DEGREE SETS FOR HOMogeneously TRACEABLE NONHAMILTONIAN Graphs

by

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A graph $G$ is homogeneously traceable if for each vertex $v$ of $G$ there exists a hamiltonian path of $G$ with initial vertex $v$. While every hamiltonian graph is homogeneously traceable, not every homogeneously traceable graph is hamiltonian. For example, the Petersen graph is a homogeneously traceable nonhamiltonian graph. In [1] it was shown that homogeneously traceable nonhamiltonian graphs exist for all orders $p$ except $3 < p < 8$. In the construction presented, every homogeneously traceable nonhamiltonian graph of order 9 and greater contained a vertex of degree 3.

R. Frucht (personal communication) asked if there exist homogeneously traceable nonhamiltonian graphs with only large degrees. Of course, the Petersen graph is cubic. It is the object of this paper to give a complete answer to this question.

The following result was established in [3] and will be useful.

**Lemma.** If $G$ is homogeneously traceable of order $p \geq 3$, then $G$ is 2-connected.

It is convenient to construct a class of graphs for use in our main result. Define the graphs $H_{n+1}$, $n \geq 1$, to consist of 2 disjoint cycles, $C$: $u_1, u_2, \ldots, u_{2n+1}$ and $C'$: $v_1, v_2, \ldots, v_{2n+1}$, and for each $i = 1, 2, \ldots, 2n+1$ join $u_i$ and $v_i$ by a path $P_i$ of length 2. Denote the vertex of degree 2 on $P_i$ by $t_i$. These graphs are homogeneously traceable and nonhamiltonian for each $n \geq 1$.

The degree set of a graph $G$ is the set of degrees of the vertices of $G$.

**Theorem.** Suppose $S = \{n_0, n_1, \ldots, n_k\}$ is a set of $k+1$ ($\geq 1$) positive integers and $n_i \geq 2$ for all $i$ ($0 \leq i < k$). Then $S$ is the degree set of a homogeneously traceable nonhamiltonian graph unless $S = (2)$.

**Proof.** Without loss of generality we assume that $n_0 < n_1 < \ldots < n_k$. Suppose $S = (2)$. Then, by the Lemma, $G$ is 2-connected. Since $G$...
is 2-regular of order at least 3, then $G$ is a cycle, and hence $G$ is hamiltonian. Thus (3) is not the degree set of a homogeneously traceable nonhamiltonian graph. We now consider the converse. Suppose $S \neq (2)$. We distinguish three cases.

Case 1. Suppose $3 \in S$. If $S = (3)$, the Petersen graph satisfies the Theorem. If $S = (2, 3)$, the graph $H_i$ suffices. Now, if $3 \in S$ but $S \neq (3)$ and $S \neq (2, 3)$, consider the graph $H_l$, where $l$ is odd and $l > \max(3, k)$. We now construct a graph $H$ from $H_l$.

If $n_k = 2$, then $n_2 = 3$ and each $u_i$ and $v_i$ ($i = 1, 2, \ldots, k$) has degree 3. For each $i = 1, 2, \ldots, k$ replace the vertex $t_i$ (and its incident edges) by $M_i = K_{n_i - 2}$, where $e = x_iy_i \in E(K_{n_i - 1})$, that is, a copy of the complete graph on $n_i - 1$ vertices minus 1 edge. Then insert the edges $u_i x_i$ and $v_i y_i$. If $l > k$, repeat this argument with $M_i$, replacing each $t_i$, $k + 1 < i < l$. Then $\deg_{H_i} u_i - \deg_{H_i} v_i = 3$ and $\deg_{H_i} x_i = n_k$ for each $x_i \in V(M_i), i = 1, 2, \ldots, k$; for $l > k$ we have $\deg_{H_i} x_i = n_k$ if $x_i \in V(M_l), k + 1 < i < l$. Thus $H$ has the degree set $S$.

To see that $H$ is homogeneously traceable note that each $M_i$ is hamiltonian connected as $\deg_{M_i} < (|V(M_i)| - 1)/2$ for each $v \in V(M_i), i = 1, 2, \ldots, l$. Thus, by Ore's theorem [2], $M_i$ is hamiltonian connected. To find a hamiltonian path beginning with vertex $u_i$ or $v_i$ ($i = 1, 2, \ldots, l$) consider the path $P_i$ in $H_i$ beginning at $u_i$ or, respectively, at $v_i$ with vertex $t_i$ replaced by a hamiltonian path through $M_i, i = 1, 2, \ldots, l$. Further, we can find a hamiltonian path with initial vertex $x_i \in V(M_i), i = 1, 2, \ldots, l$, by beginning with the hamiltonian path $P_i$ in $H_i$ with initial vertex $t_i$. If $t_i$ is followed by $u_i$ on $P_i$, then replace $t_i$ by a hamiltonian $x_i - y_i$ path in $M_i$; similarly, if $t_i$ is followed by $v_i$, then replace $t_i$ by a hamiltonian $x_i - y_i$ path in $M_i$. Replace each $t_j$ ($j \neq i$) by a hamiltonian $x_j - y_j$ on $P_i$, the replacement matching the order of $u_j$ and $v_j$ on $P_i$. Since each $M_i (i = 1, 2, \ldots, l)$ is hamiltonian connected and since there are hamiltonian paths in $H_i$ with initial vertex $t_i$ and second vertex either $u_i$ or $v_i$, such substitutions yield a hamiltonian path in $H$ with initial vertex $t_i$. Thus $H$ is homogeneously traceable.

To see that $H$ is not hamiltonian, suppose to the contrary that $H$ is hamiltonian. Then we could start and end some hamiltonian cycle with some vertex $x \in V(M_i)$. Note that the vertices of any $M_i$ must be consecutive (although their particular order may vary) in any hamiltonian cycle of $H_i$, as the edges $u_i x_i$ and $v_i y_i$ must be used. However, replacing the subsequence of vertices in $M_i$ with $t_i$, we produce a hamiltonian cycle in $H_i$, which is impossible since $H_i$ is nonhamiltonian. Thus $H$ is homogeneously traceable nonhamiltonian with degree set $S$.

If $n_k = 2$, we repeat the last argument with vertices $t_j (j = 2, 3, \ldots, l)$, leaving $t_1$ unchanged. Then $\deg_{H_i} t_i = 2$ and again $H$ has the degree set $S$. 

An analogous argument shows $H$ is homogeneously traceable and non-hamiltonian.

Case 2. Suppose $S = \{n_1, n_2, \ldots, n_k\}$ and $n_i \geq 4$ for $i = 0, 1, \ldots, k$. Again consider the graph $H_l$, where $l$ is odd and $l \geq \max(S, k)$. We next construct a graph $H$ from $H_l$.

Remove vertex $u_i$ ($i = 1, 2, \ldots, l$) and its incident edges and in each case insert a copy of the graph $L_i = K_{n_{i+1}} - \{x_i, f_i, w_i\}$, where $x_i, f_i, w_i \in V(L_i)$. Then remove each vertex $v_i$ ($i = 1, 2, \ldots, l$) and replace it with a copy of $M_i = K_{n_{i+1}} - \{s_i, t_i\}$ for $r_i, s_i, t_i \in V(M_i)$. Now insert the edges $x_i, s_i, t_i$ for $i = 1, 2, \ldots, l-1$ and $x_l, s_l, t_l$.

Remove each vertex $v_i$ and its incident edges ($i = 1, 2, \ldots, l$) and insert a copy of $G_i = K_{n_{i+1}} - \{a_i, b_i, c_i, d_i\}$, where $a_i, b_i, c_i, d_i \in V(G_i)$. Then add the edges $f_i, a_i, f_i, b_i, s_i, t_i, c_i, d_i$ ($i = 1, 2, \ldots, l$). If $l > h$, then let $G_i = G_h$ for each $i = k+1, k+2, \ldots, l$.

As before, the graphs $G_i, M_i$, and $L_i$ ($i = 1, 2, \ldots, l$) are hamiltonian connected. An argument analogous to that in the last case shows that $H$ is homogeneously traceable. Further, since at most one of the edges $f_i, a_i, f_i, v_i$ (and similarly $s_i, t_i, c_i, d_i$) can appear on any hamiltonian path or cycle for each $i$, an analogous argument shows that $H$ is not hamiltonian. Since $H$ has the degree set $S$, case 2 is completed.

Case 3. Suppose $S = \{2, n_1, n_2, \ldots, n_k\}$ and $n_i \geq 4$ for $i = 1, 2, \ldots, k$. Here we proceed exactly as in case 2 except vertex $u_i$ is not changed and the additional edge $f_i, u_i$ is inserted. The graph $H$ (see Fig. 3) so constructed has the degree set $S$. However, the edge $f_i, u_i$ can appear on no hamiltonian path, so the argument of case 2 shows $H$ to be homogeneously traceable and non-hamiltonian.

Fig. 2. The graph $H$ (dotted lines represent missing edges in a complete graph)
REFERENCES


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