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On 2-factors containing 1-factors in bipartite graphs

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Abstract

Moon and Moser (Israel J. Math. 1 (1962) 163–165) showed that if G is a balanced bipartite graph of order $2n$ and minimum degree $\delta \geqslant (n+1)/2$, then G is hamiltonian. Recently, it was shown that their well-known degree condition also implies the existence of a 2-factor with exactly k cycles provided $n \geqslant \max\{52, 2k^2+1\}$. In this paper, we show that a similar degree condition implies that for each perfect matching M, there exists a 2-factor with exactly k cycles including all edges of M.

1. Introduction

All graphs considered are simple, without loops or multiple edges. An m-factor of a graph G is an m-regular subgraph of G that spans the vertex set $V(G)$. From time to time, we call a 1-factor *a perfect matching*. It is readily seen that a 1-factor of G is a collection of independent edges that covers all vertices of G and a 2-factor is a collection of independent cycles that covers all vertices of G. In 1952, Dirac [4] determined how large the minimum degree must be to guarantee the existence of a hamiltonian cycle, a 2-factor with exactly one cycle.

Theorem 1 (Dirac [4]). *Let G be a graph of order n ($n \geqslant 3$). If the minimum degree $\delta(G) \geqslant n/2$, then G has a hamiltonian cycle.*

Häggkvist [5] showed that when n is even, a similar hypothesis implies something much stronger.

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Theorem 2 (Häggkvist [5]). *Let G be a graph on n vertices, in which the degree sum of any two nonadjacent vertices is at least* $n+1$, *where* $n\geqslant 3$. *Then each perfect matching is contained in a hamiltonian cycle.*

Later, stronger results were obtained by Berman [1] and Jackson and Wormald [6]. Recently, Dirac's result has been generalized as follows.

Theorem 3 (Brandt et al. [2]). *Let k be a positive integer and G be a graph of order n* ($n\geqslant 4k$). *If the minimum degree* $\delta(G)\geqslant n/2$, *then G contains a 2-factor with exactly k components.*

We believe that similar hypothesis can also imply that each perfect matching is contained in a 2-factor with exactly k components, for every $k\leqslant n/4$. The purpose of this paper is to support this thought by proving a similar result for bipartite graphs. A bipartite graph $(X,Y;E)$ is called balanced if $|X|=|Y|$. A bipartite graph has a 2-factor only if it is balanced. Moon and Moser [7] obtained the following hamiltonian result for balanced bipartite graphs using a degree sum condition.

Theorem 4 (Moon and Moser [7]). *Let G be a balanced bipartite graph on* $2n$ *vertices. If* $d(u)+d(v)>n$ *for every two nonadjacent vertices u and v in different parts of G, then G is hamiltonian. Hence, if* $\delta(G)\geqslant(n+1)/2$, *then G is hamiltonian.*

Theorem 4 was recently generalized in [3].

Theorem 5 (Chen et al., preprint). *Let k be a positive integer and let G be a balanced bipartite graph of order* $2n$ *where* $n\geqslant\max\{52,2k^2+1\}$. *Then, if* $\delta(G)\geqslant(n+1)/2$, *G contains a 2-factor with exactly k cycles.*

Las Vergnas proved the following in [8].

Theorem 6 (Las Vergnas [8]). *Let G be a balanced bipartite graph of order* $2n$. *If*

$$d(u)+d(v)\geqslant n+2$$

for every pair of nonadjacent vertices u and v (in different parts), then each perfect matching of G is contained in a hamiltonian cycle.

The purpose of this paper is to prove the following related result.

Theorem 7. *Let k be a positive integer and let G be a balanced bipartite graph of order* $2n$ *where* $n\geqslant 9k$. *If* $\delta(G)\geqslant(n+2)/2$, *then for every perfect matching M, G has a 2-factor with exactly k components including every edge of M.*

Remark. Since the conclusion is that G contains at least k vertex-disjoint cycles, it is readily seen that $n \geqslant 2k$ is necessary. The condition $n \geqslant 9k$ comes from our proof techniques. The following example shows that $n > 3k$ is necessary.

Example. Form a bipartite graph H as follows: Take independent sets of vertices of cardinality $k = |V_i| = |W_i|$ for $i = 0, 1, 2$. Now place all edges between V_i and W_{i+1} as well as between V_i and W_i (subscripts taken mod 3). In addition place a matching between the sets V_1 and W_0, V_2 and W_1, and between V_0 and W_2. These edges form the matching M. It is now easily seen that any cycle containing alternating matching and nonmatching edges must have length at least 6. Thus, the full range of possible cycles is not available, hence $n > 3k$. □

It is not difficult to see that the minimum condition $\delta \geqslant (n+2)/2$ is best possible for $k = 1$. However, for $k \geqslant 2$, the minimum degree $\delta \geqslant n/2$ is necessary. When $k > 2$, $\delta = n/2$ is not sufficient. For example, the graph $G = 2K_{r,r}$ (for r odd) fails to have a 2-factor with exactly r cycles. It is unknown whether $(n+1)/2$ is sufficient when $k \geqslant 2$.

In the following we will reserve the graph $G = (X, Y; E)$ to be a balanced bipartite graph of order $2n$. Let G be a balanced bipartite graph and M a perfect matching of G. A cycle C is called an M-cycle if every other edge of C belongs to M, a path $P[u, v]$ is called an M-path if the cycle $P[u, v]u$ is an M-cycle, and a 2-factor of G is called an M-2-factor if every component of the 2-factor is an M-cycle. For any two disjoint subgraphs A and B of G, let $E(A, B)$ denote the set of edges with one endvertex in A and the other endvertex in B and set $e(A, B) = |E(A, B)|$. In the case $A \subseteq X$ and M is a matching, we define

$$\underline{A} = \{y \in Y\colon\ xy \in M \text{ and } x \in A\}.$$

If $A \subset Y$ then $\underline{A}$ is defined analogously. Further, for any $W \subseteq V(G)$, we let $\langle W \rangle$ denote the subgraph induced by W. For each vertex $v \in V(G)$, we let $N_H(v) = N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$.

2. The proof of Theorem 7

The proof will be divided into lemmas. It is readily seen that the main theorem follows from Lemmas 1 and 5.

Lemma 1. *Let k be a positive integer. If $n \geqslant 9k$ and the minimum degree $\delta(G) \geqslant n/2$, then for every perfect matching M, G contains k vertex-disjoint M-cycles.*

Lemma 2. *Let M be a perfect matching in G and suppose $C_1 = x_1y_1x_2y_2\ldots x_sy_sx_1$ is a longest M-cycle in G with $x_iy_i \in M$ for $i = 1, 2, \ldots, s$ and $G - V(C)$ has hamiltonian cycle $C_2 = u_1v_1u_2v_2\ldots u_tv_tu_1$ with $u_jv_j \in M$ for $j = 1, 2, \ldots, t$. If $N(u_i) \cap V(C_1) \neq \emptyset$ and*

$N(v_{i-1}) \cap V(C_1) \neq \emptyset$, *then*

$$d(u_i) + d(v_{i-1}) \leqslant n + 1.$$

Lemma 3. *Let M be a perfect matching of G and let C be a longest M-cycle and let $P = u_1v_1u_2v_2\ldots u_tv_t$ be an M-path in $G - V(C)$, then*

$$d_C(u_1) + d_C(v_t) \leqslant |V(C)|/2.$$

Lemma 4. *Let M be a perfect matching in G. If for every M-path $u_1v_1u_2v_2\ldots u_tv_t$, we have $d(u_1) + d(v_t) \geqslant n + 2$, then G has a hamiltonian cycle which includes every edge of M.*

Lemma 5. *Let M be a perfect matching of G. If the minimum degree $\delta \geqslant (n+2)/2$ and G contains k vertex-disjoint M-cycles, then G contains an M-2-factor with exactly k cycles.*

2.1. Proof of Lemma 1

In fact, we will show that G has k vertex-disjoint M-cycles, which are either 4-cycles or 6-cycles. To the contrary, we assume that G has t vertex-disjoint M-cycles of length 4 or 6 with $t \leqslant k - 1$ and G does not contain $t + 1$ vertex-disjoint M-cycles of lengths 4 or 6. Note that t may be zero. Let $C_1, C_2, \ldots, C_t$ be t vertex-disjoint cycles such that $\sum |V(C_i)|$ is minimum under the constraint $|V(C_i)| \leqslant 6$. Without loss of generality, we assume that $C_1, C_2, \ldots, C_s$ are 4-cycles and $C_{s+1}, \ldots, C_t$ are 6-cycles. Let $H = G - \bigcup_{i=1}^{t} V(C_i)$.

Claim 1. *Let uv be an edge of M in H and C_i $(s + 1 \leqslant i \leqslant t)$ a 6-cycle, then $e(\{u, v\}, C_i) \leqslant 3$.*

Proof. Let $C_i = u_1v_1u_2v_2u_3v_3u_1$ with $u_1v_1, u_2v_2, u_3v_3 \in M$. For each j $(j = 1, 2, 3)$, we cannot have both $uv_j \in E$ and $vu_j \in E$, otherwise we can use the 4-cycle $u_jv_juvu_j$ to replace C_i, which contradicts the minimality of $\sum_{j=1}^{t} |V(C_i)|$. □

Claim 2. *There is an edge $u_0v_0 \in M \cap H$ such that*

$$e(\{u_0, v_0\}, V(C_i)) \leqslant 3 \quad \textit{for each } i = 1, 2, \ldots, s.$$

Proof. To the contrary, assume that for every edge $uv \in M \cap H$ there is a cycle C_i, $i = 1, 2, \ldots, s$ such that $e(\{u, v\}, C_i) \geqslant 4$, that is, $V(C_i) \cup \{u, v\}$ induces a complete bipartite graph $K_{3,3}$ (or we could swap cycles to find the needed edge). Since

$$|M \cap H| \geqslant |V(H)|/2 \geqslant \frac{2n - 6t}{2} \geqslant \frac{9k - 6t}{2} \geqslant 3k > t,$$

by the Pigeonhole principle, $M\cap H$ contains two edges u_1v_1 and u_2v_2 and there is a 4-cycle C_i $(i\leqslant s)$ such that

$$e(\{u_1,v_1\},C_i)=4 \quad\text{and}\quad e(\{u_2,v_2\},C_i)=4.$$

Then, it is readily seen that the induced subgraph $\langle\{u_1,v_1,u_2,v_2\}\cup V(C_i)\rangle$ contains two M-cycles of length 4 in H, which contradicts the maximality of the number of 4-cycles (M-cycles) and 6-cycles (M-cycles). This contradiction completes the proof of the claim. □

Now suppose $u_0v_0\in M\cap H$ such that

$$e(\{u_0,v_0\},C_i)\leqslant 3 \quad \text{for } i=1,2,\ldots,t.$$

For convenience, let

$$n_1=\left(\sum_{i=1}^{t}|V(C_i)|\right)\Big/2$$

and

$$n_2=|N_H(u_0)-\{v_0\}|$$

and

$$n_3=|N_H(v_0)-\{u_0\}|$$

and

$$n_4=\frac{|V(H)|}{2}-|N_H(u_0)\cup N_H(v_0)|.$$

Since H contains no M-cycle of length 4, $N_H(u_0)\cap N_H(v_0)=\{u_0\}$. In particular, $n=n_1+n_2+n_3+n_4+1$. Note that $n_1=3t-s$ and $n_2+n_3+2=d_H(u_0)+d_H(v_0)\geqslant n-3t$, that is, $n_2+n_3\geqslant n-3t-2$. Thus,

$$n_1+n_4\leqslant n-(n-3t-2)-1=3t+1.$$

Without loss of generality, in the remainder of the proof we assume that $n_2\leqslant n_3$.

Claim 3. *For every $x\in N_H(u_0)$, the inequality $|N(x)\cap N_H(u_0)-\{u_0\}|\geqslant(n_2+2)/2$ holds.*

Proof. To the contrary, we assume $|N(x)\cap N_H(u_0)-\{v_0\}|\leqslant(n_2+1)/2$. Since H contains no M-cycle of length 6, $N_H(x)\cap(N(v_0)-\{v_0\})=\emptyset$, which implies

$$|N_H(x)\cap N_H(u_0)-\{v_0\}|\geqslant d(x)-(n_1+n_4+1)\geqslant\frac{n+2}{2}-(n_1+n_4+1).$$

Thus,

$$(n+2)/2-(n_1+n_4+1)\leqslant(n_2+1)/2. \tag{1}$$

Since $n_2 \leqslant n_3$, we have

$$n_2 \leqslant ((n-(n_1+n_4+1))/2. \tag{2}$$

Substituting Eq. (2) into Eq. (1) we obtain $n \leqslant (n-(n_1+n_4+1))/2+2(n_1+n_4)+1$. Upon solving we see that

$$n \leqslant 3(n_1+n_4)+2<9t+5<9k,$$

which contradicts the assumption $n \geqslant 9k$, completing the proof of Claim 3. □

We consider the subgraph $G(X_1 \cup Y_1)$ induced by the union of $Y_1 = N_H(u_0) - \{v_0\}$ and $X_1 = \underline{Y_1}$. Clearly, $|X_1| = |Y_1| = n_2$. Then Claim 3 shows that $|N(x) \cap Y_1| \geqslant (|Y_1|+2)/2 = (n_2+2)/2$ for each $x \in X_1$. By the Pigeonhole principle, there is a $y_0 \in Y_1$ such that $|N(y_0) \cap X_1| \geqslant (|X_1|+2)/2 = (n_2+2)/2$. Assume $x_0 y_0 \in M$. Then, $|\underline{N(x_0)} \cap N(y_0)| \geqslant 2$. Thus, $G(X_1 \cup Y_1)$ contains an M-cycle of length 4 in H, a contradiction to our choice of $C_1, \ldots, C_t$. This contradiction completes the proof of Lemma 1. □

2.2. *Proof of Lemma 2*

First we note that $s+t=n$. Now, without loss of generality, we assume that $i=1$ (and in this case that $i-1$ is s). Since $N(u_1) \cap V(C_1) \neq \emptyset$ and $N(v_t) \cap V(C_1) \neq \emptyset$, we may assume that $u_1 y_s \in E$ and that the closest neighbor of y_s along C_1 from v_t is x_{r+1}. That is, we assume that $u_1 y_s, v_t x_{r+1} \in E$ and that

$$N(u_1) \cap \{y_1, \ldots, y_r\} = \emptyset,$$

$$N(v_s) \cap \{x_1, \ldots, x_r\} = \emptyset.$$

Since C_1 is a longest M-cycle, $u_1 y_i \in E$ implies that $v_t x_{i+1} \notin E$ or a longer cycle is formed. For that same reason, we have $r \geqslant t$. Thus,

$$d_{C_1}(v_t) \leqslant \frac{|V(C_1)|}{2} - r - (d_{C_1}(u_1) - 1)$$

or

$$d_{C_1}(u_1) + d_{C_1}(v_t) \leqslant |V(C_1)|/2 - (r-1) \leqslant n - 2t + 1,$$

which implies that

$$d(u_1) + d(v_t) \leqslant (n-2t+1) + 2t = n+1. \quad \square$$

2.3. *Proof of Lemma 3*

Assume $C = x_1 y_1 x_2 y_2 \ldots x_s y_s x_1$ with $x_i y_i \in M$ for $i = 1, 2, \ldots, s$. Since C is one of the longest M-cycles, $u_i y_i \in E$ implies $v_t x_{i+1} \notin E$. Then, $d_C(u_i) + d_C(v_t) \leqslant s = |V(C)|/2$. □

2.4. Proof of Lemma 4

We prove Lemma 4 by induction on n. Since $d(u_1)+d(v_t)\geqslant n+2$ implies $n\geqslant 2$, and for $n=2$, $G=K_{2,2}$, Lemma 4 is clearly true when $n=2$. Assume that Lemma 4 is true for balanced bipartite graphs with order less than $2n$. Let $G=(X,Y;E)$ be a balanced bipartite graph of order $2n$ and let $C_1=x_1y_1x_2y_2\dots x_sy_sx_1$ be a longest M-cycle of G. Further, we assume that $s<n$. Now let $H=G-V(C_1)$. For every M-path $u_1v_1\dots u_tv_t$ in H, by Lemma 3, $d_H(u_1)+d_H(v_t)\geqslant(n+2)-s=|V(H)|/2+2$. By the induction hypothesis, H has a hamiltonian cycle $C_2=u_1v_2u_2v_2\dots u_mv_mu_1$ with $u_iv_i\in M$ for each $i=1,2,\dots,m$. By Lemma 2, either we have $N(u_i)\cap V(C_1)=\emptyset$ or $N(v_{i-1})\cap V(C_1)=\emptyset$. Also, since C is a longest M-cycle in G, $s\geqslant m$. Furthermore, for any two vertices u_i and v_j, either $N(u_i)\cap V(C_1)\neq\emptyset$ or $N(u_j)\cap V(C_1)\neq\emptyset$. Otherwise

$$n+2\leqslant d(u_i)+d(v_j)\leqslant|V(H)|\leqslant n,$$

a contradiction to our degree condition. Therefore, either

$$N(u_i)\cap V(C_1)\neq\emptyset \text{ and } N(v_j)\cap V(C_1)=\emptyset \quad \text{for all } i \text{ and } j$$

or

$$N(u_i)\cap V(C_1)=\emptyset \text{ and } N(v_j)\cap V(C_1)\neq\emptyset. \quad \text{for all } i \text{ and } j.$$

Without loss of generality, assume $u_1y_1\in E$, then $N(x_1)\cap N(H)=\emptyset$ or a cycle longer than C is formed and $N(v_1)\cap V(C_1)=\emptyset$ follows by the above conditions. This implies that $d(x_1)+d(v_1)\leqslant n$, a contradiction. □

2.5. Proof of Lemma 5

Let $C_1,C_2,\dots,C_k$ be k vertex-disjoint M-cycles in G such that $\sum_{i=1}^{k}|V(C_i)|$ is maximum over all such possible choices. Assume $|V(C_i)|=2n_i$ for each $i=1,2,\dots,k$ and $n_1\geqslant n_2\geqslant\cdots\geqslant n_k$. Let $n_{k+1}=n-\sum_{i=1}^{k}n_i$, that is, $|V(G-\bigcup_{i=1}^{k}V(C_i))|=2n_{k+1}$. Let $H=G-V(\bigcup_{i=1}^{k}C_i)$. By Lemma 3, for each M-path $P[u,v]$ in H and each cycle C_i, we have that $d_{C_i}(u)+d_{C_i}(v)\leqslant n_i$. In particular then,

$$d_H(u)+d_H(v)\geqslant n_{k+1}+2.$$

By Lemma 4, H has a hamiltonian cycle C_{k+1} which is also an M-cycle. From the choice of C_1, $C_2,\dots,C_k$, we can assume that $n_1\geqslant n_2\geqslant\cdots\geqslant n_{k+1}$. Let $C_{k+1}=u_1v_1u_2v_2\dots u_{n_{k+1}}v_{n_{k+1}}u_1$ with $u_iv_i\in M$ for each $i=1,2,\dots,n_{k+1}$. By Lemma 3, for any two vertices u_l and v_m, we have that

$$d_{C_i}(u_l)+d_{C_i}(v_m)\leqslant n_i$$

for each $i=1,2,\dots,k$. Thus, from our minimum degree condition, for each $i=1,2,\dots,k$,

$$|N(u_l)\cap(V(C_i\cup C_{k+1}))|+|N(v_m)\cap V(C_i\cup C_{k+1}))|\geqslant n_i+n_{k+1}+2.$$

Since C_k is the longest M-cycle in $\langle V(C_k \cup C_{k+1})\rangle$, by Lemma 2, we have that either $N(v_j)\cap V(C_k)=\emptyset$ or $N(u_{j+1})\cap V(C_k)=\emptyset$ for each $j=1,2,\ldots,t$. Since the above statements are true for all $j=1,2,\ldots,n_{k+1}$, without loss of generality, we assume that

$$N_{C_k}(u_j)\neq\emptyset \text{ and } N_{C_k}(v_l)=\emptyset \quad \text{for all } j \text{ and } l.$$

Claim 4. *We have that $n_{k+1}=2$ and for each C_m $(1\leqslant m\leqslant k)$, either $N(u_i)\supseteq V(C_m)\cap Y$ for both $i=1$ and 2 or $N(v_j)\supseteq V(C_m)\cap X$ for both $j=1$ and 2, but, not both.*

Proof. To the contrary, we assume $n_{k+1}\geqslant 3$. Then, $d_H(u_1)+d_H(v_1)\geqslant n_{k+1}+2\geqslant 5$. Without loss of generality, we assume $d_H(u_1)\geqslant 3$. Assume $u_1v_s\in E$ with $1<s<n_{k+1}$. Then, $V(H)$ can be partitioned into an M-cycle $C^*=u_1v_1u_2v_2\ldots u_sv_su_1$ and an M-path $P=u_{s+1}v_{s+1}\ldots u_{n_{k+1}}v_{n_{k-1}}$.

Assume $C_k=x_1y_1x_2y_2\ldots x_{n_k}y_{n_k}x_1$ with $x_iy_i\in M$ and, without loss of generality, assume $u_{s+1}y_{n_k}\in E$ (or we would relabel vertices). We consider the M-path

$$Q=x_1y_1x_2y_2\ldots x_{n_k}y_{n_k}u_{s+1}v_{s+1}\ldots u_{n_{k+1}}v_{n_{k+1}}.$$

Since $N(x_1)\cap V(C_{k+1})=\emptyset$ (or we contradict our choice of cycles $C_1,\ldots,C_k$) and $N(v_{n_{k+1}})\cap V(C_k)=\emptyset$, we have

$$|N(x_1)\cap V(C_k\cup C_{k+1})|+|N(v_{k+1})\cap V(C_k\cup C_{k+1})|\leqslant n_k+n_{k+1}.$$

However, since $d(x_1)+d(v_{n_{k-1}})\geqslant n+2$, there must exist a cycle C_i such that

$$d_{C_i}(x_1)+d_{C_i}(v_{n_{k+1}})\geqslant n_i+2.$$

By Lemma 2, C_i can be extended to a hamiltonian M-cycle in $\langle V(C_i\cup Q)\rangle$, which implies that G has an M-2-factor with exactly k cycles, a contradiction to our assumptions. Hence, $n_{k+1}=2$.

Now, for each M-path $x_1y_1\ldots x_jy_j$ in $\langle V(C_k)\rangle$, we have

$$d_{G-V(C_{k+1})}(x_1)+d_{G-V(C_{k+1})}(y_j)\geqslant (n-n_{k+1})+2.$$

In the same manner as above, we can show that $n_k=2$ and for each C_m with $1\leqslant m\leqslant k-1$ either

$$N(V(C_k))\cap V(C_i)\cap X=\emptyset$$

or

$$N(V(C_k))\cap V(C_i)\cap Y=\emptyset.$$

Continuing in this manner, we can show that $n_3=n_4=\cdots=n_k=n_{k+1}=2$ and for all $i=1,2$ and $j=3,4,\ldots,k+1$, either

$$N(V(C_j))\cap V(C_i)\cap X=\emptyset \quad \text{or } N(V(C_j))\cap V(C_i)\cap Y=\emptyset, \quad \text{but not both hold.}$$

Further, we assume that

$$C_1 = x_1y_1x_2y_2\dots x_{n_1}y_{n_1}x_1, \quad \text{where } x_iy_i \in M \text{ for each } i,$$

$$C_2 = u_1v_1u_2v_2\dots u_{n_2}v_{n_2}u_1, \quad \text{where } u_jv_j \in M \text{ for each } j.$$

For any two vertices $u_i \in V(C_2)$ and $v_j \in V(C_2)$, since for each $m \geqslant 3$ either $d_{C_m}(u_i) = 0$ or $d_{C_m}(v_j) = 0$ holds, we have

$$d_{C_1 \cup C_2}(u_i) + d_{C_1 \cup C_2}(v_j) \geqslant n_1 + n_2 + 2.$$

In particular, we obtain that either $d_{C_1}(u_i) \neq 0$ or $d_{C_1}(v_j) \neq 0$. Now by Lemma 2, either $d_{C_1}(u_{i+1}) = 0$ or $d_{C_1}(v_i) = 0$ for each $i = 1, 2, \dots, n_2$. Combining the above two statements, we obtain that either

$$N_{C_1}(V(C_2) \cap X) \neq \emptyset \quad \text{and} \quad N_{C_1}(V(C_2) \cap Y) = \emptyset$$

or

$$N_{C_1}(V(C_2) \cap X) = \emptyset \quad \text{and} \quad N_{C_1}(V(C_2) \cap Y) \neq \emptyset.$$

Without loss of generality, assume

$$N_{C_1}(V(C_2) \cap X) \neq \emptyset \quad \text{and} \quad N_{C_1}(V(C_2) \cap Y) = \emptyset.$$

Note that for each $x_i \in V(C_1) \cap X$ and each $y_j \in V(C_2) \cap Y$, $d_{C_2}(x_i) = 0$ and $d_{C_1}(y_j) = 0$, which gives

$$d_{G-V(C_1 \cup C_2)}(x_i) + d_{G-V(C_1 \cup C_2)}(v_j) \geqslant n_3 + n_4 + \cdots n_{k+1} + 2.$$

Without loss of generality, we assume that $C_3 = w_1z_1w_2z_2w_1$ and assume $N(x_i) \supseteq \{z_1, z_2\}$ for every $x_i \in V(C_1) \cap X$ and $N(v_j) \supseteq \{w_1, w_2\}$ for each $v_j \in V(C_2) \cap Y$. Since $N(v_j) \cap V(C_1) = \emptyset$ for each $v_j \in V(C_2) \cap Y$ and $d_{C_1 \cup C_2}(u_j) + d_{C_1 \cup C_2}(v_j) \geqslant n_1 + n_2 + 2$, we obtain $d_{C_1}(u_j) \geqslant 2$. Without loss of generality, we assume $u_1y_s \in E$ $(s \neq n_1)$ and $u_2y_{n_1} \in E$. Then, $\langle V(C_1 \cup C_2 \cup C_3) \rangle$ contains the following 2-factor with two M-cycles:

$$C_1^* = x_1y_1x_2y_2\dots x_sy_su_1v_1w_1z_1x_1,$$

$$C_2^* = x_{s+1}y_{s+1}x_{s+2}y_{s+2}\dots x_{n_1}y_{n_1}u_2v_2u_3v_3\dots u_{n_2}v_{n_2}w_2z_2x_{s+1}.$$

Then, C_1^*, C_2^*, C_4, $C_5, \dots, C_{k+1}$ form a 2-factor of G with exactly k M-cycles. □

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