induced subgraphs of N: $CI(N) = \{N, B, Z_1, K_3, P_4\}$ (see Figure 1. Hence, we have the following corollary.

Corollary 1.2 *Let G be a connected graph and let $S \in CI(N)$. If G is $\{K_{1,3}, S\}$-free, then G is traceable.*

Thus we know of five pairs of subgraphs that imply traceability when forbidden. Faudree and Gould [2] have recently shown that these five are the only pairs with this property.

Having characterized the pairs of subgraphs that imply traceability, it is a natural step to ask about triples of forbidden subgraphs. Once again, it is clear that any triple that has P_3 as a member will automatically imply traceabilty, regardless of what the other two members are. Furthermore, any triple that contains one of the five pairs of graphs referred to in the above Corollary will also immediately imply traceability. In the sections that follow, we identify two families of triples of subgraphs that, when forbidden, imply traceability in sufficiently large, connected graphs. These families do not involve P_3, and they do not contain any of the five pairs mentioned earlier.

Before proceeding, we give explanations for some of the notation that will be encountered. First of all, if S is a subset of the vertices of a graph G, then $< S >$ will denote the subgraph of G that is induced by S. Furthermore, if T is a subgraph of G and $v \in V(G)$, then the set $N_T(v)$ is described by $N_T(v) = \{x \in V(T) : xv \in E(G)\}$.

The last bit of notation that we wish to explain involves paths. In a graph G, suppose we have paths P_i described as follows:

$$P_1 : v_{1,1}, v_{1,2}, \dots, v_{1,j_1}$$

$$P_2 : v_{2,1}, v_{2,2}, \dots, v_{2,j_2}$$

$$\vdots$$

$$P_k : v_{k,1}, v_{k,2}, \dots, v_{k,j_k}.$$

Assuming that the edges $v_{i,j_i}v_{i+1,1}$ exist for $i = 1, \dots, k-1$, and assuming that the vertices are all distinct, then the path P in G described by

$$P : v_{1,1}, \dots, v_{1,j_1}, v_{2,1}, \dots, v_{2,j_2}, \dots, v_{k,1}, \dots, v_{k,j_k}$$

will be denoted as follows:

$$[v_{1,1}, v_{1,j_1}]_{P_1}, [v_{2,1}, v_{2,j_2}]_{P_2}, \dots, [v_{k,1}, v_{k,j_k}]_{P_k}.$$

In a similar fashion, if $v_{1,j_1} = v_{2,1}$, then the notation given by

$$[v_{1,1}, v_{1,j_1}]_{P_1}, (v_{2,1}, v_{2,j_2}]_{P_2}$$

will represent the path S in G given by

$$S : v_{1,1}, \dots, v_{1,j_1}, v_{2,2}, \dots, v_{2,j_2}.$$

2 The First Family: $\{K_{1,r}, Y_l, Z_1\}$

We begin this section with several lemmas.

Lemma 2.1 ([5]) *A graph G is Z_1-free if and only if each component of G is K_3-free or complete multipartite.*

The following lemma is a well known result from introductory graph theory.

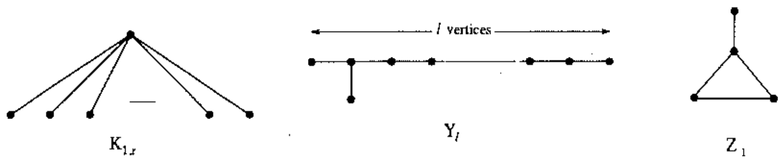

Figure 2: The First Family.

Lemma 2.2 *If* $G = K_{r_1,r_2,\ldots,r_k}$, *then* G *is traceable if and only if for each* $i \in \{1,2,\ldots,k\}$

$$r_i \leq 1 + \sum_{j \neq i} r_j.$$

Theorem 2.1 *Let* $r \geq 4$ *and* $l \geq 4$ *be fixed integers. Let* G *be a connected graph of order* n *which is* $\{K_{1,r}, Y_l, Z_1\}$*-free. Then if* n *is sufficiently large,* G *is traceable.*

Proof: We consider two cases.
Case 1: Suppose G is K_3-free.

Suppose the maximum degree of G, $\Delta(G)$, is at least r, and let v be a vertex of degree $\Delta(G)$. Since G is K_3-free, we know that $N(v)$ is an independent set. So we have an induced $K_{1,r}$ in G, a contradiction. Therefore it must be that $\Delta(G) < r$.

Let w be a vertex of degree $\Delta(G)$ and suppose that $deg(w) \geq 3$. For each $i \in \{1,2,\ldots\}$ define

$$N_i(w) = \{u \in V(G) : d(u,w) = i\}.$$

Since $deg(w) < r$ we know that $|N_1(w)| < r$. Similarly, since the degree of any vertex in $N_1(w)$ is bounded by r, we know that $|N_2(w)| < r^2$. By the same token, we can argue that $|N_3(w)| < r^3, \ldots, |N_i(w)| < r^i, \ldots$. Thus, if n is large enough, it must be that $N_l(w) \neq \emptyset$. We assume that n is large enough for this to be true.

Let a_l be a vertex of $N_l(w)$. Then there must exist vertices $a_1, a_2, \ldots, a_{l-1}$ such that $a_i \in N_i(w)$ for each i and such that

$$\{w, a_1, a_2, \ldots, a_{l-1}, a_l\}$$

is a path in G. In fact, due to the nature of the sets N_i, this path is induced. Let b_1, c_1 be vertices of $N_1(w)$ different from a_1 (see Figure 3).

Since G is K_3-free, there are no adjacencies among a_1, b_1, and c_1. If one of b_1 or c_1 is adjacent to a_2, (say b_1), then

$$< \{a_1, b_1, a_2, a_3, a_4, \ldots, a_l\} >$$

is an induced Y_l, a contradiction. If neither b_1 nor c_1 is adjacent to a_2, then we have that

$$< \{b_1, c_1, w, a_1, a_2, a_3, \ldots, a_{l-2}\} >$$

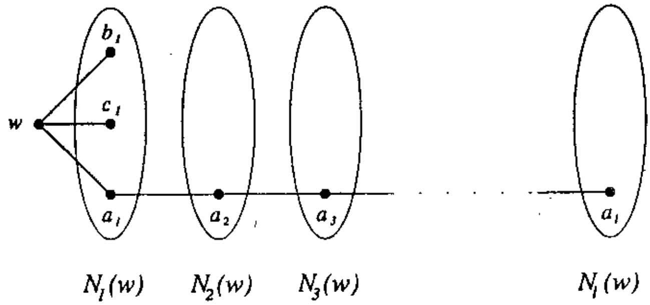

Figure 3: The sets $N_i(w)$.

is an induced Y_l, another contradiction. Therefore it must be that $deg(w) = \Delta(G) \leq 2$.

Hence, G is a connected graph with maximum degree 2. That is, G is either a path or a cycle, and both of these are traceable.

Case 2: Suppose G is not K_3-free.

Since G is Z_1-free, we know from Lemma 2.1 that G is complete multipartite. Further, since G is $K_{1,r}$-free, the size of each partite set must be less than r. Thus, if n is large enough, no one partite set can be larger than the sum of the sizes of the other partite set. That is, the condition of Lemma 2.2 holds, and G must be traceable.

This completes the proof. ∎

Note here that $n \geq 1 + \sum_{i=0}^{l-1}(r-1)^i$ suffices in the proof. Also notice that the result holds for $r < 4$ and/or $l < 4$ from the pairs work in [2].

Corollary 2.2 *Let $r \geq 4$ and $l \geq 4$ be fixed integers. Let $R \in CI(K_{1,r}), S \in CI(Y_l)$, and $T \in CI(Z_1)$. If G is a connected graph of order n that is $\{R, S, T\}$-free, and if n is sufficiently large, then G is traceable.* ∎

3 The Second Family: $\{K_{1,r}, P_4, V_m\}$

The proof in this section will make use of Ramsey numbers, a well-known topic in graph theory. Given positive integers m and n, the Ramsey number, denoted $R(m,n)$, is the least positive integer p such that *every* graph on p vertices contains either K_m or $\overline{K_n}$ as an induced subgraph. That is, either G contains a clique of size m (or more) or G contains an independent set of vertices of size n.

We now state another well known result from Hall [4].

Theorem 3.1 ([4]) *Let $G = (X \cup Y, E)$ be a bipartite graph. Then X can be matched to a subset of Y if and only if $|N(X')| \geq |X'|$ for all subsets X' of X.*

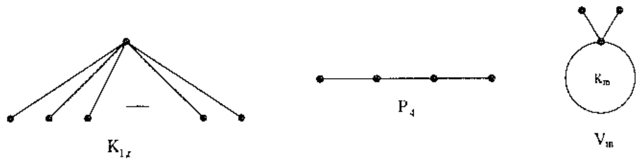

Figure 4: The Second Family.

Theorem 3.2 *Let $r \geq 4$ and $m \geq 3$ be fixed intergers. Let G be a connected graph of order n that is $\{K_{1,r}, P_4, V_m\}$-free. If n is sufficiently large, then G is traceable.*

Proof: Let C be the largest clique in G, and let

$$P = \{v \in V(G \setminus C) : cv \in E(G) \text{ for some } c \in V(C)\}.$$

That is, P is the set of neighbors of the vertices of C that are not themselves in C. We will call the vertices of P *peripherals.* We now prove several preliminary facts.

Fact 3.1 *For the graph G, $V(G) = V(C) \cup P$.*

Proof: Suppose this is not the case. Then there must be some vertex of G, say v, that is neither in $V(C)$ nor P, and such that $d(v, C) = 2$. We know then that v must be adjacent to some vertex, say w of P. Further, it must be that w is adjacent to some $c \in V(C)$, and since C is maximal, there is some $c' \in V(C)$ such that $wc' \notin E(G)$. But this means that $< \{v, w, c, c'\} >$ is an induced P_4, a contradiction. Hence, no such v can exist, and so $V(G) = V(C) \cup P$. ∎

Fact 3.2 *For the clique C, $|V(C)| \geq 4r + m$.*

Proof: Since n is sufficiently large, we can suppose $n > R(4r + m, r(4r + m) + 1)$. If $|V(C)| < 4r + m$, there must be an independent set of vertices I of size $r(4r + m) + 1$. At most one of these vertices is in C, so there are at least $r(4r + m)$ independent peripherals. Thus some vertex $c \in C$ is adjacent to at least

$$\frac{r(4r+m)}{|V(C)|} > \frac{r(4r+m)}{4r+m} > r$$

of these independent peripherals in I. Denote r of these vertices by $v_1, v_2, \ldots, v_r$. Then we have that

$$\{c, v_1, v_2, \ldots, v_r\}$$

is an induced $K_{1,r}$. Therefore it must be that $|V(C)| \geq 4r + m$. ∎

Fact 3.3 *For the set P, $\alpha(P) < r$, where $\alpha(P)$ is the independence number of P.*

Proof: Suppose this is not the case, and let $\{v_1, v_2, \ldots, v_r\}$ be an independent set of peripherals. We suppose without loss of generality that these vertices are labeled so that $|N_C(v_1)| \geq |N_C(v_2)| \geq \ldots \geq |N_C(v_r)|$.

We now claim that if x and y are nonadjacent peripherals, then either $N_C(x) \subseteq N_C(y)$ or $N_C(y) \subseteq N_C(x)$. If this is not the case, then both $N_C(x) \setminus N_C(y)$ and $N_C(y) \setminus N_C(x)$ are nonempty. Let $a \in N_C(x) \setminus N_C(y)$ and let $b \in N_C(y) \setminus N_C(x)$. Then we have that $< \{x, a, b, y\} >$ is an induced P_4 (see Figure 5), which is a contradiction.

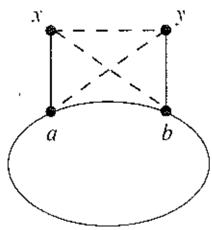

Figure 5: An induced P_4.

Thus, the claim holds, and therefore we have that $N_C(v_1) \supseteq N_C(v_2) \supseteq \ldots \supseteq N_C(v_r)$, and since $N_C(v_r) \neq \emptyset$, there exists a vertex $w \in \bigcap_{i=1}^{r} N_C(v_i)$. But then we have that $< \{w, v_1, v_2, \ldots, v_r\} >$ is an induced $K_{1,r}$, and this is a contradiction. Therefore, it must be that $\alpha(P) < r$. ∎

Now we partition P into two sets:

$$P^+ = \{v \in P : |N_C(v)| > 2r\}$$

$$P^- = \{v \in P : |N_C(v)| \leq 2r\}$$

Fact 3.4 *The graph $< P^- >$ is complete.*

Proof: Suppose that x and y in P^- are nonadjacent. We have that $|N_C(x) \cup N_C(y)| \leq 2r + 2r = 4r$, and since $|V(C)| \geq 4r + m$, we have that $|V(C) \setminus N_C(x) \cup N_C(y)| \geq m$. Now, from the claim within the proof of Fact 3.3, we know that x and y must have a common neighbor in C, say d (see Figure 6).

Thus if $x_1, x_2, \ldots, x_{m-1}$ are distinct vertices of $C \setminus (N_C(x) \cup N_C(y))$, then

$$< \{x, y, d, x_1, x_2, \ldots, x_{m-1}\} >$$

is an induced V_m, a contradiction. Thus, $< P^- >$ must be complete. ∎

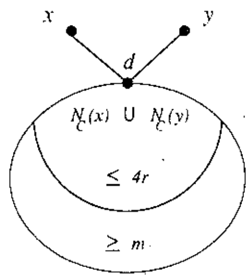

Figure 6: An induced V_m.

We will now partition the vertices of P^+ into disjoint paths. Let S_1 be a longest path (not necessarily induced) in $< P^+ >$. Say that a_1 and b_1 are the endpoints of S_1. Let S_2 be a longest path in $< P^+ \setminus V(S_1) >$, say with endpoints a_2 and b_2. We can continue this process until we have paths $S_1, S_2, \ldots, S_l$ where $V(S_1) \cup V(S_2) \cup \ldots \cup V(S_l) = P^+$, and where for each $i \in \{1, 2, \ldots, l\}$, the path S_i has endpoints a_i and b_i. Now, from this construction, we see that $V(S_1), V(S_2), \ldots, V(S_l)$ are necessarily disjoint. Further, $\{a_1, a_2, \ldots, a_l\}$ must be an independent set, for if a_i were adjacent to a_j where $i < j$, then the path S_i could have been extended by one vertex. Hence, from Fact 3.3, it must be that $l < r$.

Now, it might be the case that some of these paths are single vertices. Suppose that the paths $S_1, S_2, \ldots, S_t$ are the paths with more than one vertex, and that $S_{t+1}, S_{t+2}, \ldots, S_l$ are the single vertex paths. That is, $a_i = b_i$ for each $i \in \{t+1, t+2, \ldots, l\}$. For $t + 1 \leq i \leq l$, split the vertex a_i into two distinct vertices a_i and b_i, and connect them with an edge. Further, place edges between b_i and all vertices of $N_C(a_i)$ (see Figure 7).

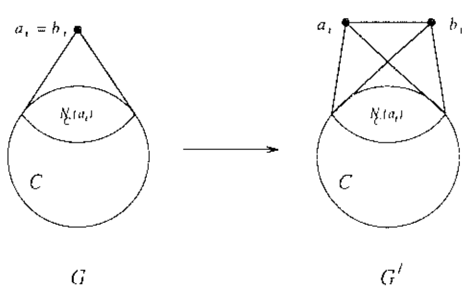

Figure 7: The splitting of vertex a_i.

This splitting creates a new graph G' (if $t = l$, then $G' = G$), and we have converted the single vertex paths $S_{t+1}, \ldots, S_l$ into two-vertex paths, while the other paths $S_1, \ldots, S_t$ remain unchanged.

Consider the bipartite subgraph of G' defined as follows: $B = (X \cup Y, E(B))$ where $X = \{a_1, b_1, a_2, b_2, \ldots, a_l, b_l\}$ and $Y = \bigcup_{i=1}^{l}(N_C(a_i) \cup N_C(b_i))$ and where $E(B) = \{xy : x \in X, y \in Y, \text{ and } xy \in E(G')\}$. Let $R \subseteq X$. Then in B we have

$$|R| \leq |X| = 2l < 2r < |N(R)|$$

since in G, $|N_C(x)| > 2r$ for any $x \in R$. Thus by Lemma 3.1, X can be matched to a subset of Y. For each i, say that a_i and b_i in X are matched to a_i^* and b_i^* in Y, respectively. Figure 8 illustrates the situation in the graph G'.

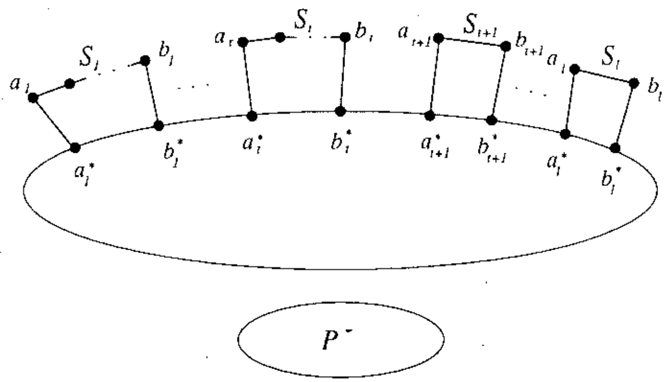

Figure 8: The graph G'.

We now show that G' is traceable by constructing a hamiltonian path. Since $< P^- >$ is complete, it is certainly traceable. Let T_1 be a hamiltonian path in $< P^- >$, say with endpoints x and y. Now, let $c \in V(C)$ be a neighbor of y.

Case 1: Suppose $c = a_i^*$ or $c = b_i^*$ for some i.

Suppose without loss of generality that $c = a_1^*$. Let T_2 be the following path in G':

$$a_1^*, [a_1, b_1]_{S_1}, b_1^*, a_2^*, [a_2, b_2]_{S_2}, b_2^*, \ldots, a_l^*, [a_l, b_l]_{S_l}, b_l^*.$$

Finally, let T_3 be a path that begins at $b_l^* \in V(C)$ and contains all vertices of C that are not on the path T_2, and suppose the other endpoint of T_3 is $z \in V(C)$. Then the path given by

$$[x, y]_{T_1}, c, (a_1^*, b_l^*]_{T_2}, (b_l^*, z]_{T_3}$$

is a hamiltonian path in G'.

Case 2: Suppose $c \neq a_i^*, b_i^*$ for all i.

Let T_2 be as described in Case 1. Let T_3 be a hamiltonian path in C with endpoints c and a_1^*. Then the path given by

$$[x, y]_{T_1}, [c, a_1^*]_{T_3}, (a_1^*, b_l^*]_{T_2}$$

is a hamiltonian path in G'.

Therefore, we can conclude that G' is traceable. (Note: If either of P^+ or P^- is empty, G' is still clearly traceable.) If G is isomorphic to G', the proof is complete. So, assume that G is not isomorphic to G', and say that vertex $a_i \in G$ was split to form vertices a_i and b_i in G'. In our construction of a hamiltonian path in G', the edge a_ib_i was used (it was the path S_i). Hence, by identifying the vertex $a_i \in V(G')$ with the vertex $b_i \in V(G')$, we do not affect the existence of a spanning path. Therefore, if we identify all pairs of vertices of G' that were a result of splitting, we obtain the graph G, and we see that G is also traceable. ∎

Note here that $n > R(4r + m, r(4r + m) + 1)$ suffices. Again, if $r < 4$ and/or $m < 3$, the result follows from the pairs result in [2].

Corollary 3.3 *Let $r \geq 4$ and $m \geq 3$ be fixed integers. Let $R \in CI(K_{1,r})$, $S \in CI(P_4)$, and $T \in CI(V_m)$. If G is a connected graph of order n that is $\{R, S, T\}$-free, and if n is sufficiently large, then G is traceable.* ∎

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