Potential G-graphical degree sequences

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Abstract

In this paper we consider a variation of the classical Turán-type extremal problems. Let $S$ be an $n$-element graphical sequence, and $\sigma(S)$ be the sum of the terms in $S$. Let $G$ be a graph. The problem is to determine the smallest $m$ such that any $n$-term graphical sequence $S$ having $\sigma(S) \geq m$ has a realization containing $G$. Denote this value $m$ by $\sigma(G, n)$. We present several results for this parameter for various graphs $G$. In particular, we show $\sigma(K_4, n) = 4n - 4$ for $n \geq 9$.

1 Introduction.

There are several famous results, Havel and Hakimi [6, 5] and Erdős and Gallai [3], which give necessary and sufficient conditions for a sequence $S = (d_1, d_2, \ldots, d_n)$ to be the degree sequence of a simple graph $G$. Unfortunately, knowing that a sequence has a realization gives no information about the properties that such a graph might have. In this paper, we explore this question of properties of graphs with a given degree sequence which is related to work originally introduced by A. R. Rao [7].

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For convenience, we employ the following terminology. If \( S = (d_1, d_2, \ldots, d_n) \) is a sequence of non-negative integers then it is called \textit{graphical} if there is a (simple) graph of order \( n \), whose degrees are precisely the terms in \( S \). If \( G \) is such a graph then \( G \) is said to \textit{realize} \( S \) or be a \textit{realization} of \( S \). A graphical sequence \( S \) is \textit{potentially} \( G \)-\textit{graphical} if there is a realization of \( S \) containing \( G \), while \( S \) is \textit{forcibly} \( G \)-\textit{graphical} if every realization of \( S \) contains \( G \). Throughout the paper subgraph means non-induced subgraph. For any undefined terms, refer to [1].

One of the classical extremal problems is to determine the minimum even integer \( m \) such that every \( n \)-term graphical sequence \( S \) with \( \sigma(S) \geq m \) is forcibly \( G \)-graphical; this \( m \) is denoted \( \text{ex}(G, n) \). Here we consider the following variant: determine the minimum even integer \( m \) such that every \( n \)-term graphical sequence \( S \) with \( \sigma(S) \geq m \) is potentially \( G \)-graphical. We denote this minimum \( m \) by \( \sigma(G, n) \).

This problem was considered by Erdős, Jacobson and Lehel [4] where they showed the following:

**Theorem A.** For \( n \geq 6 \), \( \sigma(K_3, n) = 2n \).

They also gave a construction that gave a lower bound for \( \sigma(K_k, n) \), which also would be a bound for \( \sigma(G, n) \) for any graph \( G \) of order \( k \).

**Theorem B.** \( \sigma(K_k, n) \geq (k - 2)(2n - k + 1) + 2 \).

This result can easily be seen, by noting that \( H = K_k - 2 + K_n - k + 2 \) gives a uniquely realizable degree sequence and \( H \) clearly does not contain \( K_k \). Also note, this degree sequence only contains \( k - 2 \) vertices of degree at least \( k - 1 \), but a \( K_k \) would require \( k \) vertices of degree at least \( k - 1 \).

In Section 2, we show that Theorem B achieves the correct value for the case \( k = 4 \) and \( n \geq 9 \). In Section 3 we also find \( \sigma(G, n) \) for various other graphs \( G \).

2. \( \sigma(K_4, n) \)

We begin with a useful result which extends a theorem of S.B. Rao [8].

**Lemma 1.** Let \( H \) be a degree sequence \( S = (d) \), \( G \subset H \), and \( x \in V(G) \). \( \text{deg}_H(x) \), then there exist \( V(H) \) and \( \text{deg}_{H'}(v_i) = d_i \) such that:

1. \( H - \{x, y\} = H' - \{x, \} \)
2. the subgraph of \( H' \) in \( G' \) isomorphic.

**Proof.** Suppose the seq vertices \( v_i \) and \( x \) are as above. Take \( H = H' \) and clearly there exist non-empty set \(- N_H(x)\). Since \( \text{deg}_H(y) \) choose any subset \( C \) realization \( H' \) of \( S \) by inter at \( x \) with endvertices in \( A \) and \( x \) with endvertices in \( C \), centered at \( y \) with endvertex centered at \( y \) with endvertex.

It is easy to see that these are met. \( \square \)

**Corollary.** [Rao [7]]. If \( S \) \( H \) containing \( K_k \), then \( k \) \( K_k \) with the \( k \) vertices have the

**Note.** In fact, Lemma 1 \( G \subset H \), then there is a the vertices of \( G \) have the

**Proposition 2.** If \( S \) is a 28 then either there is \((4, 4, 4, 4, 4, 4, 4, 0)\).

Before proceeding with there is no realization of \( S \).
Lemma 1. Let $H$ be a graph having $V(H) = \{v_1, \ldots, v_n\}$ and degree sequence $S = (d_1, \ldots, d_n)$ where $\deg_H(v_i) = d_i$. If $G \subseteq H$, and $x \in V(G)$, $y \in V(H) \setminus V(G)$ with $\deg_H(y) \geq \deg_H(x)$, then there exists a realization $H'$ of $S$ with $V(H') = V(H)$ and $\deg_{H'}(v_i) = d_i$ such that

1. $H - \{x, y\} = H' - \{x, y\}$
2. the subgraph of $H'$ induced by $(V(G) \setminus \{x\}) \cup \{y\}$ has a subgraph $G'$ isomorphic to $G$.

Proof. Suppose the sequence $S$, the graphs $H$ and $G$, and the vertices $v_{ij}$ and $y$ are as above. If $N_G(x) \subseteq N_G(y)$ then simply take $H = H'$ and clearly (1) and (2) above hold. Hence, assume there exist non-empty sets $A = N_G(x) - N_G(y)$ and $B = N_H(y) - N_H(x)$. Since $\deg_H(y) \geq \deg_H(x)$, it follows that $|B| \geq |A|$. Now choose any subset $C \subseteq B$ having $|C| = |A|$. Now form a new realization $H'$ of $S$ by interchanging the edges of the star centered at $x$ with endvertices in $A$ with the non-edges of the star centered at $x$ with endvertices in $C$, and interchanging the edges of the star centered at $y$ with endvertices in $C$ with the non-edges of the star centered at $y$ with endvertices in $A$.

It is easy to see that this is a realization $H'$ of $S$ and (1) and (2) are met. □

Corollary (Rao [7]). If $S$ is a graphical sequence with a realization $H$ containing $K_k$, then there is a realization $H'$ of $S$ containing $K_k$ with the $k$ vertices having the $k$ largest degrees.

Note, in fact, Lemma 1 shows that if $H$ is a realization of $S$ with $G \subseteq H$, then there is a realization $H'$ of $S$ with $G \subseteq H'$ so that the vertices of $G$ have the largest degrees of $S$.

Proposition 2. If $S$ is an 8-term graphical sequence with $\sigma(S) \geq 28$ then either there is a realization of $S$ containing $K_4$ or $S = (4, 4, 4, 4, 4, 4, 4, 0)$.

Before proceeding with the proof of this proposition, we note that there is no realization of $S = (4, 4, 4, 4, 4, 4, 4, 0)$ containing a $K_4$. 

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Consider the complementary sequence \( \bar{S} = (3, 3, 3, 3, 3, 3, 3, 7) \). Clearly in any realization of \( \bar{S} \), there is a dominating vertex and a 2-regular subgraph on the remaining 7 vertices. There is no independent set of 4 vertices in this 2-regular subgraph and thus no realization of \( \bar{S} \) can contain 4 independent vertices. Therefore, no realization of \( S \) can contain a \( K_4 \).

**Proof of Proposition 2.** Let \( S = (d_1, d_2, \ldots, d_8) \) be a graphical sequence with \( \sigma(S) \geq 28 \). Assume \( d_1 \geq d_2 \geq \ldots \geq d_8 \); it must be the case that \( 4 \leq d_1 \leq 7 \). Applying the Havel-Hakimi characterization of realizable sequences it follows that \( S' = (d_2 - 1, d_3 - 1, \ldots, d_1 + 1 - 1, d_{d_1 + 2}, \ldots, d_8) \) is realizable and \( \sigma(S') \geq 14 \). By Theorem A, \( \sigma(K_3, 7) = 14 \), thus there is a realization of \( S' \) containing \( K_3 \). Furthermore, by Lemma 2, there is a realization having \( K_3 \) on the largest three degrees of \( S' \). If these three highest degrees are obtained from \( S \) by subtracting 1, then we simply reinset the vertex of degree \( d_1 \), producing a realization of \( S \) containing \( K_4 \). This implies we may assume \( 4 \leq d_1 \leq 6 \). In addition, the general form for \( S \) must be one of the following three types.

**Case 1:** \( S = (6, d_2, d_3, a, a, a, a, a) \) where \( 3 \leq a \leq 6 \). If \( a = 5 \) or 6 then \( S = (6, 6, 6, 6, 6, 6, 6) \) or \( (6, 6, 5, 5, 5, 5, 5, 5) \). If \( a = 4 \) then \( S \) is one of \((6, 6, 6, 4, 4, 4, 4, 4), (6, 6, 5, 4, 4, 4, 4, 4), (6, 6, 4, 4, 4, 4, 4, 4), (6, 4, 4, 4, 4, 4, 4, 4)\). If \( a = 3 \) then \( S \) must be one of \((6, 6, 5, 3, 3, 3, 3, 3), (6, 5, 4, 3, 3, 3, 3, 3), (6, 6, 3, 3, 3, 3, 3, 3)\). Each of these sequences are potentially \( K_4 \)-graphical.

**Case 2:** \( S = (5, d_2, d_3, a, a, a, a, d_8) \) where \( 3 \leq a \leq 5 \). If \( a = 5 \), then \( S = (5, 5, 5, 5, 5, 5, d_8) \) where \( d_8 = 1, 3, \) or 5. If \( a = 4 \), there are seven possible sequences: \((5, 5, 4, 4, 4, 4, 4, 4), (5, 5, 4, 4, 4, 4, 4, 2), (5, 5, 4, 4, 4, 4, 4, 0), (5, 4, 4, 4, 4, 4, 4, 3), (5, 4, 4, 4, 4, 4, 4, 1), (5, 5, 4, 4, 4, 4, 4, 3), and (5, 5, 5, 4, 4, 4, 4, 1)\). If \( a = 3 \), there are four possible sequences \((5, 5, 5, 3, 3, 3, 3, 3), (5, 5, 4, 3, 3, 3, 3, 2), (5, 5, 5, 3, 3, 3, 3, 1)\) and \((5, 4, 4, 3, 3, 3, 3, 3)\). Each of these sequences are potentially \( K_4 \)-graphical.

**Case 3:** \( S = (4, 4, 4, 4, 4, (4, 4), (4, 2), (3, 3), (3, 1), (3, 1) \) except the last pair yield a re.

With all cases exhausted, the

**Proposition 3.** If \( S \) is an \( n \)-sequence \( \sigma(S) \leq 4n - 6 \) with smallest two \( 2 \)-s, then there exists a realization

**Proof.** Assume \( \sigma(S) = \) induction on \( n \) and \( k \) (upward)

For \( n = 8 \), there exist exact check that each has an appropriate \( k \geq \frac{n-1}{2} \), the result is vacuously 3 vertices having degree less.

We now assume the \( n \)-term graphic sequence with most two \( 2 \)-s. As noted, implies (vacuously) a realization downward on \( k \) we may assume that for every \( k \geq k_0 \) if \( S \) exists a realization of \( S \) assume \( k_0 \geq 1 \).

Now let \( S^* \) be an \((n+1)\)-sequence containing \( K_4 \) at least 2 containing at most \( n+2 \). Note that \( \sigma(S^*) \geq 3n + 2 \) that either all terms are equal greater than 3.

If all terms equal 3 (or containing \( K_4 \) is easily obtained a 3-regular graph on \( n \)-has a term strictly greater than 3).

Subcase A. Suppose \( S^* \) is
since \( S = (3, 3, 3, 3, 3, 3, 3, 7) \)

**Case 3:** \( S = (4, 4, 4, 4, 4, 4, d, d) \) with \((d, d)\) being one of 
\((4, 4), (4, 2), (3, 3), (3, 1), (2, 2)\) or \((4, 0)\). The sequence in all cases except the last pair yields a realization containing \( K_4 \).

With all cases exhausted, the result follows. \( \square \)

**Proposition 3.** If \( S \) is an \( n \)-term graphical sequence \((n \geq 8)\) having \( \sigma(S) \leq 4n - 6 \) with smallest term at least 2 and containing at most two 2's, then there exists a realization of \( S \) containing a \( K_4 \).

**Proof.** Assume \( \sigma(S) = 4n - 4 - 2k \) \((k \geq 1)\). We proceed by induction on \( n \) and \( k \) (upward on \( n \) and downward on \( k \) for each \( n \)).

For \( n = 8 \), there exist exactly 15 such sequences and it is easy to check that each has an appropriate realization. Also, for any \( n \) and \( k \geq \frac{n - 1}{2} \), the result is vacuous, since \( \sigma(S) \leq 3n - 3 \) implies at least 3 vertices having degree less than three.

We now assume the \( n \) case holds for all \( k \) and let \( S \) be an \( n + 1 \) term graphical sequence with smallest term at least 2 and containing at most two 2's. As noted, if \( k \geq \frac{(n+1) - 1}{2} \), then \( \sigma(S) = 4n - 2k \) implies (vacuously) a realization containing \( K_4 \). Thus, inducting downward on \( k \) we may assume there exists a \( k_0 \) \((1 \leq k_0 \leq \frac{n - 1}{2})\) so that for every \( k \geq k_0 \) if \( S \) is as above and \( \sigma(S) = 4n - 4 - 2k \) there exists a realization of \( S \) containing \( K_4 \). If \( k_0 = 1 \) we are done, so assume \( k_0 > 1 \).

Now let \( S^* \) be an \((n+1)\)-term graphical sequence with smallest term at least 2 containing at most two 2's and having \( \sigma(S^*) = 4n - 2(k_0 - 1) \). Note that \( \sigma(S^*) \geq 3n + 3 \) (using the bound on \( k_0 \)) which implies that either all terms are equal to 3 or at least one term is strictly greater than 3.

If all terms equal 3 (and \( n + 1 \) is thus even) then a realization containing \( K_4 \) is easily obtained by considering \( K_4 \cup H \) where \( H \) is a 3-regular graph on \( n - 4 \) vertices. Thus we may assume that \( S^* \) has a term strictly greater than 3.

**Subcase A.** Suppose \( S^* \) contains a term equal to 2.
By Lemma 1 there exists a realization $G$ of $S^*$ with degree 2 vertex (call it $x$) adjacent to a vertex of degree strictly greater than 3, (call it $y$) and a vertex $z$ of degree at least 3. Then $G - x$ produces a graphic $n$-term sequences $S'$ with $\sigma(S') = 4(n - 1) - 2(k_0 - 1)$ with $S'$ having smallest term at least 2 and at most two 2's. Thus, by the induction hypothesis $S'$ has a realization containing $K_4$ and by inserting $x$ we can obtain a realization of $S^*$ with a $K_4$.

Subcase B. Suppose that $S^*$ does not contain a term equal to 2 and $S^*$ is not the sequence of all 3's.

Clearly 3 must be a term of $S^*$, and by Lemma 1 there exists a realization $G$ and a vertex $x$ of degree 3 with $x$ adjacent to a vertex $y$ of degree strictly greater than 3. Now $G - x$ produces an $n$-term graphic sequence $S'$ with smallest term at least 2 and at most two terms equal to 2. Further,

$$\sigma(S') = 4n - 2(k_0 - 1) - 6$$

$$= 4n - 4 - 2(k_0 - 1) - 2(1) = 4n - 4 - 2k_0.$$ 

Thus we are finished with this case by induction and this case completes the proof. □

Theorem 4. If $n \geq 9$, then $\sigma(K_4, n) = 4n - 4$.

Proof. Suppose $n \geq 9$, Theorem B shows that $\sigma(K_4, n) \geq 4n - 4$. We now show that any $n$-term graphical sequence $S$ having $\sigma(S) \geq 4n - 4$ has a realization containing a $K_4$. We will proceed by induction and begin with the case $n = 9$. Let $S = (d_1, \ldots, d_9)$ be a 9-term graphical sequence having $d_1 \geq d_2 \geq \ldots \geq d_9$ with $\sigma(S) \geq 32$. Let $G$ be a realization of $S$. If the smallest term in $S$ is $d_9 = 0, 1$ or 2 and $x \in V(G)$ with $\deg_G(x) = d_9$, then $G - x$ yields an 8-term sequence $S^*$ with $\sigma(S^*) \geq 28$. By Proposition 2, $S^*$ has a realization containing a $K_4$, because $S^* \neq (4, 4, 4, 4, 4, 4, 4, 0)$.

Assume that $d_9 \geq 3$; furthermore, it follows that $d_1 \geq 4$. By Lemma 1, there is a realization $G$ of $S$ with vertex $x$ having $\deg_G(x) \geq 3$ and $x$ is adjacent to a vertex $y$ with $\deg_G(y) \geq 4$. It follows that the 8-term graphical sequence $S^*$ obtained from $G - x$ is either has $\sigma(S^*) \geq 28$ or $\sigma(S^*) \leq 26$. In the first case, since $S^*$ cannot contain 0, $S^*$ has a realization containing a $K_4$ by Proposition 2. This results in a realization of $S$ with a $K_4$ after the insertion of $x$. If $\sigma(S^*) \leq 2$ and contains at most two realizations of $S$ containing $a$.

We now proceed by induction. Suppose that $S^*$ contains an $n$-term 1-1-term graphical sequence $S'$ with $\sigma(S') \geq 4n$. Let $G$ be any $d e G(x) = d_{n+1}$. If $S^* \leq -4$ then by induction an $a_{n+1}$ then the $n$-term graphical $\sigma(S^*) \leq 4n - 6$, and smaller there is a realization of $S^*$ realization of $S$ containing 1.

If $d_{n+1} \leq 2$ then the $n$-term graphical sequence $S^*$ obtained if has smallest term at least 2.

Finally, if $d_{n+1} = 3$ by having $x$ adjacent to a sequence $S^*$ obtained if has smallest term at least 2.

3 $\sigma(G, n)$

In this section we look at turning to matchings, or result for matchings obtained.

Theorem 5. For $p \geq 2$, $\sigma$

We now turn our attention to $\sigma(G, n)$.

Theorem 6. For $n \geq 4$, $\sigma$
lization $G$ of $S^*$ with degree 2 $x$ of degree strictly greater than 3, at least 3. Then $G - x$ produces the $\sigma(S') = 4(n - 1) - 2(k_0 - 1)$ and at most two 2's. Thus, by realization containing $K_4$ and by $n$ of $S^*$ with a $K_4$,
not contain a term equal to 2 and
and by Lemma 1 there exists a tree 3 with $x$ adjacent to a vertex.
Now $G - x$ produces an $n$-term term at least 2 and at most two

\[ n(k_0 - 1) - 6 \]

\[ n = 4n - 4 - 2k_0. \]

ise by induction and this case
\[ n \geq 4n - 4. \]

shows that $\sigma(K_4, n) \geq 4n - 4.$

tical sequence $S$ having $\sigma(S) \geq 3$ a $K_4$. We will proceed by $n = 9$. Let $S = (d_1, \ldots, d_9)$ be a
$d_1 \geq d_2 \geq \ldots \geq d_9$ with $\sigma(S) \geq 3$ the smallest term in $S$ is $d_9 = 0$, $(x) = d_9$, then $G - x$ yields an
28. By Proposition 2, $S^*$ has a
* $\neq (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 0)$.

e, it follows that $d_1 \geq 4$. By $G$ of $S$ with vertex $x$ having vertex $y$ with $\deg_G(y) \geq 4$. It
quence $S^*$ obtained from $G - x$ has $\sigma(S^*) \leq 4n - 6$ and $S^*$
has smallest term at least 2 and at most two 2's. Thus by Proposition 3, the appropriate realization of $S^*$ exists and this yields a realization of $S$ containing a $K_4$. $\square$

3 $\sigma(G, n)$

In this section we look at a few very special cases for this number. Turning to matchings, our first result coincides with the extremal result for matchings obtained by Chvátal and Hanson [1], hence it is immediate.

Theorem 5. For $p \geq 2$, $\sigma(pK_2, n) = (p - 1)(2n - 2) + 2$.

We now turn our attention to cycles. The first interesting case is $C_4$.

Theorem 6. For $n \geq 4$,
\[ \sigma(C_4, n) = \begin{cases} 3n - 1 & \text{if } n \text{ is odd} \\ 3n - 2 & \text{if } n \text{ is even.} \end{cases} \]

Proof. To see that \( \sigma(C_4, n) \geq 3n - 1 \) or \( 3n - 2 \), consider the uniquely realizable degree sequence obtained from \( K_1 + pK_2 \) \((n = 2p + 1)\) or \( K_1 + (pK_2 \cup K_1) \(n = 2p + 2)\). We now show the upper bound.

For \( n = 4 \), if a graph has size \( q \geq 5 \), then clearly it contains a \( C_4 \). For \( n = 5 \), we have that \( q \geq 7 \). There are exactly 4 graphs of order 5 and size 7 and each contains a \( C_4 \). Thus, we now assume the result is true for all values up to \( n \) and consider a sequence \( S \) of \( n + 1 \) terms.

If \( S \) contains a term equal to 1, then remove it and adjust the new sequence \( S' \). By induction \( S' \) must contain a \( C_4 \). We also note that there must be at least 2 vertices of degree at least 3 in our sequence (as \( n - 1 + 2(n - 1) \) is too small).

Then, by Theorem A there is a realization of \( S \) containing a \( K_3 \), which by Lemma 1, can be obtained in a realization using the two vertices of highest degree (recall each is at least 3).

Say this \( K_3 \) has vertices \( w_1, w_2, \) and \( w_3 \). Further suppose the two vertices of highest degree in the graph are \( w_1 \) and \( w_2 \). Thus, \( w_1 \) and \( w_2 \) have at least one more adjacency in the graph. Say \( w_1 \) is adjacent to \( x \) and \( w_2 \) is adjacent to \( y \).

If \( x = y \) we are done as a \( C_4 \) is formed. If \( x \neq y \) we consider two cases.

Case 1. Suppose \( x \) and \( y \) have a common adjacency, say \( z \).

Then we see that the edges \( zw_1 \) and \( zw_3 \) are not in \( G \) or a \( C_4 \) would exist. But then the edge interchange which removes \( w_1w_3 \) and \( xz \) and inserts the independent edges \( zw_1 \) and \( zw_3 \) produces a realization containing a \( C_4 \).

Case 2. Suppose \( x \) and \( y \) have no common adjacencies off \( K_3 \).

Since both \( x \) and \( y \) have adjacencies off \( K_3 \), suppose that \( x \) is adjacent to \( x_1 \) and \( y \) is adjacent to \( y_1 \) and \( x_1 \neq y_1 \). Further suppose that \( x_1 \) and \( y_1 \) are not adjacent. Then, the interchange that removes the independent edges \( xx_1 \) and \( yy_1 \) and inserts the independent edges \( xy \) and \( x_1y_1 \) produces a realization of \( S \) containing a \( C_4 \). If \( x_1 \) and \( y_1 \) are adjacent, then again it is easy to see that the edges \( w_1x_1 \) already exist. But again the \( xx_1 \) and inserts \( w_1x_1 \) and containing a \( C_4 \).

In all cases a \( C_4 \) was produced therefore the result is proved.

Clearly, \( \sigma(C_4, n) \leq \sigma(K_4) \) see where in the range from lies.

4 Conclusion

This extension of the classic question but presently suffice. We feel that the complete gi following.

Conjecture: For \( n \) sufficiently large, this number is linear this value and \( ex(K_p, n) \).

References

[4] ERDÖS, P., JACOBSON, Same Degree Sequence and Theory, Combinatorics a Chartrand, Oellerman and:
see that the edges $w_1x_1$ and $xw_3$ are not in $G$ or a $C_4$ would already exist. But again the interchange that removes $w_1w_3$ and $x_1$ and inserts $w_1x_1$ and $xw_3$ produces a realization of $S$ containing a $C_4$.

In all cases a $C_4$ was produced in some realization of $S$ and therefore the result is proved. □

Clearly, $\sigma(C_4, n) \leq \sigma(K_4 - e, n) \leq \sigma(K_4, n)$. It would be nice to see where in the range from $3n - 2$ to $4n - 4$, the value $\sigma(K_4 - e, n)$ lies.

4 Conclusion

This extension of the classical extremal problem seems like a natural question but presently sufficient techniques have not been developed. We feel that the complete graph case is the most tractable so we give the following.

Conjecture: For $n$ sufficiently large

$$\sigma(K_k, n) = (k-2)(2n-k+1) + 2.$$ 

As a weakening of this, it would be nice to see that for $n$ sufficiently large, this number is linear in $n$, explaining the difference between this value and $ex(K_k, n)$.

References


ON DEFINING SETS
THE CARTESIAN PROBLEM

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In a given graph $G$, a defining set is a minimum set of colors that exists a unique extension of the vertices of $G$. A defining set is called a minimum defining set. The minimizing cardinality is denoted by $d$. Mahmoodian et al. [3] proved that among all defining sets each given $m$ and for all $k$, each given $m$ and for all $k$, we have

1. $d(C_m \times K_3, x) = k$
2. $m \leq d(C_m \times K_4, x)$
3. $d(C_m \times K_5, x) = 2$

$2m \leq d(C_m \times K_5, x)$

1 Introduction

A $k$-coloring of a graph $G$ is a function $f: V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(v) \neq f(u)$ for all $v, u \in V(G)$ with $uv \in E(G)$.