On 2-factors containing 1-factors in bipartite graphs

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Abstract

Moon and Moser (Israel J. Math. 1 (1962) 163–165) showed that if \( G \) is a balanced bipartite graph of order \( 2n \) and minimum degree \( \delta \geq (n + 1)/2 \), then \( G \) is hamiltonian. Recently, it was shown that their well-known degree condition also implies the existence of a 2-factor with exactly \( k \) cycles provided \( n \geq \max\{52, 2k^2 + 1\} \). In this paper, we show that a similar degree condition implies that for each perfect matching \( M \), there exists a 2-factor with exactly \( k \) cycles including all edges of \( M \). © 1999 Published by Elsevier Science B.V. All rights reserved

1. Introduction

All graphs considered are simple, without loops or multiple edges. An \( m \)-factor of a graph \( G \) is an \( m \)-regular subgraph of \( G \) that spans the vertex set \( V(G) \). From time to time, we call a 1-factor a perfect matching. It is readily seen that a 1-factor of \( G \) is a collection of independent edges that covers all vertices of \( G \) and a 2-factor is a collection of independent cycles that covers all vertices of \( G \). In 1952, Dirac \[4\] determined how large the minimum degree must be to guarantee the existence of a hamiltonian cycle, a 2-factor with exactly one cycle.

Theorem 1 (Dirac [4]). Let \( G \) be a graph of order \( n \) (\( n \geq 3 \)). If the minimum degree \( \delta(G) \geq n/2 \), then \( G \) has a hamiltonian cycle.

Hägkvist [5] showed that when \( n \) is even, a similar hypothesis implies something much stronger.
Theorem 2 (Häggkvist [5]). Let \( G \) be a graph on \( n \) vertices, in which the degree sum of any two nonadjacent vertices is at least \( n + 1 \), where \( n \geq 3 \). Then each perfect matching is contained in a hamiltonian cycle.

Later, stronger results were obtained by Berman [1] and Jackson and Wormald [6]. Recently, Dirac's result has been generalized as follows.

Theorem 3 (Brandt et al. [2]). Let \( k \) be a positive integer and \( G \) be a graph of order \( n \) \((n \geq 4k)\). If the minimum degree \( \delta(G) \geq n/2 \), then \( G \) contains a 2-factor with exactly \( k \) components.

We believe that similar hypothesis can also imply that each perfect matching is contained in a 2-factor with exactly \( k \) components, for every \( k \leq n/4 \). The purpose of this paper is to support this thought by proving a similar result for bipartite graphs. A bipartite graph \((X,Y;E)\) is called balanced if \(|X| = |Y|\). A bipartite graph has a 2-factor only if it is balanced. Moon and Moser [7] obtained the following hamiltonian result for balanced bipartite graphs using a degree sum condition.

Theorem 4 (Moon and Moser [7]). Let \( G \) be a balanced bipartite graph on \( 2n \) vertices. If \( d(u) + d(v) > n \) for every two nonadjacent vertices \( u \) and \( v \) in different parts of \( G \), then \( G \) is hamiltonian. Hence, if \( \delta(G) \geq (n + 1)/2 \), then \( G \) is hamiltonian.

Theorem 4 was recently generalized in [3].

Theorem 5 (Chen et al., preprint). Let \( k \) be a positive integer and let \( G \) be a balanced bipartite graph of order \( 2n \) where \( n \geq \max\{52, 2k^2 + 1\} \). Then, if \( \delta(G) \geq (n + 1)/2 \), \( G \) contains a 2-factor with exactly \( k \) cycles.

Las Vergnas proved the following in [8].

Theorem 6 (Las Vergnas [8]). Let \( G \) be a balanced bipartite graph of order \( 2n \). If \( d(u) + d(v) \geq n + 2 \) for every pair of nonadjacent vertices \( u \) and \( v \) (in different parts), then each perfect matching of \( G \) is contained in a hamiltonian cycle.

The purpose of this paper is to prove the following related result.

Theorem 7. Let \( k \) be a positive integer and let \( G \) be a balanced bipartite graph of order \( 2n \) where \( n \geq 9k \). If \( \delta(G) \geq (n + 2)/2 \), then for every perfect matching \( M \), \( G \) has a 2-factor with exactly \( k \) components including every edge of \( M \).
Remark. Since the conclusion is that $G$ contains at least $k$ vertex-disjoint cycles, it is readily seen that $n \geq 2k$ is necessary. The condition $n \geq 9k$ comes from our proof techniques. The following example shows that $n > 3k$ is necessary.

Example. Form a bipartite graph $H$ as follows: Take independent sets of vertices of cardinality $k = |V_i| = |W_i|$ for $i = 0, 1, 2$. Now place all edges between $V_i$ and $W_{i+1}$ as well as between $V_i$ and $W_i$ (subscripts taken mod 3). In addition place a matching between the sets $V_0$ and $W_1$, $V_2$ and $W_1$, and between $V_0$ and $W_2$. These edges form the matching $M$. It is now easily seen that any cycle containing alternating matching and nonmatching edges must have length at least 6. Thus, the full range of possible cycles is not available, hence $n > 3k$. □

It is not difficult to see that the minimum condition $\delta \geq (n + 2)/2$ is best possible for $k = 1$. However, for $k \geq 2$, the minimum degree $\delta \geq n/2$ is necessary. When $k > 2$, $\delta = n/2$ is not sufficient. For example, the graph $G = 2K_{r,r}$ (for $r$ odd) fails to have a 2-factor with exactly $r$ cycles. It is unknown whether $(n + 1)/2$ is sufficient when $k > 2$.

In the following we will reserve the graph $G = (X, Y; E)$ to be a balanced bipartite graph of order $2n$. Let $G$ be a balanced bipartite graph and $M$ a perfect matching of $G$. A cycle $C$ is called an $M$-cycle if every other edge of $C$ belongs to $M$, a path $P[u, v]$ is called an $M$-path if the cycle $P[u, v]u$ is an $M$-cycle, and a 2-factor of $G$ is called an $M$-2-factor if every component of the 2-factor is an $M$-cycle. For any two disjoint subgraphs $A$ and $B$ of $G$, let $E(A, B)$ denote the set of edges with one endvertex in $A$ and the other endvertex in $B$ and set $e(A, B) = |E(A, B)|$. In the case $A \subseteq X$ and $M$ is a matching, we define

$$A = \{y \in Y: xy \in M \text{ and } x \in A\}.$$ 

If $A \subseteq Y$ then $A$ is defined analogously. Further, for any $W \subseteq V(G)$, we let $\langle W \rangle$ denote the subgraph induced by $W$. For each vertex $v \in V(G)$, we let $N_H(v) = N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$.

2. The proof of Theorem 7

The proof will be divided into lemmas. It is readily seen that the main theorem follows from Lemmas 1 and 5.

**Lemma 1.** Let $k$ be a positive integer. If $n \geq 9k$ and the minimum degree $\delta(G) \geq n/2$, then for every perfect matching $M$, $G$ contains $k$ vertex-disjoint $M$-cycles.

**Lemma 2.** Let $M$ be a perfect matching in $G$ and suppose $C_1 = x_1y_1x_2y_2 \ldots x_sy_s$ is a longest $M$-cycle in $G$ with $x_iy_i \in M$ for $i = 1, 2, \ldots, s$ and $G - V(C_1)$ has Hamiltonian cycle $C_2 = u_1v_1u_2v_2 \ldots u_tv_1u_1$ with $u_jv_j \in M$ for $j = 1, 2, \ldots, t$. If $N(u_i) \cap V(C_1) \neq \emptyset$ and
Lemma 3. Let $M$ be a perfect matching of $G$ and let $C$ be a longest $M$-cycle and let $P = u_1v_1u_2v_2 \ldots u_nv_i$ be an $M$-path in $G - V(C)$, then

$$d_C(u_1) + d_C(v_i) \leq |V(C)|/2.$$ 

Lemma 4. Let $M$ be a perfect matching in $G$. If for every $M$-path $u_1v_1u_2v_2 \ldots u_nv_i$, we have $d(u_1) + d(v_i) \geq n + 2$, then $G$ has a hamiltonian cycle which includes every edge of $M$.

Lemma 5. Let $M$ be a perfect matching of $G$. If the minimum degree $\delta \geq (n+2)/2$ and $G$ contains $k$ vertex-disjoint $M$-cycles, then $G$ contains an $M$-2-factor with exactly $k$ cycles.

2.1. Proof of Lemma 1

In fact, we will show that $G$ has $k$ vertex-disjoint $M$-cycles, which are either 4-cycles or 6-cycles. To the contrary, we assume that $G$ has $t$ vertex-disjoint $M$-cycles of length 4 or 6 with $t \leq k - 1$ and $G$ does not contain $t + 1$ vertex-disjoint $M$-cycles of lengths 4 or 6. Note that $t$ may be zero. Let $C_1, C_2, \ldots, C_t$ be $t$ vertex-disjoint cycles such that $\sum |V(C_i)|$ is minimum under the constraint $|V(C_i)| \leq 6$. Without loss of generality, we assume that $C_1, C_2, \ldots, C_s$ are 4-cycles and $C_{s+1}, \ldots, C_t$ are 6-cycles. Let $H = G - \bigcup_{i=1}^{t} V(C_i)$.

Claim 1. Let $uv$ be an edge of $M$ in $H$ and $C_i$ ($s + 1 \leq i \leq t$) a 6-cycle, then $e({u, v}, G) \geq 3$.

Proof. Let $C_i = u_1v_1u_2v_2u_3v_3u_1$ with $u_1v_1, u_2v_2, u_3v_3 \in M$. For each $j$ ($j = 1, 2, 3$), we cannot have both $uv_j \in E$ and $vu_j \in E$, otherwise we can use the 4-cycle $u_jv_juv_j$ to replace $C_i$, which contradicts the minimality of $\sum |V(C_i)|$. \(\Box\)

Claim 2. There is an edge $u_0v_0 \in M \cap H$ such that

$$e({u_0, v_0}, V(C_i)) \leq 3 \quad \text{for each } i = 1, 2, \ldots, s.$$ 

Proof. To the contrary, assume that for every edge $uv \in M \cap H$ there is a cycle $C_i$, $i = 1, 2, \ldots, s$ such that $e({u, v}, C_i) \geq 4$, that is, $V(C_i) \cup \{u, v\}$ induces a complete bipartite graph $K_{3,3}$ (or we could swap cycles to find the needed edge). Since

$$|M \cap H| \geq |V(H)|/2 \geq \frac{2n - 6t}{2} \geq \frac{9k - 6t}{2} \geq 3k > t,$$
by the Pigeonhole principle, \( M \cap H \) contains two edges \( u_1v_1 \) and \( u_2v_2 \) and there is a 4-cycle \( C_i \) (\( i \leq s \)) such that
\[
e(\{u_1, v_1\}, C_i) = 4 \quad \text{and} \quad e(\{u_2, v_2\}, C_i) = 4.
\]
Then, it is readily seen that the induced subgraph \( \langle \{u_1, v_1, u_2, v_2\} \cup V(C_i) \rangle \) contains two \( M \)-cycles of length 4 in \( H \), which contradicts the maximality of the number of 4-cycles (\( M \)-cycles) and 6-cycles (\( M \)-cycles). This contradiction completes the proof of the claim.

Now suppose \( u_0v_0 \in M \cap H \) such that
\[
e(\{u_0, v_0\}, C_i) \leq 3 \quad \text{for} \quad i = 1, 2, \ldots, t.
\]
For convenience, let
\[
n_1 = \left( \sum_{i=1}^{t} |V(C_i)| \right) / 2
\]
and
\[
n_2 = |N_H(u_0) - \{v_0\}|
\]
and
\[
n_3 = |N_H(v_0) - \{u_0\}|
\]
and
\[
n_4 = \frac{|V(H)|}{2} - |N_H(u_0) \cup N_H(v_0)|.
\]
Since \( H \) contains no \( M \)-cycle of length 4, \( N_H(u_0) \cap N_H(v_0) = \{u_0\} \). In particular, \( n = n_1 + n_2 + n_3 + n_4 + 1 \). Note that \( n_1 = 3t - s \) and \( n_2 + n_3 + 2 = d_H(u_0) + d_H(v_0) \geq n - 3t \), that is, \( n_2 + n_3 \geq n - 3t - 2 \). Thus,
\[
n_1 + n_4 \leq n - (n - 3t - 2) - 1 = 3t + 1.
\]
Without loss of generality, in the remainder of the proof we assume that \( n_2 \leq n_3 \).

**Claim 3.** For every \( x \in N_H(u_0) \), the inequality \( |N(x) \cap N_H(u_0) - \{u_0\}| \geq (n_2 + 2)/2 \) holds.

**Proof.** To the contrary, we assume \( |N(x) \cap N_H(u_0) - \{v_0\}| \leq (n_2 + 1)/2 \). Since \( H \) contains no \( M \)-cycle of length 6, \( N_H(x) \cap (N(v_0) - \{v_0\}) = \emptyset \), which implies
\[
|N_H(x) \cap N_H(u_0) - \{v_0\}| \geq d(x) - (n_1 + n_4 + 1) \geq \frac{n + 2}{2} - (n_1 + n_4 + 1).
\]
Thus,
\[
(n + 2)/2 - (n_1 + n_4 + 1) \leq (n_2 + 1)/2. \tag{1}
\]
Since \( n_2 \leq n_3 \), we have
\[
n_2 \leq \left( \frac{n - (n_1 + n_4 + 1)}{2} \right).
\] (2)

Substituting Eq. (2) into Eq. (1) we obtain
\[
n \leq \left( \frac{n - (n_1 + n_4 + 1)}{2} \right) + 2(n_1 + n_4) + 1.
\]
Upon solving we see that
\[
n \leq 3(n_1 + n_4) + 2 < 9t + 5 < 9k,
\]
which contradicts the assumption \( n \geq 9k \), completing the proof of Claim 3.  

We consider the subgraph \( G(X_1 \cup Y_1) \) induced by the union of \( Y_1 = N_H(u_0) - \{v_0\} \) and \( X_1 = Y_1 \). Clearly, \( |X_1| = |Y_1| = n_2 \). Then Claim 3 shows that \( |N(x) \cap Y_1| \geq \frac{|Y_1| + 2}{2} = (n_2 + 2)/2 \) for each \( x \in X_1 \). By the Pigeonhole principle, there is a \( y_0 \in Y_1 \) such that \( |N(y_0) \cap X_1| \geq \frac{|X_1| + 2}{2} = (n_2 + 2)/2 \). Assume \( x_0 y_0 \in M \). Then, \( |N(x_0) \cap N(y_0)| \geq 2 \). Thus, \( G(X_1 \cup Y_1) \) contains an \( M \)-cycle of length 4 in \( H \), a contradiction to our choice of \( C_1 \)....\( C_t \). This contradiction completes the proof of Lemma 1.  

2.2. Proof of Lemma 2

First we note that \( s + t = n \). Now, without loss of generality, we assume that \( i = 1 \) (and in this case that \( i - 1 \) is \( s \)). Since \( N(u_i) \cap V(C_1) \neq \emptyset \) and \( N(v_i) \cap V(C_1) \neq \emptyset \), we may assume that \( u_1 y_s \in E \) and that the closest neighbor of \( y_s \) along \( C_1 \) from \( v_t \) is \( x_{t+1} \). That is, we assume that \( u_1 y_s, v_t x_{t+1} \in E \) and that
\[
N(u_1) \cap \{y_1, \ldots, y_r\} = \emptyset,
N(v_t) \cap \{x_1, \ldots, x_r\} = \emptyset.
\]

Since \( C_1 \) is a longest \( M \)-cycle, \( u_1 y_i \in E \) implies that \( v_t x_{i+1} \notin E \) or a longer cycle is formed. For that same reason, we have \( r \geq t \). Thus,
\[
d_{C_1}(v_t) \leq \left| \frac{V(C_1)}{2} \right| - r - (d_{C_1}(u_t) - 1)
\]
or
\[
d_{C_1}(u_1) + d_{C_1}(v_t) \leq |V(C_1)| - (r - 1) \leq n - 2t + 1,
\]
which implies that
\[
d(u_1) + d(v_t) \leq (n - 2t + 1) + 2t = n + 1.
\]

2.3. Proof of Lemma 3

Assume \( C = y_1 x_2 y_2 \ldots x_s y_s x_1 \) with \( x_i y_i \in M \) for \( i = 1, 2, \ldots, s \). Since \( C \) is one of the longest \( M \)-cycles, \( u_i y_i \in E \) implies \( v_t x_{i+1} \notin E \). Then, \( d_C(u_i) + d_C(v_t) \leq s = \left| V(C) \right| / 2 \).  

2.4. Proof of Lemma 4

We prove Lemma 4 by induction on \( n \). Since \( d(u_i) + d(v_i) \geq n + 2 \) implies \( n \geq 2 \), and for \( n = 2 \), \( G = K_{2,2} \), Lemma 4 is clearly true when \( n = 2 \). Assume that Lemma 4 is true for balanced bipartite graphs with order less than \( 2n \). Let \( G = (X, Y; E) \) be a balanced bipartite graph of order \( 2n \) and let \( C_1 = x_1 y_1 x_2 y_2 \ldots x_n y_n x_1 \) be a longest \( M \)-cycle of \( G \). Further, we assume that \( s < n \). Now let \( H = G - V(C_1) \). For every \( M \)-path \( u_1 v_1 \ldots u_t v_t \) in \( H \), by Lemma 3, \( d_H(u_i) + d_H(v_i) \geq (n + 2) - s = |V(H)|/2 + 2 \). By the induction hypothesis, \( H \) has a Hamiltonian cycle \( C_2 = u_1 v_2 u_2 v_3 u_3 v_4 \ldots u_m v_m u_1 \) with \( u_i v_i \in M \) for each \( i = 1, 2, \ldots, m \). By Lemma 2, either we have \( N(u_i) \cap V(C_1) = \emptyset \) or \( N(v_{i-1}) \cap V(C_1) = \emptyset \). Also, since \( C \) is a longest \( M \)-cycle in \( G \), \( s \geq m \). Furthermore, for any two vertices \( u_i \) and \( v_j \), either \( N(u_i) \cap V(C_1) \neq \emptyset \) or \( N(v_j) \cap V(C_1) \neq \emptyset \). Otherwise

\[
|V(H)| \leq n,
\]

a contradiction to our degree condition. Therefore, either

\[
N(u_i) \cap V(C_1) \neq \emptyset \quad \text{and} \quad N(v_j) \cap V(C_1) = \emptyset \quad \text{for all} \quad i \quad \text{and} \quad j
\]
or

\[
N(u_i) \cap V(C_1) = \emptyset \quad \text{and} \quad N(v_j) \cap V(C_1) \neq \emptyset \quad \text{for all} \quad i \quad \text{and} \quad j.
\]

Without loss of generality, assume \( u_1 v_1 \in E \), then \( N(x_1) \cap N(H) = \emptyset \) or a cycle longer than \( C \) is formed and \( N(v_1) \cap V(C_1) = \emptyset \) follows by the above conditions. This implies that \( d(x_1) + d(v_1) \leq n \), a contradiction. \( \square \)

2.5. Proof of Lemma 5

Let \( C_1, C_2, \ldots, C_k \) be \( k \) vertex-disjoint \( M \)-cycles in \( G \) such that \( \sum_{i=1}^{k} |V(C_i)| \) is maximum over all such possible choices. Assume \( |V(C_i)| = 2n_i \) for each \( i = 1, 2, \ldots, k \) and \( n_1 \geq n_2 \geq \cdots \geq n_k \). Let \( n_{k+1} = n - \sum_{i=1}^{k} n_i \), that is, \( |V(G - \bigcup_{i=1}^{k} V(C_i))| = 2n_{k+1} \). Let \( H = G - \bigcup_{i=1}^{k} C_i \). By Lemma 3, for each \( M \)-path \( P[u,v] \) in \( H \) and each cycle \( C_i \), we have that \( d_{C_i}(u) + d_{C_i}(v) \leq n_i \). In particular then,

\[
d_H(u) + d_H(v) \geq n_{k+1} + 2.
\]

By Lemma 4, \( H \) has a Hamiltonian cycle \( C_{k+1} \) which is also an \( M \)-cycle. From the choice of \( C_1, C_2, \ldots, C_k \), we can assume that \( n_1 \geq n_2 \geq \cdots \geq n_{k+1} \). Let \( C_{k+1} = u_1 v_1 u_2 v_2 \ldots u_{n_{k+1}} v_{n_{k+1}} u_1 \) with \( u_i v_i \in M \) for each \( i = 1, 2, \ldots, n_{k+1} \). By Lemma 3, for any two vertices \( u_i \) and \( v_m \), we have that

\[
d_{C_i}(u_i) + d_{C_i}(v_m) \leq n_i
\]

for each \( i = 1, 2, \ldots, k \). Thus, from our minimum degree condition, for each \( i = 1, 2, \ldots, k \),

\[
|N(u_i) \cap (V(C_i \cup C_{k+1}))| + |N(v_m) \cap V(C_i \cup C_{k+1})| \geq n_i + n_{k+1} + 2.
\]
Since $C_k$ is the longest $M$-cycle in $\langle V(C_k \cup C_{k+1}) \rangle$, by Lemma 2, we have that either $N(v_j) \cap V(C_k) = \emptyset$ or $N(u_{j+1}) \cap V(C_k) = \emptyset$ for each $j = 1, 2, \ldots, t$. Since the above statements are true for all $j = 1, 2, \ldots, n_{k+1}$, without loss of generality, we assume that $N_{C_k}(u_j) \neq \emptyset$ and $N_{C_k}(v_i) = \emptyset$ for all $j$ and $i$.

**Claim 4.** We have that $n_{k+1} = 2$ and for each $C_m$ $(1 \leq m \leq k)$, either $N(u_i) \supseteq V(C_m) \cap Y$ for both $i = 1$ and $2$ or $N(v_j) \supseteq V(C_m) \cap X$ for both $j = 1$ and $2$, but, not both.

**Proof.** To the contrary, we assume $n_{k+1} \geq 3$. Then, $d_H(u_i) + d_H(v_i) \geq n_{k+1} + 2 \geq 5$. Without loss of generality, we assume $d_H(u_1) \geq 3$. Assume $u_1 v_s \in E$ with $1 < s < n_{k+1}$. Then, $V(H)$ can be partitioned into an $M$-cycle $C^* = u_1 v_1 u_2 v_2 \ldots u_t v_t u_1$ and an $M$-path $P = u_{n+1} v_{n+1} \ldots u_{n_{k+1}} v_{n_{k+1}}$.

Assume $C_k = x_1 y_1 x_2 y_2 \ldots x_{n_k + 1} y_{n_k + 1}$ with $x_i, y_i \in M$ and, without loss of generality, assume $x_{n+1} y_{n+1} \in E$ (or we would relabel vertices). We consider the $M$-path $Q = x_1 y_1 x_2 y_2 \ldots x_{n_k + 1} y_{n_k + 1} u_{n+1} v_{n+1} \ldots u_{n_{k+1}} v_{n_{k+1}}$.

Since $N(x_1) \cap V(C_{k+1}) = \emptyset$ (or we contradict our choice of cycles $C_1, \ldots, C_k$) and $N(v_{n_{k+1}}) \cap V(C_k) = \emptyset$, we have

$$|N(x_1) \cap V(C_k \cup C_{k+1})| + |N(v_{n_{k+1}}) \cap V(C_k \cup C_{k+1})| \leq n_k + n_{k+1}.$$  

However, since $d(x_1) + d(v_{n_{k+1}}) \geq n + 2$, there must exist a cycle $C_i$ such that $d_{C_i}(x_1) + d_{C_i}(v_{n_{k+1}}) \geq n_i + 2$.

By Lemma 2, $C_i$ can be extended to a hamiltonian $M$-cycle in $\langle V(C_i \cup Q) \rangle$, which implies that $G$ has an $M$-2-factor with exactly $k$ cycles, a contradiction to our assumptions. Hence, $n_{k+1} = 2$.

Now, for each $M$-path $x_1 y_1 \ldots x_j y_j$ in $\langle V(C_k) \rangle$, we have

$$d_{G \setminus V(C_{k+1})}(x_1) + d_{G \setminus V(C_{k+1})}(y_j) \geq (n - n_{k+1}) + 2.$$  

In the same manner as above, we can show that $n_k = 2$ and for each $C_m$ with $1 \leq m \leq k - 1$ either

$$N(V(C_k)) \cap V(C_i) \cap X = \emptyset$$  

or

$$N(V(C_k)) \cap V(C_i) \cap Y = \emptyset.$$  

Continuing in this manner, we can show that $n_2 = n_3 = \cdots = n_k = n_{k+1} = 2$ and for all $i = 1, 2$ and $j = 3, 4, \ldots, k+1$, either

$$N(V(C_j)) \cap V(C_i) \cap X = \emptyset$$  

or

$$N(V(C_j)) \cap V(C_i) \cap Y = \emptyset,$$  

but not both hold.
Further, we assume that
\[ C_1 = x_1 y_1 x_2 y_2 \ldots x_n y_n x_1, \quad \text{where } x_i y_i \in M \text{ for each } i, \]
\[ C_2 = u_1 v_1 u_2 v_2 \ldots u_n v_n u_1, \quad \text{where } u_i v_j \in M \text{ for each } j. \]

For any two vertices \( u_i \in V(C_2) \) and \( v_j \in V(C_2) \), since for each \( m \geq 3 \) either \( d_{C_1}(u_i) = 0 \) or \( d_{C_2}(v_j) = 0 \), we have
\[ d_{C_1 \cup C_2}(u_i) + d_{C_1 \cup C_2}(v_j) \geq n_1 + n_2 + 2. \]

In particular, we obtain that either \( d_{C_1}(u_i) \neq 0 \) or \( d_{C_2}(v_j) \neq 0 \). Now by Lemma 2, either \( d_{C_1}(u_{i+1}) = 0 \) or \( d_{C_1}(v_{i}) = 0 \) for each \( i = 1, 2, \ldots, n_2 \). Combining the above two statements, we obtain that either
\[ N_{C_1}(V(C_2) \cap X) \neq \emptyset \quad \text{and} \quad N_{C_1}(V(C_2) \cap Y) = \emptyset \]
or
\[ N_{C_1}(V(C_2) \cap X) = \emptyset \quad \text{and} \quad N_{C_1}(V(C_2) \cap Y) \neq \emptyset. \]

Without loss of generality, assume
\[ N_{C_1}(V(C_2) \cap X) \neq \emptyset \quad \text{and} \quad N_{C_1}(V(C_2) \cap Y) = \emptyset. \]

Note that for each \( x_i \in V(C_1) \cap X \) and each \( y_j \in V(C_2) \cap Y \), \( d_{C_1}(x_i) = 0 \) and \( d_{C_2}(y_j) = 0 \), which gives
\[ d_{G - V(C_1 \cup C_2)}(x_i) + d_{G - V(C_1 \cup C_2)}(v_j) \geq n_3 + n_4 + \cdots + n_{k+2} + 2. \]

Without loss of generality, we assume that \( C_3 = w_1 z_1 w_2 z_2 w_1 \) and assume \( N(x_i) \subseteq \{z_1, z_2\} \) for every \( x_i \in V(C_1) \cap X \) and \( N(v_j) \subseteq \{w_1, w_2\} \) for each \( v_j \in V(C_2) \cap Y \). Since \( N(v_i) \cap V(C_1) = \emptyset \) for each \( v_i \in V(C_2) \cap Y \), and \( d_{C_1 \cup C_2}(u_i) + d_{C_1 \cup C_2}(v_j) \geq n_1 + n_2 + 2 \), we obtain \( d_{C_1}(u_i) \geq 2 \). Without loss of generality, we assume \( u_1 y_1 \in E \) \( (s \neq n_1) \) and \( u_2 y_2 \in E \). Then, \( V(C_1 \cup C_2 \cup C_3) \) contains the following 2-factor with two \( M \)-cycles:
\[ C_1^* = x_1 y_1 x_2 y_2 \ldots x_n y_n u_1 v_1 w_1 z_1 x_1, \]
\[ C_2^* = x_{i+1} y_{i+1} x_{i+2} y_{i+2} \ldots x_n y_n u_2 v_2 u_3 v_3 \ldots u_{n_2} v_{n_2} w_2 z_2 x_1. \]

Then, \( C_1^*, C_2^*, C_4, C_5, \ldots, C_{k+1} \) form a 2-factor of \( G \) with exactly \( k \) \( M \)-cycles.

References