›isson variable with *independent*
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ishops, or the number Q_k of k-
ı be shown to be approximately
proof of Theorem 2.3. It turns
ınd $\mathrm{E}(B_k) \sim \frac{2}{3}\mathrm{E}(X_k)$, so that

$$+ \mathrm{E}(B_k)$$

$$\Xi(X_k) \sim \mathrm{P}(X_k = 0)\mathrm{P}(B_k = 0).$$

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Potentially G-graphical degree sequences

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Abstract

In this paper we consider a variation of the classical Turán-type extremal problems. Let S be an n-element graphical sequence, and $\sigma(S)$ be the sum of the terms in S. Let G be a graph. The problem is to determine the smallest m such that any n-term graphical sequence S having $\sigma(S) \geq m$ has a realization containing G. Denote this value m by $\sigma(G, n)$. We present several results for this parameter for various graphs G. In particular, we show $\sigma(K_4, n) = 4n - 4$ for $n \geq 9$.

1 Introduction

There are several famous results, Havel and Hakimi [6, 5] and Erdös and Gallai [3], which give necessary and sufficient conditions for a sequence $S = (d_1, d_2, \ldots, d_n)$ to be the degree sequence of a simple graph G. Unfortunately, knowing that a sequence has a realization gives no information about the properties that such a graph might have. In this paper, we explore this question of properties of graphs with a given degree sequence which is related to work originally introduced by A. R. Rao [7].

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For convenience, we employ the following terminology. If $S = (d_1, d_2, \ldots, d_n)$ is a sequence of non-negative integers then it is called *graphical* if there is a (simple) graph of order n, whose degrees are precisely the terms in S. If G is such a graph then G is said to *realize* S or be a *realization of* S. A graphical sequence S is *potentially G-graphical* if there is a realization of S containing G, while S is *forcibly G-graphical* if every realization of S contains G. Throughout the paper subgraph means non-induced subgraph. For any undefined terms, refer to [1].

One of the classical extremal problems is to determine the minimum even integer m such that every n-term graphical sequence S with $\sigma(S) \geq m$ is forcibly G-graphical; this m is denoted $ex(G, n)$. Here we consider the following variant: determine the minimum even integer m such that every n-term graphical sequence S with $\sigma(S) \geq m$ is potentially G-graphical. We denote this minimum m by $\sigma(G, n)$.

This problem was considered by Erdös, Jacobson and Lehel [4] where they showed the following:

Theorem A. For $n \geq 6$, $\sigma(K_3, n) = 2n$.

They also gave a construction that gave a lower bound for $\sigma(K_k, n)$, which also would be a bound for $\sigma(G, n)$ for any graph G of order k.

Theorem B. $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$.

This result can easily be seen, by noting that $H = K_{k-2} + \overline{K_{n-k+2}}$ gives a uniquely realizable degree sequence and H clearly does not contain K_k. Also note, this degree sequence only contains $k-2$ vertices of degree at least $k-1$, but a K_k would require k vertices of degree at least $k-1$.

In Section 2, we show that Theorem B achieves the correct value for the case $k = 4$ and $n \geq 9$. In Section 3 we also find $\sigma(G, n)$ for various other graphs G.

2 $\sigma(K_4, n)$

We begin with a useful result which extends a theorem of S. B. Rao [8].

Lemma 1. Let H be a
degree sequence $S = ($
$G \subset H$, and $x \in V(G$
$\deg_H(x)$, then there exi
$V(H)$ and $\deg_{H'}(v_i) = d_i$ s
(1) $H - \{x, y\} = H' - \{x,$
(2) the subgraph of H' in
subgraph G' isomorphic

Proof. Suppose the seq
vertices v_{i_j} and y are as
take $H = H'$ and clearly (
there exist non-empty se
$- N_H(x)$. Since $\deg_H($
Now choose any subset C
realization H' of S by int
at x with endvertices in A
x with endvertices in C,
centered at y with endver
centered at y with endverti
It is easy to see that thi
are met. □

Corollary (Rao [7]). If S
H containing K_k, then t
K_k with the k vertices havi
Note, in fact, Lemma 1
$G \subseteq H$, then there is a
the vertices of G have the

Proposition 2. If S is a
28 then either there is
(4, 4, 4, 4, 4, 4, 4, 0).

Before proceeding with
there is no realization of

: following terminology. If $S =$
-negative integers then it is called
ph of order n, whose degrees are
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S. A graphical sequence S is
a realization of S containing G,
every realization of S contains
ih means non-induced subgraph.
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Erdős, Jacobson and Lehel [4]

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$\cdot k + 1) + 2.$

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equence and H clearly does not
sequence only contains $k-2$
a K_k would require k vertices of

B achieves the correct value for
ion 3 we also find $\sigma(G, n)$ for

xtends a theorem of S. B. Rao [8].

Lemma 1. Let H be a graph having $V(H) = \{v_1, \ldots, v_n\}$ and degree sequence $S = (d_1, \ldots, d_n)$ where $\deg_H(v_i) = d_i$. If $G \subset H$, and $x \in V(G)$, $y \in V(H) \setminus V(G)$ with $\deg_H(y) \geq \deg_H(x)$, then there exists a realization H' of S with $V(H') = V(H)$ and $\deg_{H'}(v_i) = d_i$ such that

(1) $H - \{x, y\} = H' - \{x, y\}$

(2) the subgraph of H' induced by $(V(G) \setminus \{x\}) \cup \{y\}$ has a subgraph G' isomorphic to G.

Proof. Suppose the sequence S, the graphs H and G, and the vertices v_{i_j} and y are as above. If $N_G(x) \subseteq N_G(y)$ then simply take $H = H'$ and clearly (1) and (2) above hold. Hence, assume there exist non-empty sets $A = N_G(x) - N_G(y)$ and $B = N_H(y) - N_H(x)$. Since $\deg_H(y) \geq \deg_H(x)$, it follows that $|B| \geq |A|$. Now choose any subset $C \subseteq B$ having $|C| = |A|$. Now form a new realization H' of S by interchanging the edges of the star centered at x with endvertices in A with the non-edges of the star centered at x with endvertices in C, and interchanging the edges of the star centered at y with endvertices in C with the non-edges of the star centered at y with endvertices in A.

It is easy to see that this is a realization H' of S and (1) and (2) are met. □

Corollary (Rao [7]). If S is a graphical sequence with a realization H containing K_k, then there is a realization H' of S containing K_k with the k vertices having the k largest degrees.

Note, in fact, Lemma 1 shows that if H is a realization of S with $G \subseteq H$, then there is a realization H' of S with $G \subseteq H'$ so that the vertices of G have the largest degrees of S.

Proposition 2. If S is an 8-term graphical sequence with $\sigma(S) \geq 28$ then either there is a realization of S containing K_4 or $S = (4, 4, 4, 4, 4, 4, 4, 0)$.

Before proceeding with the proof of this proposition, we note that there is no realization of $S = (4, 4, 4, 4, 4, 4, 4, 0)$ containing a K_4.

Consider the complementary sequence $\overline{S} = (3, 3, 3, 3, 3, 3, 3, 7)$. Clearly in any realization of $\overline{S}$, there is a dominating vertex and a 2-regular subgraph on the remaining 7 vertices. There is no independent set of 4 vertices in this 2-regular subgraph and thus no realization of $\overline{S}$ can contain 4 independent vertices. Therefore, no realization of S can contain a K_4.

Proof of Proposition 2. Let $S = (d_1, d_2 \ldots d_8)$ be a graphical sequence with $\sigma(S) \geq 28$. Assume $d_1 \geq d_2 \geq \ldots \geq d_8$; it must be the case that $4 \leq d_1 \leq 7$. Applying the Havel-Hakimi characterization of realizable sequences it follows that $S' = (d_2-1, d_3-1, \ldots d_{d_1+1}-1, d_{d_1+2}, \ldots, d_8)$ is realizable and $\sigma(S') \geq 14$. By Theorem A, $\sigma(K_3, 7) = 14$, thus there is a realization of S' containing K_3. Furthermore, by Lemma 2, there is a realization having K_3 on the largest three degrees of S'. If these three highest degrees are obtained from S by subtracting 1, then we simply reinsert the vertex of degree d_1, producing a realization of S containing K_4. This implies we may assume $4 \leq d_1 \leq 6$. In addition, the general form for S must be one of the following three types.

Case 1: $S = (6, d_2, d_3, a, a, a, a, a)$ where $3 \leq a \leq 6$. If $a = 5$ or 6 then $S = (6, 6, 6, 6, 6, 6, 6, 6)$ or $(6, 6, 5, 5, 5, 5, 5, 5)$. If $a = 4$ then S is one of $(6, 6, 6, 4, 4, 4, 4, 4)$, $(6, 5, 5, 4, 4, 4, 4, 4)$, $(6, 6, 4, 4, 4, 4, 4, 4)$ $(6, 4, 4, 4, 4, 4, 4, 4)$. If $a = 3$ then S must be one of $(6, 6, 5, 3, 3, 3, 3, 3)$, $(6, 5, 4, 3, 3, 3, 3, 3)$, $(6, 6, 3, 3, 3, 3, 3, 3)$. Each of these sequences are potentially K_4-graphical.

Case 2: $S = (5, d_2, d_3, a, a, a, a, d_8)$ where $3 \leq a \leq 5$. If $a = 5$, then $S = (5, 5, 5, 5, 5, 5, 5, d_8)$ where $d_8 = 1, 3$, or 5. If $a = 4$, there are seven possible sequences: $(5, 5, 4, 4, 4, 4, 4, 4)$, $(5, 5, 4, 4, 4, 4, 4, 2)$, $(5, 5, 4, 4, 4, 4, 4, 0)$, $(5, 4, 4, 4, 4, 4, 4, 3)$, $(5, 4, 4, 4, 4, 4, 4, 1)$, $(5, 5, 5, 4, 4, 4, 4, 3)$, and $(5, 5, 5, 4, 4, 4, 4, 1)$. If $a = 3$, there are four possible sequences $(5, 5, 5, 3, 3, 3, 3, 3)$, $(5, 5, 4, 3, 3, 3, 3, 2)$, $(5, 5, 5, 3, 3, 3, 3, 1)$ and $(5, 4, 4, 3, 3, 3, 3, 3)$. Each of these sequences are potentially K_4-graphical.

Case 3: $S =$ (4, 4, 4, 4, 4,
(4, 4), (4, 2), (3, 3), (3, 1), (2
except the last pair yield a re
With all cases exhausted, th

Proposition 3. If S is an n-
$\sigma(S) \leq 4n - 6$ with smallest
two 2's, then there exists a re

Proof. Assume $\sigma(S) =$
induction on n and k (upward
For $n = 8$, there exist exa
check that each has an app
$k \geq \frac{n-1}{2}$, the result is vac
3 vertices having degree les
We now assume the n c
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most two 2's. As noted,

implies (vacuously) a rea

downward on k we may as

that for every $k \geq k_0$ if S
exists a realization of S c
assume $k_0 > 1$.
Now let S^* be an $(n+1)$-
at least 2 containing at mo
Note that $\sigma(S^*) \geq 3n +$
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If all terms equal 3 (a
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a 3-regular graph on $n -$
has a term strictly greater
Subcase A. Suppose S^*

ence $\bar{S} = (3, 3, 3, 3, 3, 3, 3, 7)$.
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e $d_1 \geq d_2 \geq \ldots \geq d_8$; it must be
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ws that $S' = (d_2-1,\ d_3-1, \ldots$
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realization of S' containing K_3.
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$a, d_8)$ where $3 \leq a \leq 5$. If $a =$
where $d_8 = 1, 3$, or 5. If $a = 4$,
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4, 4, 4, 4, 3), (5, 4, 4, 4, 4, 4, 4, 1),
4, 4, 4, 1). If $a = 3$, there are four
, 3), (5, 5, 4, 3, 3, 3, 3, 2), (5, 5, 5,
, 3). Each of these sequences are

Case 3: $S = (4, 4, 4, 4, 4, 4, d_7, d_8)$ with (d_7, d_8) being one of (4, 4), (4, 2), (3, 3), (3, 1), (2, 2) or (4, 0). The sequence in all cases except the last pair yield a realization containing K_4.

With all cases exhausted, the result follows. □

Proposition 3. If S is an n-term graphical sequence ($n \geq 8$) having $\sigma(S) \leq 4n - 6$ with smallest term at least 2 and containing at most two 2's, then there exists a realization of S containing a K_4.

Proof. Assume $\sigma(S) = 4n - 4 - 2k$ ($k \geq 1$). We proceed by induction on n and k (upward on n and downward on k for each n).

For $n = 8$, there exist exactly 15 such sequences and it is easy to check that each has an appropriate realization. Also, for any n and $k \geq \frac{n-1}{2}$, the result is vacuous, since $\sigma(S) \leq 3n - 3$ implies at least 3 vertices having degree less than three.

We now assume the n case holds for all k and let S be an $n + 1$ term graphic sequence with smallest term at least 2 and containing at most two 2's. As noted, if $k \geq \frac{(n+1)-1}{2}$, then $\sigma(S) = 4n - 2k$ implies (vacuously) a realization containing K_4. Thus, inducting downward on k we may assume there exists a $k_0 \left(1 \leq k_0 \leq \frac{n-1}{2}\right)$ so that for every $k \geq k_0$ if S is as above and $\sigma(S) = 4n - 4 - 2k$ there exists a realization of S containing K_4. If $k_0 = 1$ we are done, so assume $k_0 > 1$.

Now let S^* be an $(n+1)$-term graphic sequence with smallest term at least 2 containing at most two 2's and having $\sigma(S^*) = 4n - 2(k_0 - 1)$. Note that $\sigma(S^*) \geq 3n + 3$ (using the bound on k_0) which implies that either all terms are equal to 3 or at least one term is strictly greater than 3.

If all terms equal 3 (and $n + 1$ is thus even) then a realization containing K_4 is easily obtained by considering $K_4 \cup H$ where H is a 3-regular graph on $n - 4$ vertices. Thus we may assume that S^* has a term strictly greater than 3.

Subcase A. Suppose S^* contains a term equal to 2.

By Lemma 1 there exists a realization G of S^* with degree 2 vertex (call it x) adjacent to a vertex of degree strictly greater than 3, (call it y) and a vertex z of degree at least 3. Then $G - x$ produces a graphic n-term sequences S' with $\sigma(S') = 4(n-1) - 2(k_0 - 1)$ with S' having smallest term at least 2 and at most two 2's. Thus, by the induction hypothesis S' has a realization containing K_4 and by inserting x we can obtain a realization of S^* with a K_4.

Subcase B. Suppose that S^* does not contain a term equal to 2 and S^* is not the sequence of all 3's.

Clearly 3 must be a term of S^*, and by Lemma 1 there exists a realization G and a vertex x of degree 3 with x adjacent to a vertex y of degree strictly greater than 3. Now $G - x$ produces an n-term graphic sequence S' with smallest term at least 2 and at most two terms equal to 2. Further,

$$\begin{aligned}\sigma(S') &= 4n - 2(k_0 - 1) - 6\\ &= 4n - 4 - 2(k_0 - 1) - 2(1) = 4n - 4 - 2k_0.\end{aligned}$$

Thus we are finished with this case by induction and this case completes the proof. □

Theorem 4. If $n \geq 9$, then $\sigma(K_4, n) = 4n - 4$.

Proof. Suppose $n \geq 9$, Theorem B shows that $\sigma(K_4, n) \geq 4n - 4$. We now show that any n-term graphical sequence S having $\sigma(S) \geq 4n - 4$ has a realization containing a K_4. We will proceed by induction and begin wtih the case $n = 9$. Let $S = (d_1, \ldots, d_9)$ be a 9-term graphical sequence having $d_1 \geq d_2 \geq \ldots \geq d_9$ with $\sigma(S) \geq 32$. Let G be a realization of S. If the smallest term in S is $d_9 = 0$, 1 or 2 and $x \in V(G)$ with $\deg_G(x) = d_9$, then $G - x$ yields an 8-term sequence S^* with $\sigma(S^*) \geq 28$. By Proposition 2, S^* has a realization containing a K_4, because $S^* \neq (4, 4, 4, 4, 4, 4, 4, 0)$.

Assume that $d_9 \geq 3$; furthermore, it follows that $d_1 \geq 4$. By Lemma 1, there is a realization G of S with vertex x having $\deg_G(x) \geq 3$ and x is adjacent to a vertex y with $\deg_G(y) \geq 4$. It follows that the 8-term graphical sequence S^* obtained from $G - x$ either has $\sigma(S^*) \geq 28$ or $\sigma(S^*) \leq 26$. In the first case, since S^* cannot contain 0, S^* has a realization containing a K_4 by Proposition 2. This results in a realization of S with a K_4 after the

insertion of x. If $\sigma(S^*) \leq 2$
and contains at most two
realization of S containing a

We now proceed by indu
values from 9 to n, and w
+ 1)-term graphical seque
$\sigma(S) \geq 4n$. Let G be any
$\deg_G(x) = d_{n+1}$. If S^*
-4 then by induction an ap
then the n-term graphical
$\sigma(S^*) \leq 4n - 6$, and smalle
there is a realization of S^*
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If $d_{n+1} \leq 2$ then the n-te
$G - x$ has $\sigma(S^*) \geq 4n - 4$,
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containing K_4.

Finally, if $d_{n+1} = 3$ by
having x adjacent to a
sequence S^* obtained fr
has smallest term at least 2
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of S containing a K_4. □

3 $\sigma(G, n)$

In this section we look at
Turning to matchings, o
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immediate.

Theorem 5. For $p \geq 2$, σ

We now turn our attenti

Theorem 6. For $n \geq 4$,

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and by Lemma 1 there exists a
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Now $G - x$ produces an n-term
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$k_0 - 1) - 6$
$(1) = 4n - 4 - 2k_0$.
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shows that $\sigma(K_4, n) \geq 4n - 4$.
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$= 9$. Let $S = (d_1, \ldots, d_9)$ be a
$d_1 \geq d_2 \geq \ldots \geq d_9$ with $\sigma(S) \geq$
the smallest term in S is $d_9 = 0$,
$(x) = d_9$, then $G - x$ yields an
28. By Proposition 2, S^* has a
$* \neq (4, 4, 4, 4, 4, 4, 4, 0)$.
e, it follows that $d_1 \geq 4$. By
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26. In the first case, since S^*
lization containing a K_4 by
zation of S with a K_4 after the

insertion of x. If $\sigma(S^*) \leq 26$, S^* has the smallest term of at least 2 and contains at most two 2's. Thus by Proposition 3, again a realization of S containing a K_4 is obtained.

We now proceed by induction, assuming the result is true for all values from 9 to n, and we consider $S = (d_1, d_2, \ldots d_{n+1})$ an $(n+1)$-term graphical sequence having $d_1 \geq d_2 \geq \ldots \geq d_{n+1}$ and $\sigma(S) \geq 4n$. Let G be any realization of S, and $x \in V(G)$ having $\deg_G(x) = d_{n+1}$. If S^* obtained from $G - x$ has $\sigma(S^*) \geq 4n - 4$ then by induction an appropriate realization results. If $d_{n+1} \geq 4$ then the n-term graphical sequence S^* obatined from $G - x$ has $\sigma(S^*) \leq 4n - 6$, and smallest term at least 3. Thus by Proposition 3, there is a realization of S^* containing K_4 and thus by inserting x, a realization of S containing K_4.

If $d_{n+1} \leq 2$ then the n-term graphical sequence S^* obtained from $G - x$ has $\sigma(S^*) \geq 4n - 4$, and as noted above there is a realization containing K_4 and by inserting x, we obtain a realization of S containing K_4.

Finally, if $d_{n+1} = 3$ by Lemma 1, there is a realization G^* of S having x adjacent to a vertex y with $\deg_G(y) \geq 4$. Now the sequence S^* obtained from $G^* - x$ has $\sigma(S^*) \leq 4n - 6$ and S^* has smallest term at least 2 and at most two 2's. Thus by Proposition 3, the appropraite realization of S^* exists and this yields a realization of S containing a K_4. □

3 $\sigma(G, n)$

In this section we look at a few very special cases for this number. Turning to matchings, our first result coincides with the extremal result for matchings obtained by Chvátal and Hanson [1], hence it is immediate.

Theorem 5. For $p \geq 2$, $\sigma(pK_2, n) = (p-1)(2n-2) + 2$.

We now turn our attention to cycles. The first interesting case is C_4.

Theorem 6. For $n \geq 4$,

$$\sigma(C_4, n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. To see that $\sigma(C_4, n) \geq 3n - 1$ or $3n - 2$, consider the uniquely realizable degree sequence obtained from $K_1 + pK_2$ ($n = 2p + 1$) or $K_1 + (pK_2 \cup K_1)$ ($n = 2p + 2$). We now show the upper bound.

For $n = 4$, if a graph has size $q \geq 5$, then clearly it contains a C_4. For $n = 5$, we have that $q \geq 7$. There are exactly 4 graphs of order 5 and size 7 and each contains a C_4. Thus, we now assume the result is true for all values up to n and consider a sequence S of $n + 1$ terms.

If S contains a term equal to 1, then remove it and adjust the new sequence S'. By induction S' must contain a C_4. We also note that there must be at least 2 vertices of degree at least 3 in our sequence (as $n - 1 + 2(n - 1)$ is too small).

Then, by Theorem A there is a realization of S containing a K_3, which by Lemma 1, can be obtained in a realization using the two vertices of highest degree (recall each is at least 3).

Say this K_3 has vertices w_1, w_2 and w_3. Further suppose the two vertices of highest degree in the graph are w_1 and w_2. Thus, w_1 and w_2 have at least one more adjacency in the graph. Say w_1 is adjacent to x and w_2 is adjacent to y.

If $x = y$ we are done as a C_4 is formed. If $x \neq y$ we consider two cases.

Case 1. Suppose x and y have a common adjacency, say z.

Then we see that the edges zw_1 and xw_3 are not in G or a C_4 would exist. But then the edge interchange which removes w_1w_3 and xz and inserts the independent edges zw_1 and xw_3 produces a realization containing a C_4.

Case 2. Suppose x and y have no common adjacencies off K_3.

Since both x and y have adjacencies off K_3, suppose that x is adjacent to x_1 and y is adjacent to y_1 and $x_1 \neq y_1$. Further suppose that x_1 and y_1 are not adjacent. Then, the interchange that removes the independent edges xx_1 and yy_1 and inserts the independent edges xy and x_1y_1 produces a realization of S containing a C_4. If x_1 and y_1 are adjacent, then again it is easy to

see that the edges w_1x_1 a
already exist. But again the
xx_1 and inserts w_1x_1 an
containing a C_4.

In all cases a C_4 was p
therefore the result is proved.

Clearly, $\sigma(C_4, n) \leq \sigma(K_4$
see where in the range from
lies.

4 Conclusion

This extension of the classic
question but presently suffic
We feel that the complete g
the following.

Conjecture: For n sufficien

$\sigma(K_k, n) =$

As a weakening of this, it w
large, this number is linear
this value and $ex(K_k, n)$.

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-1 if n is odd
-2 if n is even.

: $3n-1$ or $3n-2$, consider the
nce obtained from $K_1 + pK_2$ (n =
ı = $2p + 2$). We now show the

≥ 5, then clearly it contains a C_4.
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Thus, we now assume the result is
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then remove it and adjust the new
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degree at least 3 in our sequence (as

realization of S containing a K_3,
ned in a realization using the two
ch is at least 3).
and w_3. Further suppose the two
raph are w_1 and w_2. Thus, w_1 and
icency in the graph. Say w_1 is
y.
formed. If $x \neq y$ we consider two

mmon adjacency, say z.
v_1 and xw_3 are not in G or a C_4
interchange which removes w_1w_3
at edges zw_1 and xw_3 produces a

common adjacencies off K_3.
encies off K_3, suppose that x is
nt to y_1 and $x_1 \neq y_1$. Further
jacent. Then, the interchange that
s xx_1 and yy_1 and inserts the
v_1 produces a realization of S
adjacent, then again it is easy to

see that the edges w_1x_1 and xw_3 are not in G or a C_4 would already exist. But again the interchange that removes w_1w_3 and xx_1 and inserts w_1x_1 and xw_3 produces a realization of S containing a C_4.

In all cases a C_4 was produced in some realization of S and therefore the result is proved. □

Clearly, $\sigma(C_4, n) \leq \sigma(K_4 - e, n) \leq \sigma(K_4, n)$. It would be nice to see where in the range from $3n-2$ to $4n-4$, the value $\sigma(K_4 - e, n)$ lies.

4 Conclusion

This extension of the classical extrémal problem seems like a natural question but presently sufficient techniques have not been developed. We feel that the complete graph case is the most tractable so we give the following.

Conjecture: For n sufficiently large

$$\sigma(K_k, n) = (k-2)(2n-k+1) + 2.$$

As a weakening of this, it would be nice to see that for n sufficiently large, this number is linear in n, explifying the difference between this value and $ex(K_k, n)$.

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ON DEFINING SETS
THE CARTESIAN PR
A COM

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In a given graph G, a
of colors is a *defining se*
exists a unique extension
of the vertices of G. A d
is called a *minimum def*
cardinality is denoted by
Mahmoodian et al [3
each given m and for all
Among our results are

(1) $d(C_m \times K_3, \chi) = \lfloor$

(2) $m \leq d(C_m \times K_4, \chi$

(3) $d(C_m \times K_5, \chi) = 2$
$2m \leq d(C_m \times K_5,$

1 Introduction

A *k-coloring* of a graph G
the vertices of G such that