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Traceability in graphs with forbidden triples of subgraphs

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Abstract

If $\mathscr{F}$ is a collection of connected graphs, and if a graph G does not contain any member of $\mathscr{F}$ as an induced subgraph, then G is said to be $\mathscr{F}$-free. The members of $\mathscr{F}$ in this situation are called forbidden subgraphs. In a previous paper (Gould and Harris, 1995) the authors demonstrated two families of triples of subgraphs which imply traceability when forbidden. In this paper the authors identify two additional families that enjoy this same property.

Keywords: Graph; Traceable; Forbidden subgraph; Triples

1. Background and definitions

All graphs considered in this paper are simple graphs — no loops or multiple edges. For definitions of terms not defined here, see [3].

If G and S are connected graphs, and if no induced subgraph of G is isomorphic to S, then G is said to be *S-free*. Moreover, if $\mathscr{F}$ is a family of connected graphs, and if no induced subgraph of G is isomorphic to any graph in $\mathscr{F}$, then G is said to be *$\mathscr{F}$-free*. In these cases, the graph S and the graphs in $\mathscr{F}$ are called *forbidden subgraphs*. Several common forbidden subgraphs are shown in Fig. 1.

Several results are known regarding the relationship of forbidden subgraphs to traceability, the existence of a Hamiltonian path. For instance, a connected, P_3-free graph is complete, and therefore traceable. For this reason, in this paper we will only consider forbidden subgraphs that are neither P_3 nor any subgraph of P_3.

Another result of this type is from Duffus et al. [1]. The graphs $K_{1,3}$ (often called the 'claw') and N are seen in Fig. 1.

Theorem A (Duffus [1]). *If G is a connected $\{K_{1,3},N\}$-free graph, then G is traceable.*

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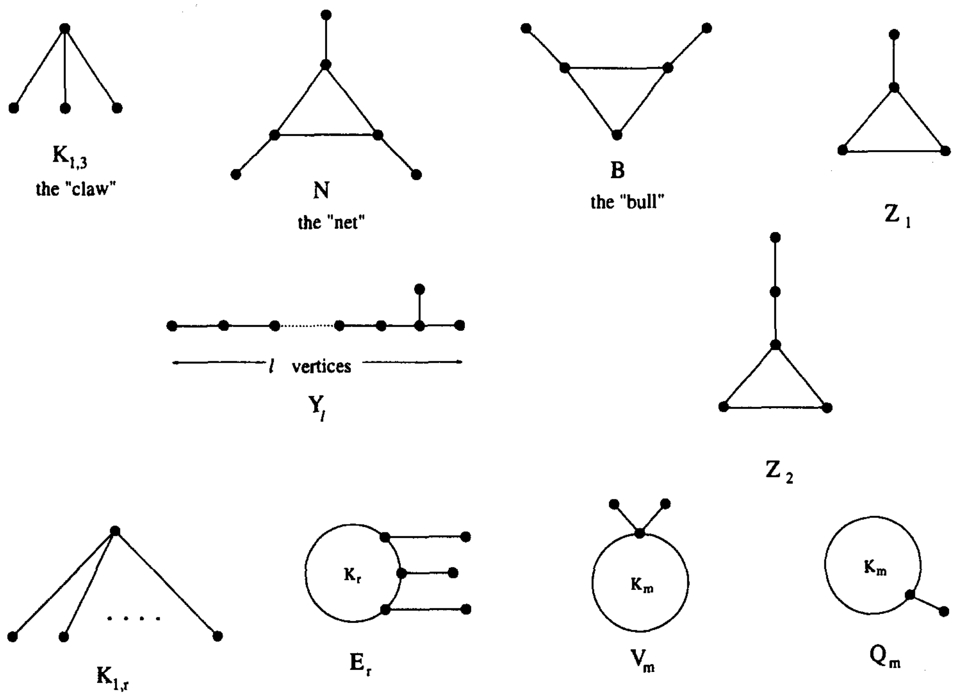

Fig. 1. Common forbidden subgraphs.

It is clear that if G is $\{K_{1,3},X\}$-free, where X is a connected induced subgraph of N, then G is also $\{K_{1,3},N\}$-free, and thus traceable. Thus, each of the pairs $\{K_{1,3},N\}$, $\{K_{1,3},B\}$, $\{K_{1,3},Z_1\}$, $\{K_{1,3},K_3\}$, and $\{K_{1,3},P_4\}$ imply traceability when forbidden in connected graphs.

In [2], Faudree and Gould demonstrate that P_3 is the only single graph, and that the five pairs listed above are the only pairs of graphs, that imply traceability when forbidden. Therefore, since the singles and pairs have been characterized, it is natural to consider triples of subgraphs.

In [4] two families of triples of subgraphs were considered.

Theorem B (Gould and Harris [4]). *Let $r\geqslant 4$ and $l\geqslant 4$ be fixed integers. Let G be a connected graph of order n which is $\{K_{1,r},Y_l,Z_1\}$-free. Then if n is sufficiently large, G is traceable.*

Theorem C (Gould and Harris [4]). *Let $r\geqslant 4$ and $m\geqslant 3$ be fixed intergers. Let G be a connected graph of order n that is $\{K_{1,r},P_4,V_m\}$-free. If n is sufficiently large, then G is traceable.*

In this paper we will prove similar results for two additional families.

Theorem 1. *Let $r \geqslant 4$ be a fixed integer. If G is a connected graph of order $n \geqslant 2r$ which is $\{K_{1,3}, E_r, Z_2\}$-free, then G is traceable.*

Theorem 2. *Let $r \geqslant 4$, $l \geqslant 5$, and $m \geqslant 3$ be fixed integers. Let G be a connected graph of order n which is $\{K_{1,r}, P_l, Q_m\}$-free. Then if n is sufficiently large, G is traceable.*

Before beginning, let us review some of the notation that will be used. This notation is consistent with that used in [4]. First of all, if S is a subset of the vertices of a graph G, then $\langle S \rangle$ will denote the subgraph of G that is induced by S. Furthermore, if T is a subgraph of G and $v \in V(G)$, then the set $N_T(v)$ is described by $N_T(v) = \{x \in V(T)\colon xv \in E(G)\}$. Let P_1 and P_2 be internally disjoint paths with end vertices $\{x_1, y_1\}$ and $\{x_2, y_2\}$, respectively. If $y_1 x_2 \in E(G)$, then the path P obtained by joining P_1 to P_2 in the natural way will be denoted by $P\colon [x_1, y_1]_{P_1}, [x_2, y_2]_{P_2}$. If $y_1 = x_2$, then the notation given by $[x_1, y_1]_{P_1}, (x_2, y_2]_{P_2}$ will represent the path formed by joining the two paths at the common end vertex.

2. Proof of Theorem 1

Lemma 1. *Suppose G is a connected, nontraceable, $\{K_{1,3}, Z_2\}$-free graph. Let u and v be distinct vertices of G satisfying at least one of the following conditions:*

(i) *u, v are the end vertices of a longest path in G;*

(ii) *either $\deg(u) = 1$ and v is the end vertex of a longest path extending from u, or $\deg(v) = 1$ and u is the end vertex of a longest path extending from v;*

(iii) *both u and v have degree 1.*

If P is a longest path joining u to v, and if $Q = V(G) \backslash V(P)$, then there exists a vertex in Q that has degree 1 in G.

Proof. Order the vertices of P from u to v. For $c \in V(P)$, $(c \neq u, v)$, let c^{-1} and c^{+1} refer to the predecessor and successor of c on P, respectively. Similarly, let c^{-i} be the ith vertex previous to c, and let c^{+i} be the ith vertex following c.

Since G is nontraceable, there exists a vertex $q \in Q$ which is adjacent to a vertex c on P. Since G is claw-free and since P is maximal, the edge $c^{-1}c^{+1}$ must be present. The maximality of P also implies that $c \neq u, v$. Suppose for the moment that $c^{-1} = u$. This implies that $uc^{+1} \in E(G)$, so condition (iii) is immediately contradicted. Further, if condition (ii) holds, it must be that $\deg(v) = 1$ and u is the endpoint of a longest path extending from v. Our path P is one such path. But we see that the path P' given by $q, c, u, c^{+1}, [c^{+2}, v]_P$ is a longer such path. So condition (ii) cannot hold, and neither can condition (i). We have therefore proved the following claim. □

Claim 2.1. *If $q \in Q$ is adjacent to $c \in V(P)$, then $c \notin \{u, v\}$, $c^{-1}c^{+1} \in E(G)$, $c^{-1} \neq u$, and $c^{+1} \neq v$.*

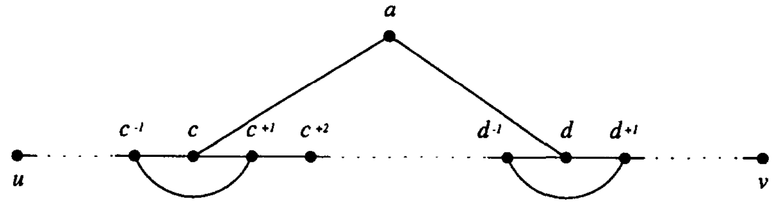

Fig. 2.

Claim 2.2. *No vertex of Q is simultaneously adjacent to a vertex of P and another vertex of Q.*

Proof. Suppose that $a \in Q$ is adjacent to $c \in V(P)$ and $b \in Q$. Now, since $\langle\{b,a,c,c^{-1},c^{+1}\}\rangle$ is a potential induced Z_2, one of the edges $bc, bc^{-1}, bc^{+1}, ac^{-1}, ac^{+1}$ must be present. Each of $bc^{-1}, bc^{+1}, ac^{-1}, ac^{+1}$ would produce a longer u–v path, so it must be that $bc \in E(G)$. Now $\langle\{b,a,c,c^{+1},c^{+2}\}\rangle$ is a potential Z_2, so at least one of the edges bc^{+1}, bc^{+2}, ac^{+1}, ac^{+2}, cc^{+2} must be present. Each of the first four produce longer u–v paths, so the edge cc^{+2} must be present. Now, by considering the potential Z_2 induced by $\{b,a,c,c^{+2},c^{+3}\}$, we see (through a similar argument) that $cc^{+3} \in E(G)$. Continuing in this fashion we obtain that c is adjacent to each c^{+i}, including v. A similar argument shows that c is adjacent to each c^{-i}, including u. From Claim 2.1, we know that $u \neq c^{-1}$ and $v \neq c^{+1}$, so the fact that both u and v are adjacent to c immediately contradicts both conditions (ii) and (iii). Further, the path $P' = a,b,c,u,(u,c^{-1}]_P,[c^{+1},v]_P$ is a longer path than P, contradicting condition (i). Hence, no such vertex a can exist. □

Claim 2.3. *No vertex of Q is adjacent to more than one vertex of P.*

Proof. Suppose $a \in Q$ is adjacent to vertices c and d on P. We can suppose, without loss, that c is between u and d on P, and that a is nonadjacent to all vertices that are between c and d on P. Recall from Claim 2.1 that $c^{-1}c^{+1}$ and $d^{-1}d^{+1}$ must both be edges of G. Let i be such that $d = c^{+i}$.

If $i \leqslant 3$, then the u–v path can be easily lengthened, which is a contradiction. So we suppose that $i > 3$ (see Fig. 2).

Our strategy at this point is to examine several potential induced Z_2's, and to reveal the edges that must be present to prevent them from existing.

First, consider the vertices $\{c^{-1},c^{+1},c,a,d\}$. They form a potential induced Z_2, and so one of the edges $ac^{-1}, ac^{+1}, dc^{-1}, dc^{+1}, cd$ must be present. Each of the first two trivially produces a longer u–v path, while for the second two the paths are

$$dc^{-1}\colon\ [u,c^{-1}],d,a,c,c^{+1},[c^{+2},d^{-1}]_P,[d^{+1},v]_P$$

$$dc^{+1}\colon\ [u,c^{-1}],c,a,d,c^{+1},[c^{+2},d^{-1}]_P,[d^{+1},v]_P.$$

Hence, the edge cd must be present.

Now, consider the vertices $\{a,d,c,c^{+1},c^{+2}\}$. They, too, form a potential induced Z_2. We know from above that $ac^{+1},dc^{+1}\notin E(G)$, so one of ac^{+2},dc^{+2}, or cc^{+2} must be present. It can be shown that both ac^{+2} and dc^{+2} yield longer u–v paths, and so the edge cc^{+2} must be present.

Finally, consider the potential Z_2 formed by the vertices c^{+1}, c^{+2}, c, d, and d^{+1}. We know from above that $dc^{+1},dc^{+2}\notin E(G)$, so one of $c^{+1}d^{+1},c^{+2}d^{+1}$, or cd^{+1} must be present. The edge $c^{+1}d^{+1}$ yields a longer u–v path: $[u,c^{-1}]_P,c,a,d,[d^{-1},c^{+2}]_P,c^{+1}$, $[d^{+1},v]_P$. The other two edges produce longer paths as well. This being a contradiction, we see that no vertex of Q can be adjacent to more than one vertex of P. □

Now, since G is nontraceable, there must be some vertex of Q, say q, that is adjacent to a vertex of P, say p. From Claim 2.2, we know that q is nonadjacent to every other vertex in Q, and from Claim 2.3 we know that q is nonadjacent to every vertex of P except p. Thus, $\deg(q)=1$, and the proof of the lemma is complete. □

Proof of Theorem 1. Suppose that G is nontraceable. Let S_1 be a longest path in G, and say the end vertices are u and v. Then u,v satisfy condition (i) of Lemma 1, and so there exists some $a_0\in V(G)\backslash V(S_1)$ such that $\deg(a_0)=1$. Let S_2 be a longest path in G extending from a_0. If we let w be the other end vertex of S_2, then a_0,w satisfy condition (ii) of the lemma. Hence, there exists some $b_0\in V(G)\backslash V(S_2)$ with degree 1 in G. Let S_3 be a longest path in G that has a_0 and b_0 as its end vertices. Then since a_0,b_0 satisfy condition (iii) of Lemma 1, there exists a vertex $c_0\in V(G)\backslash V(S_3)$ such that $\deg(c_0)=1$. Note that a_0,b_0, and c_0 are *distinct* vertices of degree 1 in G, and since G is connected, it must be that a_0, b_0, and c_0 are pairwise nonadjacent.

Now, let a_1,b_1, and c_1 be the neighbors of a_0,b_0, and c_0, respectively. Suppose that $a_1=b_1$ and consider $\langle\{a_0,b_0,a_1,x\}\rangle$ where x is some vertex of $V(G)\backslash\{a_0,b_0\}$ that is adjacent to a_1 (such an x must exist since G is connected and nontraceable). Then since G is claw-free, one of the edges a_0b_0,a_0x,b_0x must be present; but this is a contradiction since a_1 is the unique neighbor of both a_0 and b_0. Therefore, $a_1\neq b_1$ and by a similar argument, we see that a_1, b_1, and c_1 are all distinct. Thus, we have that a_0,b_0,c_0,a_1,b_1,c_1 are six distinct vertices of G.

Claim 2.4. *The vertex a_1 is adjacent to b_1.*

Proof. Suppose that a_1 is not adjacent to b_1, and let P be a shortest path in G that has a_1 and b_1 as end vertices. Note that $|V(P)|>2$ and that $a_0,b_0,c_0\notin V(P)$. Order P from a_1 to b_1. For convenience define a_1^{-1} to be a_0 and define b_1^{+1} to be b_0.

Case 1: Suppose $c_1\in V(P)$. We know that $c_1^{-1},c_1^{+1}\neq c_0$, so consider $\langle\{c_1,c_1^{-1},c_1^{+1}$, $c_0\}\rangle$. Since G is claw-free, one of the edges $c_1^{-1}c_1^{+1}$, $c_1^{-1}c_0$, $c_1^{+1}c_0$ must be present. But since $\deg(c_0)=1$, only $c_1^{-1}c_1^{+1}$ can be present. But then $[a_1,c_1^{-1}]_P,[c_1^{+1},b_1]_P$ is a path from a_1 to b_1 that is shorter than P, a contradiction.

Case 2: Suppose $c_1\notin V(P)$. Let P' : $g_0,g_1,g_2,\ldots,g_k(=c_1)$ be a shortest path that connects c_1 to the path P ($g_0\in V(P)$, $g_i\notin V(P)$ for $1\leqslant i\leqslant k$). For convenience, define g_{k+1} to be c_0.

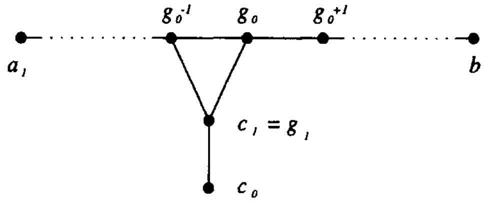

Fig. 3.

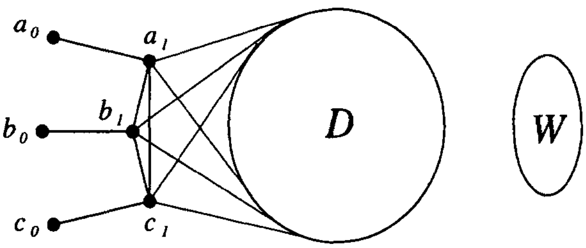

Fig. 4.

Suppose $g_0 = a_1$ and consider $\langle\{a_1, a_0, a_1^{+1}, g_1\}\rangle$. This is a potential claw, and since $\deg(a_0) = 1$, we see that $g_1 a_1^{+1} \in E(G)$. Thus, we may assume for convenience that $g_0 \in V(P)\backslash\{a_1, b_1\}$. Considering $\langle\{g_0, g_0^{-1}, g_0^{+1}, g_1\}\rangle$, we can see that one of $g_0^{-1}g_1$ and $g_0^{+1}g_1$ must be present (the edge $g_0^{-1}g_0^{+1}$ produces a shorter a_1–b_1 path). Without loss of generality, assume that $g_0^{-1}g_1 \in E(G)$.

If $k \geqslant 2$, we can consider $\langle\{g_0^{-1}, g_0, g_1, g_2, g_3\}\rangle$, a potential induced Z_2. To prevent this Z_2 from existing, one of the edges $g_0^{-1}g_3$, $g_0^{-1}g_2$, g_0g_3, g_0g_2, g_1g_3 must be present. However, each of these edges can be shown to yield a shorter path from c_1 to P. Therefore, it must be that $k = 1$, that is, $g_1 = c_1$ (see Fig. 3).

Consider the potential Z_2 formed by the vertices c_1, g_0^{-1}, g_0, g_0^{+1}, and g_0^{+2}. One of the edges $g_0^{-1}g_0^{+2}$, $g_0^{-1}g_0^{+1}$, $c_1g_0^{+2}$, $c_1g_0^{+1}$, $g_0g_0^{+2}$ must be present. We can exclude $c_1g_0^{+2}$ and $c_1g_0^{+1}$ from consideration since each would yield an induced $K_{1,3}$: $\langle\{c_1, g_0^{+2}, g_0^{-1}, c_0\}\rangle$ and $\langle\{c_1, g_0^{+1}, g_0^{-1}, c_0\}\rangle$, respectively. Furthermore, we can exclude $g_0^{-1}g_0^{+1}$ since it produces a shorter a_1–b_1 path. Thus, one of $g_0^{-1}g_0^{+2}$ or $g_0g_0^{+2}$ must be present. This implies that $g_0^{+2} \neq b_0$ since b_0 has degree 1. But then each of $g_0^{-1}g_0^{+2}$ and $g_0g_0^{+2}$ provides a shorter a_1–b_1 path, which is a contradiction.

Therefore, it must be that a_1 is adjacent to b_1, proving Claim 2.4. □

The argument used in the proof of Claim 2.4 can also be used to show that $a_1c_1, b_1c_1 \in E(G)$. Thus, we have an induced E_3 $(=N)$.

Let $l = \max\{k$: G contains an induced E_k which contains a_0, b_0, and $c_0\}$, and let S be a subgraph of G that is isomorphic to E_l. Note that $3 \leqslant l < r$ and that $|V(S)| = l + 3 < r + 3$.

Define two sets: $D = V(S)\backslash\{a_0, b_0, c_0, a_1, b_1, c_1\}$ and $W = V(G)\backslash V(S)$ (see Fig. 4).

Note that $|D| = l - 3$ and also that since $n \geqslant 2r$ we have $|W| = n - |V(S)| > n - (r + 3) \geqslant r - 3 > 0$.

Thus, $W \neq \emptyset$. Let $x \in W$ be a vertex adjacent to some vertex of S. First of all, we know that x is not adjacent to any of a_0, b_0, and c_0 since they each have degree 1. Suppose x is adjacent to a_1, and consider $\langle \{a_1, a_0, x, d\} \rangle$ where d ($\neq a_1$) is an arbitrary element of $V(S) \setminus \{a_0, b_0, c_0\}$. We know $a_0x, a_0d \notin E(G)$ since $\deg(a_0) = 1$, so since G is claw-free, the edge xd must be present. Since d was chosen arbitrarily, we can conclude that x is adjacent to all vertices of $V(S) \setminus \{a_0, b_0, c_0\}$. But then, $\langle V(S) \cup \{x\} \rangle$ is isomorphic to E_{l+1}, which contradicts the maximality of S. Hence, x cannot be adjacent to a_1, and by a similar argument, x is also nonadjacent to b_1 and c_1. Therefore no vertex of W is adjacent to any of a_0, b_0, c_0, a_1, b_1, or c_1. Since $|W| > 0$, there must be vertices of W that are adjacent to vertices of D (also implying that $D \neq \emptyset$).

If any vertex of W has distance 2 from D, we will immediately have an induced Z_2. Thus each vertex of W is adjacent to a vertex in D. Further, if $d \in D$ is adjacent to two vertices of W, say w and w', then $\langle \{d, w, w', a_1\} \rangle$ is a potential claw, implying that the edge ww' must be present. But then, $\langle \{w, w', d, a_1, a_0\} \rangle$ is an induced Z_2, which is a contradiction. Thus, each vertex of D can be adjacent to at most one vertex of W.

We therefore have that $|W| \leqslant |D| = l - 3 < r - 3$. This contradicts our earlier finding (p. 6) that $|W| > r - 3$.

We have reached the desired contradiction. Thus, G must be traceable. □

Corollary 1. *Let $r \geqslant 4$ be a fixed integer. Let R, S, and T be connected induced subgraphs of $K_{1,3}$, E_r, and Z_2, respectively. If G is a connected graph of order $n \geqslant 2r$ that is $\{R, S, T\}$-free, then G is traceable.*

3. Proof of Theorem 2

The proof of Theorem 2 is similar in many ways to that of the proof of Theorem C in [4].

Let $\gamma = \max\{r(m-2), 2r + (m-2)\} + 1$. We consider two cases.

Case 1: Suppose G is K_γ-free. Let w be a vertex with max degree, $\Delta(G)$. Since the neighborhood of any vertex cannot contain a $K_{\gamma-1}$ or a $\overline{K_r}$, it must be that $\Delta(G) < R(\gamma - 1, r)$, the Ramsey number associated with the integers $\gamma - 1$ and r.

For $i = 1, 2, \ldots$, let the set $N_i(w)$ be defined by $N_i(w) = \{v \in V(G)\colon d(w, v) = i\}$. Since the degree of every vertex in $N_1(w)$ is bounded by $R(\gamma - 1, r)$, we know that $|N_2(w)| < (R(\gamma - 1, r))^2$. By a similar argument, we can conclude that $|N_i(w)| < (R(\gamma - 1, r))^i$ for $i = 1, 2, \ldots$. We assume that n is large enough for $N_{l-1}(w)$ to be nonempty. But then if $a_{l-1} \in N_{l-1}(w)$, there must exist vertices $a_{l-2}, a_{l-3}, \ldots, a_1$ where $a_i \in N_i(w)$ such that $\{w, a_1, a_2, \ldots, a_{l-1}\}$ induces a P_l, which is a contradiction.

Therefore, G is not K_γ-free, and Case 1 cannot occur.

Case 2: Suppose G is not K_γ-free. Let C be a largest clique in G, and let $t = |V(C)|$ (thus $t \geqslant \gamma > r(m-2)$). Let $P = \{v \in V(G)\colon d(v, C) = 1\}$, where $d(v, C)$ denotes the distance from v to some vertex of C.

Claim 3.1. (i) *If* v *is a vertex of* P, *then* $|N_C(v)| \geqslant t-(m-2)$.
(ii) $V(G)=V(C)\cup P$.
(iii) $\alpha(P)<r$, *where* $\alpha(P)$ *is the independence number of* P.

Proof. Let $v \in P$, say it is adjacent to $a_0 \in V(C)$, and suppose that $|N_C(v)| \leqslant t-(m-2)-1=t-(m-1)$. Then v is nonadjacent to at least $m-1$ vertices of C. If we let $a_1,\dots,a_{m-1}$ be vertices of C that are not adjacent to v, then $\langle\{v,a_0,a_1,\dots,a_{m-1}\}\rangle$ is an induced Q_m, which is a contradiction.

Therefore, it must be that for every $v \in P$, $|N_C(v)| \geqslant t-(m-2)$, and (i) is proven.

If we suppose that (ii) is not true, then there must exist some vertex w such that $d(w,C)=2$ and such that w is adjacent to some vertex, say v, of P.

From (i) we know that v is adjacent to at least $t-(m-2)$ vertices of C. Moreover, we see that $t-(m-2)>r(m-2)-(m-2)=(m-2)(r-1)>m-2$ since $r \geqslant 4$. Thus v is adjacent to at least $m-1$ vertices of C. But then, if $a_1,\dots,a_{m-1}$ are vertices of C that are adjacent to v, we have that $\langle\{w,v,a_1,\dots,a_{m-1}\}\rangle$ is an induced Q_m, a contradiction.

Therefore, no such w can exist, and (ii) is proven.

Now, suppose $\alpha(P) \geqslant r$, and let $\{p_1,\dots,p_r\} \subseteq P$ be an independent set of vertices.

Consider the sets $N_C(p_1),\dots,N_C(p_r)$. They are all subsets of $V(C)$, and for each i,

$$|N_C(p_i)| \geqslant t-(m-2) > \left(1-\frac{1}{r}\right)t$$

since $t>r(m-2)$. It now follows (from a straightforward set systems argument) that

$$\left|\bigcap_{i=1}^{r} N_C(p_i)\right| > 0.$$

But then if we let a_0 be a member of this intersection, we have that $\langle\{a_0,p_1,\dots,p_r\}\rangle$ is an induced $K_{1,r}$, which contradicts our assumptions.

Therefore, it must be that $\alpha(P)<r$, and (iii) is proven. □

Now, partition the vertices of P into disjoint paths in the following manner. Let S_1 be a longest path (not necessarily induced) in $\langle P\rangle$, and say its end vertices are a_1 and b_1. Let S_2 be a longest path in $\langle P\backslash V(S_1)\rangle$, say with end vertices a_2 and b_2. Continue this process until the paths $S_1,S_2,\dots,S_k$ are obtained where $V(S_1)\cup V(S_2)\cup\cdots\cup V(S_k)=P$, and where for each $i\in\{1,2,\dots,k\}$, the path S_i has endpoints a_i and b_i. From this construction we see that $V(S_1),V(S_2),\dots,V(S_k)$ are necessarily disjoint. Further, due to the maximality of the paths, $\{a_1,a_2,\dots,a_k\}$ must be an independent set. Hence, from Claim 3.1, it must be that $k<r$.

Now, it might be the case that some of these paths are single vertices. Suppose that the paths $S_1,S_2,\dots,S_p$ are the paths with more than one vertex, and that $S_{p+1},S_{p+2},\dots,S_k$ are the single vertex paths. That is, $a_i=b_i$ for each $i\in\{p+1,p+2,\dots,k\}$. For $p+1\leqslant i\leqslant k$, split the vertex a_i into two distinct vertices a_i and b_i, and connect them with an edge. Further, place edges between b_i and all vertices of $N_C(a_i)$.

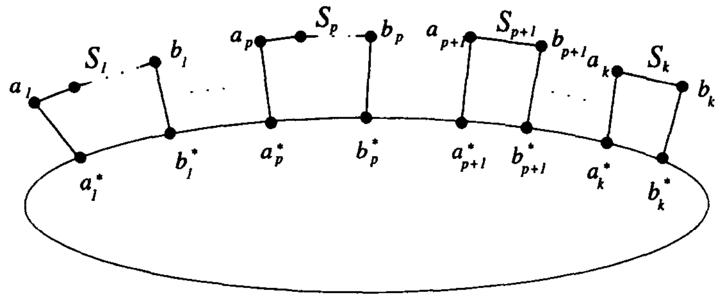

Fig. 5. The situation in G'.

This splitting creates a new graph G' (if $p=k$, then $G'=G$), and we have converted the single vertex paths $S_{p+1},\dots,S_k$ into two-vertex paths, while the other paths $S_1,\dots,S_p$ remained unchanged.

Consider the bipartite subgraph of G' defined as follows: $B=(X\cup Y, E(B)$ where $X=\{a_1,b_1,a_2,b_2,\dots,a_k,b_k\}$ and $Y=\bigcup_{i=1}^{k}(N_C(a_i)\cup N_C(b_i))$ and where $E(B)=\{xy\colon x\in X,\ y\in Y,$ and $xy\in E(G')\}$. Let $R\subseteq X$. Then in B we have

$$|R|\leqslant|X|=2k<2r<|N(R)|$$

since in G we have

$$|N_C(x)|>t-(m-2)>(2r+(m-2))-(m-2)=2r$$

for any $x\in R$. Thus, by a well-known theorem from Hall [6], X can be matched to a subset of Y. For each i, say that a_i and b_i in X are matched to a_i^* and b_i^* in Y, respectively (see Fig. 5).

We now show that G' is traceable by constructing a hamiltonian path. Let T_1 be the following path in G':

$$a_1^*,[a_1,b_1]_{S_1},b_1^*,a_2^*,[a_2,b_2]_{S_2},b_2^*,\dots,a_k^*,[a_k,b_k]_{S_k},b_k^*.$$

Let C' be the set of vertices of C that are not on the path T_1. It is clear that $\langle C'\rangle$ is complete, so let T_2 be a hamiltonian path for $\langle C'\rangle$, say with endpoints a_0 and b_0. Then a hamiltonian path for G' is given by

$$[a_1^*,b_k^*]_{T_1},[a_0,b_0]_{T_2}.$$

Therefore, we can conclude that G' is traceable. (Note: If P is empty, G' is still clearly traceable.) If $G'=G$ the proof is complete. So, assume that $G'\neq G$, and say that vertex $a_i\in G$ was split to form vertices a_i and b_i in G'. In our construction of a hamiltonian path in G', the edge a_ib_i was used (it was the path S_i). Hence, by identifying the vertex $a_i\in V(G')$ with the vertex $b_i\in V(G')$, we do not affect the existence of a spanning path. Identifying all pairs of vertices of G' that were a result of splitting, we obtain the graph G, and we see that G is also traceable. □

Note here that if we let $\Phi = R(\gamma - 1, r)$, then $n > 1 + \sum_{i=0}^{l-2}(\Phi - 1)(\Phi - 2)^i$ suffices in the proof. Also, notice that the result holds for $r < 3$ and/or $l < 4$ and/or $m < 3$ from the work in [2]. Moreover, if $r = 3$, the result follows from a theorem in [5], and if $l = 4$, the result follows from the theorem in Section 3 of [4].

Corollary 2. *Let $r \geqslant 4$, $l \geqslant 5$, and $m \geqslant 3$ be fixed integers. Let R, S, and T be connected induced subgraphs of $K_{1,r}$, P_l, and Q_m, respectively. If G is a connected graph of order n that is $\{R, S, T\}$-free, and if n is sufficiently large, then G is traceable.*

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