On $k$-Ordered Graphs


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Abstract: Ng and Schultz [J Graph Theory 1 (1997), 45–57] introduced the idea of cycle orderability. For a positive integer $k$, a graph $G$ is $k$-ordered if for every ordered sequence of $k$ vertices, there is a cycle that encounters the vertices of the sequence in the given order. If the cycle is also a Hamiltonian cycle, then $G$ is said to be $k$-ordered Hamiltonian. We give sum of degree conditions for nonadjacent vertices and neighborhood union conditions that imply a graph is $k$-ordered Hamiltonian. © 2000 John Wiley & Sons, Inc. J Graph Theory 35: 69–82, 2000

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1. INTRODUCTION

Over the years Hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example Hamiltonian connectedness. Recently a new strong Hamiltonian property was introduced in [7].

We say a graph $G$ on $n$ vertices, $n \geq 3$ is $k$-ordered for an integer $k$, $1 \leq k \leq n$, if for every sequence $S = (x_1, x_2, \ldots, x_k)$ of $k$ distinct vertices in $G$, there exists a cycle that contains all the vertices of $S$ in the designated order. A graph is $k$-ordered Hamiltonian if for every sequence $S$ of $k$ vertices there exists a Hamiltonian cycle which encounters $S$ in its designated order. We will always let $S = (x_1, x_2, \ldots, x_k)$ denote the ordered $k$-set. If we say a cycle $C$ contains $S$, we mean $C$ contains $S$ in the designated order under some orientation.

Ng and Schultz [7] showed the following:

**Proposition 1** [7]. Let $G$ be a Hamiltonian graph on $n$ vertices, $n \geq 3$. If $G$ is $k$-ordered, $3 \leq k \leq n$, then $G$ is $(k - 1)$-connected.

**Theorem 2** [7]. Let $G$ be a graph of order $n \geq 3$ and let $k$ be an integer with $3 \leq k \leq n$. If

$$
\deg u + \deg v \geq n + 2k - 6
$$

for every pair $u$, $v$ of nonadjacent vertices of $G$, then $G$ is a $k$-ordered Hamiltonian graph.

**Corollary 3** [7]. Let $G$ be a graph of order $n \geq 3$ and let $k$ be an integer such that $3 \leq k \leq n$. If

$$
\deg v \geq \frac{n}{2} + k - 3
$$

for every vertex $v$ of $G$, then $G$ is a $k$-ordered Hamiltonian graph.

The degree condition in the preceding corollary was improved by Kierstead, Sarkozy, and Selkow as follows:

**Theorem 4** [6]. Let $k \geq 2$ be a positive integer and let $G$ be a graph of order $n$, where $n \geq 11k - 3$. Then $G$ is $k$-ordered Hamiltonian if $\delta(G) \geq \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil - 1$.

One goal of this paper is to improve upon the results obtained by Ng and Schultz in [7]. We obtain a sharp lower bound on the degree sum of nonadjacent vertices that imply a graph is $k$-ordered Hamiltonian. In particular, we prove the following:

**Theorem 5.** Let $k \geq 3$ be a positive integer and let $G$ be a graph of order $n \geq 53k^2$. If for any two nonadjacent vertices $x$ and $y$, $\deg x + \deg y \geq n + \frac{3k - 9}{2}$, then $G$ is $k$-ordered Hamiltonian.
We can see that the degree sum condition in Theorem 5 is sharp by considering the following example which was mentioned in [7]. The graph \(G\) on \(n\) vertices is composed of the three parts: \(K_{k-1}, K_k - C_k,\) and \(K_{n-2k+1}\) containing all the edges between \(K_{k-1}\) and \(K_k - C_k\) and all edges between \(K_{k-1}\) and \(K_{n-2k+1}\). Between \(K_{n-2k+1}\) and \(K_k - C_k\), \(G\) contains only the edges incident to the even indexed vertices of \(C_k\). This graph is not \(k\)-ordered because there is no cycle containing the vertices of \(C_k\) in order. For \(x \in V(K_{n-2k+1})\) and \(y \in V(K_k - C_k), y\) an odd indexed vertex on \(C_k, \deg x + \deg y = n + \frac{3k-10}{2}\) for \(k\) even. Another interesting characteristic in the graph \(G\) above is that for all nonadjacent pairs of vertices \(x,y \in V(G), |N(x) \cup N(y)| \geq n - 2\). Thus, for a graph to be \(k\)-ordered, we need \(|N(x) \cup N(y)| > n - 2\) which forces the graph to be complete. So there is no nontrivial sufficient condition on unions of neighborhoods of nonadjacent pairs of vertices.

Further, we obtain the following result concerning neighborhood unions of pairs of vertices. The bounds in this theorem are motivated by the following example. Let \(k\) be a positive integer and let \(G\) be a \(k\)-connected graph of order \(n\). We say \(x\) is \(k\)-insertible if there exist \(y\) and \(\hat{x}\) such that \(N(x) \cup N(y) \geq n - 2\). This graph is not \(k\)-insertible if \(x\) and \(y\) are both odd or both even. Another interesting characterization in the graph \(G\) above is that for all nonadjacent pairs of vertices \(x,y \in V(G), |N(x) \cup N(y)| \geq n - 2\). Thus, for a graph to be \(k\)-insertible, we need \(|N(x) \cup N(y)| > n - 2\) which forces the graph to be complete. So there is no nontrivial sufficient condition on unions of neighborhoods of nonadjacent pairs of vertices.

Theorem 6. Let \(k\) be a positive integer and let \(G\) be a \(k\)-connected graph of order \(n \geq 18k^2\). If \(|N(x) \cup N(y)| \geq \frac{n+k}{2}\) for all pairs of distinct vertices \(x,y \in V(G), then G\) is \(k\)-ordered Hamiltonian.

Before beginning the proofs of these theorems, we make one general observation. If \(G\) is a Hamiltonian graph, then \(G\) is \(k\)-ordered Hamiltonian for \(k = 1, 2,\) and \(3\). By a result in [8], we know the degree sum condition in Theorem 5 implies the graph is Hamiltonian. By results in [5] and [3], we know the neighborhood condition in Theorem 6 implies the graph is Hamiltonian. Thus, in the proofs of these theorems we will immediately assume \(k \geq 4\).

2. DEGREE CONDITIONS

In this section we will prove Theorem 5. However, the following result and its corollary, which give sufficient conditions for \(k\)-ordered to imply \(k\)-ordered Hamiltonian, will make the proof much easier. We say a vertex \(x\) is \(k\)-insertible in the cycle \(C\) if \(N(x)\) contains consecutive vertices on \(C\). We say a subgraph \(H \subseteq G\) is \(k\)-insertible in the cycle \(C\) if there exists a Hamiltonian path \(P\) in \(H\) with endpoints \(x\) and \(y\) such that there exist \(a_x \in N(x), a_y \in N(y)\) and \(a_x\) and \(a_y\) are consecutive vertices on \(C\). If the cycle \(C\) is obvious, we say simply the vertex \(x\) (or the subgraph \(H\)) is \(k\)-insertible. Also, we define \(N(x,y) = N(x) \cup N(y)\) and \(N[x] = N(x) \cup \{x\}\).
Theorem 7. Let \( k \) be a positive integer and let \( G \) be a \( k \)-connected, \( k \)-ordered graph of order \( n \geq 8k^2 \). If for every pair of nonadjacent vertices \( u \) and \( v \) in \( V(G) \)

\[
\deg u + \deg v \geq n,
\]

then \( G \) is \( k \)-ordered Hamiltonian.

Proof. Let \( S = \{x_1, x_2, \ldots, x_k\} \) be an ordered subset of the vertices of \( G \). Let \( C \) by a cycle of maximum order containing all vertices of \( S \) in appropriate order. The \( k \)-vertices of \( S \) split the cycle \( C \) into \( k \) intervals: \([x_1, \ldots, x_2], [x_2, \ldots, x_3], \ldots, [x_k, \ldots, x_1]\). Let \( L = G - C \). Notice that no vertex of \( L \) has more than \( |V(C)|/2 \) adjacencies to \( C \). This implies that any nonadjacent pair of vertices in \( L \) have degree sum at least \( |V(L)| \) in \( L \). Thus \( L \) is Hamiltonian if it has at least three vertices and complete otherwise. Assume there are vertices \( x, y \in L \) with distinct neighbors in one of the intervals of \( C \) determined by \( S \), say \([x_i, x_{i+1}]\). Note that we allow \( x = y \). Let \( z_1 \) and \( z_2 \) be the immediate successor and predecessor on \( C \) to the neighbors of \( x \) and \( y \), respectively, according to the orientation of \( C \). (See Figure 1.) Observe that we can choose \( x \) and \( y \) and their neighbors in \( C \) such that none of the vertices on the interval \([z_1, z_2]\) have neighbors in \( L \). We can also assume that \( z_1 \neq z_2 \), because \( z_1 = z_2 \) implies \( x = y \) or \( C \) is not of maximal order. But neither \( z_1 \) nor \( x \) can be adjacent to more than half the vertices of \( C \) which forces

\[
\deg z_1 + \deg x \leq 2\left(\frac{|C|}{2}\right) + |L| - 1 = n - 1,
\]

a contradiction.

Let \( s = |[z_1, z_2]| \) and \( t = |L| \). Because \( x \) and \( y \) have no neighbors in \([z_1, z_2]\),

\[
\deg x + \deg y \leq 2\left((t - 1) + \frac{n - s - t + 1}{2}\right).
\]

Similarly, if \( z_1 \) is adjacent to a vertex, say \( w \), on \( C - [z_1, z_2] \), \( z_2 \) cannot be adjacent to the successor, \( w^+ \), on \( C \) or else the segment \([z_1, z_2]\) could be inserted between \( w \)
and \( w^+ \), while replacing \( [z_1, z_2] \) with a path from \( x \) to \( y \). Hence,
\[
\deg z_1 + \deg z_2 \leq 2(s - 1) + n - s - t + 1.
\]
Since \( x \) and \( y \) are both nonadjacent to \( z_1 \) and \( z_2 \), the initial degree condition forces 
\[
\deg x + \deg y + \deg z_1 + \deg z_2 \geq 2n.
\]
But, by the previous two inequalities
\[
\deg x + \deg y + \deg z_1 + \deg z_2 \leq 2n - 2,
\]
which is a contradiction. Thus on any interval \([x_i, x_{i+1}]\) of \( C \), there exists at most one vertex with neighbors in \( L \). The connectivity, then, requires each segment
\([x_i, x_{i+1}]\) to have exactly one vertex with a neighbor in \( L \). Also, we know \(|C| \geq n/2\) by observing that if \( y_1 \) and \( y_2 \) are adjacencies of \( L \) in consecutive intervals on \( C \), then the successors of these vertices, say \( w_1 \) and \( w_2 \), are not adjacent (otherwise \( C \) could be extended). Neither \( w_1 \) nor \( w_2 \) has any adjacencies in \( L \), but one of these vertices, say \( w_1 \), has degree at least \( n \) which forces \( C \) to have at least \( n/2 \) vertices. This forces \( L \) to be Hamiltonian connected.

Thus, the order of at least one of these \( k \) intervals of \( C \) must be a function of \( n \), say \( f(n) \geq \frac{n}{2k} \). Assume the interval \([x_1, x_2]\) is such a segment. Let \( z \) be the unique vertex in this interval with a neighbor in \( L \). Without loss of generality we can assume the interval \([x_1, z]\) contains \( f(n)/2 \) vertices. Let \( y \) be the unique vertex in \([x_2, x_3]\) with a neighbor in \( L \). The connectivity guarantees that if \(|L| \geq k\) we can find distinct neighbors of \( z \) and \( y \) in \( L \). Let \( y_1 \) and \( z_1 \) be the immediate predecessors of \( y \) and \( z \), respectively, on \( C \). Note that if \( y_1 \) and \( z_1 \) are adjacent, \( L \) can be inserted. (See Figure 2).

If \(|L| \geq \frac{n}{2} - k\), then \( y_1 \) has no neighbors in the interval \([x_1, z_1]\) or \( C \) would not be of maximum order. So \( \deg y_1 + \deg z_1 \leq 2(n/2 + k - 2) - f(n)/2 < n \), which is a contradiction since \( n \geq 8k^2 \).

If \(|L| < n/2 - k\), then all the vertices in the interval \([x_1, z_1]\) must have degree at least \( n - (|L| - 1 + k) = |C| - k + 1 \). Thus each is adjacent to all but possibly \( k - 1 \) of the vertices of \( C \), and, therefore, are insertible. In particular, they are insertible on \( C - [x_1, x_2] \). If \( y_1 \) and \( z_1 \) are not adjacent, we insert \( z_1 \) and compare \( y_1 \) to the predecessor of \( z_1 \), say \( z_2 \). If an edge exists here, \( L \) can be inserted. If not, we insert \( z_2 \), and continue. Thus insertion process must end before reaching \( x_1 \) since \( y_1 \) also must be adjacent to all but at most \( k - 2 \) vertices of \( C \).

An immediate corollary to Theorem 7 is the following:

![Figure 2](image-url)
Corollary 8. Let $k$ be a positive integer and let $G$ be a $k$-connected, $k$-ordered graph of order $n \geq 8k^2$. If $\delta(G) \geq \frac{n}{2}$, then $G$ is $k$-ordered Hamiltonian.

Before proving theorem 5, we restate it for reference.

Theorem 5. Let $k \geq 3$ be a positive integer and let $G$ be a graph of order $n \geq 53k^2$. If for any two nonadjacent vertices $x$ and $y$, $\deg x + \deg y \geq n + \frac{3k-9}{2}$, then $G$ is $k$-ordered Hamiltonian.

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be an ordered set of vertices of $G$. Note that by Theorem 7 it is enough to show that $G$ is $k$-ordered. The proof will be split into cases according to the connectivity of the graph. Observe that the degree condition forces $K(G) \geq \left\lceil \frac{3k-5}{2k} \right\rceil$ for $k \geq 4$.

Case 1: $K(G) \geq 5k$.

Find a cycle in $G$ by finding a shortest $x_1-x_2$ path $P_1$, in $G - \{x_3, \ldots, x_k\}$, then the shortest $x_2-x_3$ path $P_2$, in $G - \{x_4, \ldots, x_k\} - P_1$, and continue this process to generate $k-1$ internally disjoint paths $P_i$ for $i = 1, \ldots, k-1$. Note that the shortest length path from $x_i$ to $x_{i+1}$ is no more than 5. If not, we can label a path of minimum length as: $x_i = v_1, v_2, \ldots, v_l = x_{i+1}$, where $l \geq 7$. Then $v_1, v_4$, and $v_7$ are all mutually nonadjacent and have mutually disjoint neighborhoods in $G - S - \{V(P_1) \cup \cdots \cup V(P_{i-1})\}$. Thus,

$$n \geq \deg(v_1) + \deg(v_4) + \deg(v_7) \geq \frac{3}{2} \left(n + \frac{3k-9}{2}\right) - 5i - (k-i),$$

which is a contradiction for $n \geq \frac{11k}{2}$. But $n \geq \frac{11k}{2}$, since we assume $n \geq 53k^2$. By the connectivity of $G$, and $x_kx_1$ path must exist.

Case 2: $\frac{3k-1}{2} \leq K(G) < 5k$.

Subcase A: $\delta < 100k$.

Without loss of generality, we can assume that $G$ is edge-maximal with respect to the property of not being $k$-ordered (i.e., the addition of any edge makes $G$ $k$-ordered). Let $\deg v = \delta$ and $F = G - N[v]$. Also, let $L = \{v \in V(G) | \deg v < n/2\}$ (or vertices of low degree) and $H = V(G) - L$ (or vertices of high degree). We claim $F$ is complete and that every $w \in N[v] \cap H$ is adjacent to every vertex in $F$. Note that for $x \in F$, $\deg x > n - 100k$. Let $x$ and $y$ be nonadjacent vertices in $F$. Then the insertion of the edge $xy$ makes $G$ $k$-ordered. Let $C$ be the smallest cycle in $G + \{xy\}$ that contains $S$ in order. First, we claim that the cycle $C$ contains no more than half the vertices of $G$. Assume otherwise, and let $|V(C)| = \alpha n$ where $\alpha > 1/2$. Then there exists an interval, say $[x_i, x_{i+1}]$, that contains at least $\alpha n/k$ vertices. Note that no interval can have more than two vertices of $L$ or a smaller cycle is possible. Further, nonadjacent vertices a distance three or more apart on an interval can have no common neighbors off the cycle. Thus, we can find a subset of vertices of $[x_i, x_{i+1}]$, say $M$, such that $|M| \geq (\alpha n/3k) - 1, M$ is independent, and all vertices of $M$ have degree greater
than \( n/2 \). (One such \( M \) would be every third vertex on \([x_i, x_{i+1}]\).) The set \( M \) must send at least \( \left( \frac{an}{k} - 1 \right) \left( \frac{n}{2} \right) - (n - an) = m \) edges to vertices of \( C \). Thus, there exists some interval different from \([x_i, x_{i+1}]\) such that there are at least \( m/k \) edges between the two intervals. By the results in ([2] page 311) there exists a \( K_{4,4} \) between the two intervals. But such a \( K_{4,4} \) can be used to produce a smaller cycle, a contradiction (see Figure 3). Thus, the cycle \( C \) can contain at most \( n/2 \) vertices.

Let \( j \in V(C) \). Because \( c \leq n/2 \), both \( x \) and \( y \) have distinct neighbors off the cycle. Let \( x_2 \in N(x) \setminus C, y_2 \in N(y) \setminus C \). Then, \( x^* \) and \( y^* \) are nonadjacent, have no common neighbors off the cycle, and can each have at most three adjacencies in each interval of the cycle for otherwise the length of \( C \) will not be maximal. So,

\[
\frac{n + 3k - 9}{2} \leq \deg x^* + \deg y^* \leq (n - c) + 6k.
\]

This forces \( c < 6k. \) But this is impossible since

\[
2(n - 100k) \leq \deg x + \deg y \leq n + c. \tag{*}
\]

(This right hand side of the inequality follows from the fact that \( x \) and \( y \) can have no common neighbors off the cycle.) We now have shown that \( F \) is complete. Note that the same argument applies if we choose \( x \in V(F) \) and \( y \in N(v) \cap H \).

The only difference is the inequality \((*)\) above becomes

\[
(n - 100k) + \frac{n}{2} + \frac{3k - 9}{4} \leq \deg x + \deg y \leq n + c,
\]

which produces the same contradiction.

Partition \( S \) into \( S_L \) and \( S_H \) where \( S_H = S \cap H \) and \( S_L = S - H \). Note that \( S_L \subseteq N[v] \) and \( \langle S_L \rangle \) must be complete. Also note that every vertex in \( S_H \) is either a vertex in the complete subgraph \( F \) or is adjacent to every vertex of \( F \). Assume \( |S_L| = l, \) so \( |S_H| = k - l. \) If \( l \leq k/2 \), then \( \kappa(G - S_H) \geq 2l \). For every \( x_i \in S_L \) create a vertex \( x_i^* \) such that \( N(x_i^*) = N(x_i) \). Let \( S_L^* = S_L \cup \{x_i^* : i = 1, \ldots, l\} \). Add a vertex \( v \) such that \( N(v) = S_L^* \). Let \( G^* \) be the graph that results from adding \( v \) and \( \{x_i^* : x_i \in S_L\} \). Then \( G^* - S_H \) is still \( 2l \)-connected. Let \( M \) be a set of \( 2l \) distinct
vertices of $V(F - S_H) \subset V(G^*)$. A generalization of Whitney’s Theorem [4] implies that there exist $2l$ internally vertex disjoint paths, each starting at the vertex $v$ and ending at a distinct vertex of $M$. But this implies that, if we return to the graph $G$, for every vertex $x_i$ in $S_L$, we can find a pair of internally vertex disjoint paths, starting at $x_i$ and ending at distinct vertices of $M$, say an $x_i - v_{i,1}$ path and an $x_i - v_{i,2}$ path where $v_{i,j} \in M$ and $v_{i,j} \neq v_{i,j'}$. Now all vertices of $S$ are either in the complete subgraph $F$ of $G$ or have two disjoint paths to $F$. Thus we can construct a cycle containing $S$ in the appropriate order using these paths to $M$ and edges in $F$.

If $l = \frac{k}{2} + t$ for $t > 0$, then $K(G - S_H) \geq \frac{3k-1}{2} - \left(\frac{k}{2} - t\right) = 2l - t - \frac{1}{2}$. Since $S_L$ has over half the vertices of $S$, there are at least $t$ consecutive pairs $(x_i, x_i+1)$. Paths between these are made with edges in $S_L$, leaving no more than $2l - 2t$ “endpoints” of paths. Construct vertex disjoint paths in $G - S_H$ just as was done in the case $l \leq k/2$.

Subcase B: \( \delta \geq 100k \).

Let $K$ be a minimal cut set. Let $A$ and $B$ be the components of $G - K$. Find $P_3$’s from $A$ to $B$ through $K$ with the least number of vertices from $S$. Because of the minimum degree condition and $|A| \geq |B| \geq 95k$ we know we can avoid a $P_3$ that is a triple. The number of free $P_3$’s is at least $\frac{3k-1}{2} - (k - d) \geq 2d$ where $d$ is the number of doubles. (Observe that if $k$ is even, $K(G) \geq 3k/2$ and if $k$ is odd, $d \leq (k - 1)/2$.) Finally, for any $x, y \in V(A)$ (or respectively $B$), if $x$ and $y$ are nonadjacent, $|N_A(x) \cap N_A(y)| \geq \left(n + \frac{3k-9}{2} - 10k\right) - (n - 100k) > 90k$. In particular, there are more than $k$ internally vertex disjoint $x$–$y$ paths of length 2 in $A$. Thus, if $x_i$ and $x_i+1$ are both in $A$ or both in $B$ we can find a path of length no more than 2 connecting them. Otherwise, we use $P_3$’s ending both in $A$ or both in $B$ to connect $x_i$ and $x_{i+1}$. Thus, just as in the previous arguments, it is straightforward to construct a cycle containing $S$.

Case 3: \( \frac{3k-5}{2} \leq K(G) \leq \frac{3k-2}{2} \).

Pick a minimal cut set $K$. Let $A$ and $B$ be the components of $G - K$. Then a vertex $x$ in $A$ (or respectively in $B$) is adjacent to every other vertex of $A(B)$ and $K$ except possibly one. To show this consider $x \in V(A)$ and $y \in V(B)$, such that one of them has two nonadjacencies. Then,

$$n + \frac{3k-9}{2} \leq \deg x + \deg y \leq (n - K(G) - 2) + 2K(G) - 2$$

which forces $K(G) \geq (3k - 1)/2$. Thus we can find a cycle in $G$ containing $S$. $\blacksquare$

3. NEIGHBORHOOD CONDITIONS

In this section we will prove the result concerning neighborhood conditions. As before, we will first prove sufficient conditions under which $G$ is $k$-ordered implies $G$ is $k$-ordered Hamiltonian.
Theorem 9. Let $k$ be a positive integer and let $G$ be a graph of order $n \geq 10k$ with $|N(x, y)| \geq \frac{n+k}{2}$ for all pairs of distinct vertices $x$ and $y$ and $\mathcal{K}(G) \geq k+1$. If $G$ is $k$-ordered, then $G$ is also $k$-ordered Hamiltonian.

Proof. Let $S = \{x_1, \ldots, x_k\}$ be an ordered set of vertices of $G$. Let $C$ be a maximal cycle containing $S$. (i.e., $C$ cannot be extended by insertions of a vertex or a path.) Let $H$ be the largest component of $G - C$, $t = |G - C - H|$, and let $|H| = r$. We can think of the elements of $S$ as splitting the cycle $C$ into $k$ intervals: $[x_1, \ldots, x_2], \ldots, [x_k, \ldots, x_1]$.

The proof consists of four cases that depend on the order of $H$ and the magnitude of $t$.

Case 1: $|H| = r \geq k + 1$.

By the connectivity, we know there exist $k + 1$ distinct vertices in $H$ with $k + 1$ distinct neighbors in $C$. Thus there exists two vertices in $H$, say $y_1$ and $y_2$, with distinct neighbors, say $w_1$ and $w_2$ in a single interval of $C$. We can pick $y_1$ and $y_2$ (and $w_1$ and $w_2$) so that $w_1$ and $w_2$ are as close as possible on $C$. Let $z_1, z_2, \ldots, z_s$ be the vertices of $C$ between $w_1$ and $w_2$. If $z_1$ is insertible in $C$, insert it and proceed to $z_2$. If $z_2$ is insertible, do so and proceed to $z_3$. Do the same starting at $z_s$, then moving to $z_{s-1}$, and so forth. If all of the $z_i$ are insertible or all but one are insertible, we close the cycle with a $(y_1-y_2)$-path in $H$ producing a larger cycle. If a section remains, say $z_i, z_{i+1}, \ldots, z_j$, then we know $|N(z_i, z_j)| \leq t + s + \frac{n-r-s-t+1}{2}$. Also $|N(y_1, y_2)| \leq r + \frac{n-r-s-t+1}{2}$. Thus, $|N(y_1, y_2)| + |N(z_i, z_j)| \leq n + 1$. But this is a contradiction, since by assumption $|N(y_1, y_2)| + |N(z_i, z_j)| \geq n + k$.

Case 2: $2 \leq |H| = r \leq k$.

Suppose $H$ is not complete. Then we can select nonadjacent vertices $x$ and $y$ in $H$. Thus $\frac{n+k}{2} \leq |N(x, y)| \leq |H| + \frac{n-r-t}{2} = \frac{n+r-t}{2}$ which is a contradiction unless $t = 0$, $r = k$, and $H$ is complete. Without loss of generality assume the length, $s$, of the segment of the circle from $x_1$ to $x_2$ is maximum. Then from the proof of Case 1 we know at most one vertex along this segment has more than one adjacency to $H$. Select two vertices in $H$, say $y_1$ and $y_2$, with degree at most one relative to this segment. Then $|N(y_1, y_2)| \leq k + 2 + \frac{n-k-s+t+1}{2}$ which is a contradiction since $s \geq \frac{n-k}{k}$ and $n \geq 7k$.

Case 3: $|H| = 1$, $t \geq 1$.

Pair all the vertices in the segment of $C$ from $x_t$ to $x_{t+2}$ starting with $x_t$. Let $w$ and $z$ be in $G - C$. Certainly, $N(w)$ (and $N(z)$) can have at most one neighbor in each pair. Also, $N(w, z)$ contains at most one vertex from each pair with at most two exceptions, and these two exceptions must have opposite orientation. (For example, if $z$ hit immediate predecessors of neighbors of $w$ twice, the cycle could be extended.) But now, of the pairs that sit between these two exceptional ones, $N(w, z)$ must miss one altogether or $C$ can be enlarged. Thus, for every segment $P = [x_i, x_{i+1}], |N_P(w, z)| \leq \frac{|P|}{2} + 1$. This forces $|N(w, z)| < (n + k)/2$, which is a contradiction.
Case 4: $|H| = 1, t = 0.$

Let $x$ be the single vertex not in $C.$ We will first show that we can assume $\deg x \geq \frac{n+2}{4}.$ If this is not the case, then observe that $x$ has at least $k+1$ adjacencies in $C.$ Thus, there is one interval on $C$ determined by $S$ where $x$ has two adjacencies. Let $W = \{w_1, w_2, \ldots, w_i\}$ be a section of $C$ between two such consecutive adjacencies of $x.$ Exchange the section $W$ for the vertex $x$ on the cycle $C.$ Note that all the vertices of $W,$ and in fact all the vertices of $G$ except for $x,$ have degree at least $\frac{n+2}{4}.$ Now insert as many of the vertices of $W$ as possible. If all the vertices of $W$ can be inserted, we have found a Hamiltonian cycle containing $S.$ If not, we apply the methods of the previous three cases with the one restriction that the unique vertex of small degree should always remain as a vertex on $C.$

For example, Case 1 applies directly unless there are precisely $k+1$ independent edges from $H$ to $C,$ only one pair of these edges are in a single interval on $C,$ and the vertex $x$ is between the endpoints of the two edges. If $|H| = k + 1,$ every vertex of $H$ has at least $(n - 3k)/4$ neighbors on $C.$ But at most two of the vertices of $H$ can have more than one neighbor in the same interval. Thus $|C| \geq (k-1)(n-3k)/2 > n$ since $k \geq 4.$ If $|H| > k + 1,$ then the degree condition on vertices of $H$ forces $|H| \geq (n-3k)/4.$

Thus pick two vertices of $H$ with the fewest neighbors to $C.$ Such a pair forces $|H| \geq \frac{n+2}{4} - k = (n-k)/2$ and $|C| \leq (n+k)/2.$ This means every pair of vertices on $C$ without neighbors in $H$ are adjacent. Specifically, if $e_1$ and $e_2$ are independent edges from $H$ to $C$ with end vertices $a_1$ and $a_2$ on $C,$ then their successors $a_1^+$ and $a_2^+$ are adjacent and the cycle can be enlarged.

Because all the vertices of $H$ have high degree, Case 2 occurs only if $|H| = k.$ Let $w_i, w_j \in V(H).$ Then $N(w_i, w_j)$ must include half the vertices of $C$ but no consecutive pairs of vertices on $C$ or the cycle could be extended. Consider a section of $C$ labeled $a_1, b_1, a_2, b_2, a_3$ where $a_1, a_2, a_3 \in N(w_i, w_j).$ If $b_1$ or $b_2$ are insertible, the cycle can be enlarged. If neither $b_1$ or $b_2$ are insertible, then the path $b_1, a_2, b_2$ must be insertible and $C$ can be enlarged.

No alteration is necessary to apply Case 3. Thus we can assume that the single vertex $x$ not on the cycle $C$ has degree at least $(n+k)/4.$

We can extend $C$ to include $x$ unless between every pair of consecutive adjacencies of $x$ on $C$ there is at least one uninsertible vertex, say vertex $y_i,$ between the $i$th consecutive pair of neighbors of $x$ on $C.$ We want to show that there exists some $y_i$ such that $\deg y_i > \frac{2n}{l}.$ Select $l$ vertices from the set of $y_i$’s, and assume that $d_1 n \leq \deg y_i \leq d_2 n.$ Thus,

$$\deg y_i + \deg y_j = |N(y_i, y_j)| + |N(y_i) \cap N(y_j)| \geq n/2 + \alpha_{ij} n$$

where $\alpha_{ij} = |N(y_i) \cap N(y_j)|/n.$ Let $E$ be the number of edges of $G$ from the set of $y_i$’s to the cycle $C.$ Then,

$$\frac{1}{l - 1} \left[ \binom{l}{2} \frac{n}{2} + \sum_{y_{ij}} \alpha_{ij} n \right] \leq |E| \leq ld_2 n.$$
We observe that the average degree of the vertices in $G - \{y_1, \ldots, y_l\}$ to the $y_i$'s is at least $\frac{ld_2 n}{2}$. Thus, $\sum \alpha_{ij} n \geq n \left( \frac{ld_1}{2} \right)$, so

$$\frac{1}{l - 1} \left[ \binom{l}{2} \frac{n}{2} + n \left( \frac{ld_1}{2} \right) \right] \geq ld_2 n.$$  

We simplify this to get,

$$\frac{1}{4} + \frac{d_1 (ld_1 - 1)}{2(l - 1)} \leq d_2.$$  

Thus,

$$\frac{1}{4} + \frac{1}{32} \leq \frac{1}{4} + \frac{d_2^2}{2} \leq d_2.$$  

Now repeat this again by culling the $y_i$'s to a possibly smaller set of vertices whose degree is between $d_2 n \frac{9}{32} n$ and $d_3$. By the same argument, $d_3 \geq \frac{1}{4} + \frac{(9/32)^2}{32} > \frac{2}{7}$. Note that this implies we can always adjust the vertices of $C$ so that the single vertex not on $C$ has degree $\gamma n > 2n/7$. At this point we need to show that any vertex $x = G - C$ of degree at least $\gamma n$ can be inserted into the cycle $C$. First we claim that there must be three consecutive adjacencies of $x$ within a single interval of $C$, say $[x_i, x_{i+1}]$, such that there are at most five vertices between $a_1$ and $a_3$. If not, we know there must be deg $x - 2k$ triples of neighbors of $x$ within a segment, each of which requires eight vertices. Thus, taking into account vertices that are counted twice we find

$$|C| \geq \frac{7(\gamma n - 2k)}{2} > \frac{7\gamma n}{2} - 7k > n,$$

a contradiction.

So we can assume $x$ has three consecutive adjacencies, $a_1, a_2, a_3$, in $[x_i, x_{i+1}]$ with at most five vertices between $a_1$ and $a_3$. The arguments applied to the case where there are exactly five vertices between $a_1$ and $a_3$ are easily applied to the cases where there are fewer than five vertices, so we prove the only case where there are exactly five.

First consider the subcase where there is one vertex between $a_1$ and $a_2$ and three between $a_2$ and $a_3$. We label this segment: $a_1, w, a_2, y_3, y_2, y_1, a_3$. The single vertex, $w$, must not be insertible or we could have added $x$ to the cycle. Also $w$ and $y_1$ are nonadjacent or $x$ would have been inserted. Thus $N(w, y_1)$ includes at most three of the five vertices between $a_1$ and $a_3$. So $N(w, y_1)$ must include at least $(n + k - 6)/2$ of the $n - 6$ vertices on the cycle from $a_3$ to $a_1$. Since $k \geq 3$, we know $N(w, y_1)$ must include a pair of vertices that are consecutive on $C$. If $N(w)$ and $N(y_1)$ hit a consecutive pair, then segment $w, a_2, y_3, y_2, y_1$ can be inserted. If $y_1$ is insertible, compare $w$ and $y_2$ as above. Note that inserting $y_1$ does
not effect the insertibility of \( w \). Thus, we either find a pair \( w \) and \( y_i \) that allows us to enlarge the cycle, or we exhaust the \( y_i \)'s by inserting them in another place on the cycle.

Second, we consider the case where there are two vertices between each pair of \( a_i \)'s. We label the segment: \( a_1, w_1, w_2, a_2, y_1, a_3 \). One of \( y_1 \) and \( y_2 \), (and respectively \( w_1 \) and \( w_2 \)) must be insertible and the other not insertible. Assume \( w_1 \) is not insertible. Then \( y_1 \) is insertible or the degree condition would force \( y_1 \) and \( w_1 \) to have consecutive neighbors. Insert \( y_1 \). Consider \( w_1 \) and \( y_2 \). If they have consecutive neighbors, the path \( w_1, w_2, a_2, y_1 \) is insertible. So it must be the case where \( w_1 \) is insertible as a result of the insertion of \( y_1 \). Thus \( w_1, y_1 \) is an edge. Now we add \( x \) to the cycle by inserting \( w_2 \) and using the \( w_1, y_1 \) edge. By symmetry, \( y_1 \) and \( w_1 \) must be insertible while \( y_2 \) and \( w_2 \) both are not. Insert \( w_1 \). If \( y_1 \) is no longer insertible, it must be that \( w_1 \) and \( y_1 \) must have consecutive neighbors and the path \( w_1, w_2, a_2, y_2, y_1 \) is insertible. So, insert \( y_1 \). Now, \( w_2 \) and \( y_2 \) are unaffected by these insertions (if \( w_2, y_1 \) or \( w_1, y_2 \) are edges, \( x \) can be inserted) and by the degree condition must be insertible as a pair.

Before beginning the proof of theorem 6, we restate it for reference.

**Theorem 6.** Let \( k \) be a positive integer and let \( G \) be a \( k \)-connected graph of order \( n \geq 18k^2 \). If \( |N(x, y)| \geq \frac{n+k}{2} \) for all pairs of distinct vertices \( x, y \in V(G) \), then \( G \) is \( k \)-ordered Hamiltonian.

**Proof.** Let \( S = \{x_1, x_2, \ldots, x_k\} \) be a set of ordered vertices of \( G \). We first consider the case where the connectivity is exactly \( k \). In this case, let \( K \) be a minimal cut set separating \( G \) into the two components \( A \) and \( B \). By considering any pair of distinct vertices in \( A \) (or respectively, \( B \)), the neighborhood condition forces \( |A| = |B| = \frac{n-k}{2} \), both \( A \) and \( B \) to be complete, and every vertex in \( K \) to be adjacent to all but at most one vertex in \( A \) and one vertex in \( B \). Thus it is easy to verify that \( G \) is \( k \)-ordered Hamiltonian. Thus, we can assume \( K \geq k + 1 \).

By the previous theorem, for \( K(G) > k \), we need only to show that under the neighborhood condition, \( G \) is \( k \)-orderable. Again, we split the proof into cases according to the connectivity of \( G \).

**Case 1:** \( K > 7k \).

We claim there exists a path from \( x_1 \) to \( x_k \) containing all elements of \( S \) in order such that the path from \( x_i \) and \( x_{i+1} \) contains at most eight vertices. We construct this path by choosing the shortest \( x_1 \)–\( x_2 \) path that avoids \( S \), then the shortest \( x_2 \)–\( x_3 \) path that avoids \( S \) and any vertices in the \( x_1 \)–\( x_2 \) path, and so forth. Let \( x_i = a_1, a_2, \ldots, a_t = x_{i+1} \) be the shortest \( (x_i \)–\( x_{i+1} \)) path in \( G \) that avoids \( S \) and all vertices used in previous paths. If \( t \geq 9 \), then we observe that \( |N(a_1)| > n/4 \), \( |N(a_4, a_5)| > n/2 \), and \( |N(a_8, a_9)| > n/2 \) and all these sets can intersect on vertices of \( S \) or vertices used on previous paths. Thus,

\[
n \geq |N(a_1)| + |N(a_4, a_5)| + |N(a_8, a_9)| - 2[(7(i - 1) + 1) - (k - 1)] > 5n/4 - 14k,
\]
which contradicts \( n \geq 18k^2 \). Thus, we can find an \( x_1 \rightarrow x_k \) path that contains \( S \) in order and uses at most \( 7k + 1 \) vertices. The connectivity guarantees that we can find an \( x_k \rightarrow x_1 \) path to close the cycle.

**Case 2:** \( \mathcal{K}(G) = k + t \) and \( t > 0 \).

Let \( K \) be a minimal cut set, \( A \) and \( B \) components of \( G - K \), \( |A| = n_1 \) and \( |B| = n_2 \). By Hall’s Theorem we can form a perfect matching of \( K \) into \( A \) and \( K \) into \( B \), which together form \( k + t \) \( P_3 \)’s through \( K \). If both end vertices on such a path are in \( S \), we call the path a triple. If the path has two vertices in \( S \), one of which is the center vertex, it is called a double; one vertex, a single; no vertices, free. Pick matchings such that the paths contain the fewest number of vertices in \( S \). Assume no triples exist. Let \( r_1 (r_2) \) be the number of doubles with an end vertex in \( A \) \( B \) and \( t > 0 \).

Let \( K \) be a minimal cut set, \( A \) and \( B \) components of \( G - K \), \( |A| = n_1 \) and \( |B| = n_2 \). By Hall’s Theorem we can form a perfect matching of \( K \) into \( A \) and \( K \) into \( B \), which together form \( k + t \) \( P_3 \)’s through \( K \). If both end vertices on such a path are in \( S \), we call the path a triple. If the path has two vertices in \( S \), one of which is the center vertex, it is called a double; one vertex, a single; no vertices, free. Pick matchings such that the paths contain the fewest number of vertices in \( S \). Assume no triples exist. Let \( r_1 (r_2) \) be the number of doubles with an end vertex in \( A \) \( B \) that is also in \( S \). Let \( x, y \in A - S \) and \( w, z \in B - S \). Then

\[
|N(x, y)| \leq n_1 + k + t - r_1
\]

and

\[
|N(z, w)| \leq n_2 + k + t - r_2.
\]

Thus,

\[
n_1 + n_2 + 2k + 2t - r_1 - r_2 \geq n + k,
\]

but

\[
n_1 + n_2 + 2k + t = n + k.
\]

So, \( t \geq r_1 + r_2 \). Let \( r = r_1 + r_2 \). Then there exists at least \( 2r \) free \( P_3 \)’s to provide paths between vertices in \( A \) and \( B \). To find paths in \( A \) and \( B \) we use the fact that each of these graphs is \( k \)-linked. A graph is said to be \( k \)-linked if for every set of \( k \) pairs of vertices of \( G \), say \( v_i, w_i \) for \( 1 \leq i \leq k \), \( G \) contains \( k \) internally disjoint \( v_i \rightarrow w_i \) paths for \( 1 \leq i \leq k \). In [2] it was shown that every \( 22k \)-connected graphs is \( k \)-linked. Thus, we want to show that \( A \) and \( B \) are \( 22k \)-connected. Without loss of generality assume \( |A| \leq |B| \). In order to satisfy the neighborhood condition, \( |A| \geq (n + k)/2 - (k + t) \) which forces \( |B| \leq (n - k)/2 \). Let \( K' \) be a minimal cut set of \( B \) and \( B' \) a connected components of \( B - K' \) of smallest order. Let \( x, y \in V(B') \). Then

\[
(n + k)/2 \leq |N(x, y)| \leq |B'| + |K'| + k + t \leq \frac{1}{2} \left( \frac{n-k}{2} - |K'| \right) + |K'| + k + t.
\]

Thus, \( |K'| \geq (n - k - 4t)/2 \geq 22k \) since \( n \geq 18k^2 \). The same argument holds for \( A \). Thus we construct the cycle containing \( S \) by using the linkage to build the segments between consecutive vertices within \( A \) or in \( B \) and the \( P_3 \)’s to build the segments between \( A \) and \( B \).
If a triple occurs, say the middle vertex is $v$, we know $v$ must have small degree (a function of $k$) and consequently we can have at most one triple. Let $x, y \in A - S$ and $w, z \in B - S$. Then

$$|N(x, y)| \geq n_1 + k + t - (r_1 + 1)$$

and

$$|N(w, z)| \geq n_2 + k + t - (r_2 + 1).$$

So,

$$n_1 + n_2 + 2k + 2t - (r_1 + r_2 + 2) \geq n + k.$$ 

Thus $t \geq r_1 + r_2 + 2$, which provides two additional free $P_3$’s (relative to the previous case of no triples) we can associate with $v$. If $v$ is not an element of $S$, we can identify the endvertices of the $P_3$ containing $v$ with the corresponding endvertices of two free $P_3$’s and proceed as in the previous case. If $v$ is in $S$, we know $v$ has $t + 1 \geq 2$ neighbors not in $S$. Let $w$ be such a neighbor. If $w$ is in $A$ or $B$, it must already be used in a $P_3$ or the triple would not have occurred. Also, the $P_3$ using $w$ must not have been able to exchange $w$ for another vertex outside $S$. But this forces its center to have a large degree to the side opposite $w$ making it a free $P_3$ or a single $P_3$. If $w$ is in $K$ it is either on a free $P_3$ or on a single $P_3$. If it is a single with vertex $x_i$, we assign to $x_i$ one of the two extra free $P_3$’s associated with $v$ and use this one for $v$. The paths from $v$ will use the edges to the two neighbors of $v$ not in $S$. Thus we can always find two paths from $v$ to either $A$ or $B$ avoiding vertices of $S$. This completes the proof of Theorem 6.

\[\square\]

**References**