2-Factors in Claw-free Graphs

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Abstract

We consider the question of the range of the number of cycles possible in a 2-factor of a 2-connected claw-free graph with sufficiently high minimum degree. (By claw-free we mean the graph has no induced $K_{1,3}$.) In particular, we show that for such a graph $G$ of order $n \geq 51$ with $\delta(G) \geq \frac{n-3}{2}$, $G$ contains a 2-factor with exactly $k$-cycles, for $1 \leq k \leq \frac{n-24}{3}$. We also show that this result is sharp in the sense that if we lower $\delta(G)$, we cannot obtain the full range of values for $k$.

1 Introduction

The question of determining when a graph contains a 2-factor (a 2-regular spanning subgraph) has long been an important one in graph theory. Many results deal with hamiltonian graphs, that is, graphs $G$ containing a cycle that spans the vertex set $V(G)$. (See [4].) One special class of graphs that has drawn considerable interest are the claw-free graphs. Such graphs contain no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$.

In particular, the following was shown in [5].

Theorem 1 If $G$ is a 2-connected $K_{1,3}$-free graph of order $n$ with $\delta(G) \geq \frac{n-2}{3}$, then $G$ is hamiltonian.

We can see that this result is sharp by considering the following nonhamiltonian graph $G$ on $n = 3m$ vertices. Let $V(G) = A_1 \cup A_2 \cup A_3$ such that $|A_i| = m$ and $\langle A_i \rangle \cong K_m$ and let $x_i, y_i \in A_i$, $x_i \neq y_i$ for $i = 1, 2, 3$ and so that $\langle x_1, x_2, x_3 \rangle \cong \langle y_1, y_2, y_3 \rangle \cong K_3$. Clearly, the minimum degree of $G$ is $m - 1 = \frac{n-3}{3}$.

Recently the question of determining the number of cycles possible in a 2-factor of a given 2-connected graph satisfying certain degree conditions has been considered in [2].
The purpose of this paper is to investigate this question for 2-connected claw-free graphs. In particular, we will extend Theorem 1 by showing that the same minimum degree condition implies that \( G \) contains a 2-factor with exactly \( k \)-cycles for \( 1 \leq k \leq \frac{n-24}{3} \).

We will let \( \langle S \rangle_G \) denote the subgraph of \( G \) induced by \( S \) a subset of \( V(G) \). For \( A, B \subseteq V(G) \), \( e_G(A, B) \) denotes the number of edges in \( G \) with one vertex in \( A \) and the other in \( B \). For \( H \subseteq G \) we will sometimes write \( e_G(A, H) \) as shorthand for \( e_G(A, V(H)) \). The independence number of a graph will be denoted by \( \alpha(G) \). For a cycle \( C \), we will denote by \( \overrightarrow{C} \) the cycle under some orientation and \( \overleftarrow{C} \) will denote the cycle under the opposite orientation. For a vertex, \( a \), on a cycle with some orientation, \( \overrightarrow{C} \), we define \( a^+ \) and \( a^- \) to be the immediate successor and predecessor respectively of \( a \) on \( C \) with respect to this orientation. Also, for a collection of vertex disjoint cycles \( S \) each with some orientation, we define \( N^+_S(a) \) to be the set \( \{ a^+ | a \in (N(a) \cap V(S)) \} \). Let \( I = \overrightarrow{a_1, a_2, \ldots, a_k} \) where the \( a_i \)'s are consecutive vertices on a cycle. Then \( l(I) = k \), the length of the segment of the cycle. For terms not defined here, see [3].

## 2 Main Result

In this section we will prove the theorem. However, first we prove the following proposition which gives sufficient conditions for the existence of \( k \) disjoint triangles and will lay the foundation for the proof of the theorem.

**Proposition 1** Let \( G \) be a claw-free graph of order \( n \), let \( k \) be an integer, and let \( c \geq 0 \). If \( n > 3k + 6 - f(k, c) \) where \( f(1, 1) = f(2, 0) = 0 \) and \( f(k, c) = \frac{9c - 9}{k + c - 2} \) for all other values of \( k \) and \( c \) and \( \delta(G) \geq \max \{k + c, 3\} \) then \( G \) contains \( k \) disjoint triangles.

**Proof:** If \( \delta(G) \geq 3 \), then \( n \geq 4 \) and, since \( G \) is claw-free, \( G \) must contain at least one triangle. Choose \( m \) disjoint triangles in \( G \), say \( T_1, T_2, \ldots, T_m \), so that \( m \) is as large as possible. Since \( G \) is claw-free and \( \delta(G) \geq 3 \), we know \( m \geq 1 \). Assume \( m < k \). Let

\[
A = \bigcup_{i=1}^{m} V(T_i)
\]

and \( H = G - A \).

If \( \Delta(H) \geq 3 \), say \( \deg_H a \geq 3 \) for some \( a \in V(H) \), then since \( G \) is claw-free, \( b_1b_2 \in E(H) \) for some \( b_1, b_2 \in N_H(a) \) and \( \{a, b_1, b_2\} \) forms a triangle. This contradicts the maximality of \( m \). Therefore, \( \Delta(H) \leq 2 \).
Claim: For each $x \in A$, $|N_G(x) \cap V(H)| \leq 3$.

Proof: Assume $|N_G(x) \cap V(H)| \geq 4$ for some $x \in A$. Let $x \in V(T_i)$ and $V(T_i) = \{x, y, z\}$. Let $a_1, a_2, a_3, a_4$ be distinct neighbors of $x$ in $H$.

If $N_G(a_1) \cap \{a_2, a_3, a_4\} = \emptyset$, then since $x$ and $\{a_1, a_2, a_3\}$ do not form a claw, without loss of generality, $a_2a_3 \in E(G)$. We apply the same argument to $x$ and $\{a_1, a_2, a_4\}$ and $\{a_1, a_3, a_4\}$, and we have $a_2a_4 \in E(G)$ and $a_3a_4 \in E(G)$. But then $\{a_2, a_3, a_4\}$ forms a triangle, which contradicts the maximality of $m$. Therefore, $N_G(a_1) \cap \{a_2, a_3, a_4\} \neq \emptyset$. Similarly, we have $\deg_{G}(a_1, a_2, a_3, a_4) \geq 1$ for each $i$, $1 \leq i \leq 4$. Since $\Delta(H) \leq 2$, we know $(a_1, a_2, a_3, a_4) \notin G = (a_1, a_2, a_3, a_4)$. Thus, without loss of generality, we may assume $a_1a_2, a_3a_4 \in E(G)$.

Consider the subgraph induced by $F = \{(a_1, a_2, a_3, a_4, y, z)\}$. We want to show that $F$ must contain $K_3 \cup K_2$ as a subgraph because the existence of such a subgraph in $F$ implies that $(F \cup \{x\})$ contains two independent triangles which contradicts the maximality of $A$.

In order to show that $F$ must contain $K_3 \cup K_2$, we first observe that to avoid a claw centered at $x$, $F$ cannot contain 3 independent vertices. Let $S_1 = \{(a_1, a_2)\}, S_2 = \{(a_3, a_4)\}$, and $S_3 = \{(y, z)\}$. Then there are 8 independent 3-sets of vertices in $S_1 \cup S_2 \cup S_3$. Note that the addition of any edge to $S_1 \cup S_2 \cup S_3$ can destroy at most two of the 8 independent triples of vertices. Thus, $F$ must have at least 4 more edges than $S_1 \cup S_2 \cup S_3$. Without loss of generality, we can assume there are two edges between $S_1$ and $S_2$. If these two edges share an endvertex, then $F$ contains $K_3 \cup K_2$. Thus, we may assume they are independent. By symmetry, we may further assume that they are $a_1a_3$ and $a_2a_4$. Moreover, again to avoid $K_3 \cup K_2$ in $F$, we may assume $a_1a_4, a_2a_3 \in E(G) = \emptyset$. Then by considering the triple $\{a_1, a_4, y\}$ we can, without loss of generality, assume $F$ contains the edge $a_1y$. But, by the same argument, the triple $\{a_2, a_3, y\}$ forces the edge $a_2y$ or $a_3y$, and therefore $F$ contains a triangle and an independent edge. This contradicts the maximality of $A$ and the claim follows.

Since $\Delta(H) \leq 2$ and $\delta(G) \geq k + c$, we have $e_G(x, A) \geq k + c - 2$, for each $x \in V(H)$. Thus, $e_G(H, A) \geq (k + c - 2)(n - 3m)$. On the other hand, $e_G(u, H) \leq 3$ for each $u \in A$ which implies $e_G(A, H) \leq 3|A| = 9m$. Therefore, $(k + c - 2)(n - 3m) \leq 9m$. Thus, $(k + c - 2)n \leq (3k + 3c + 3)m$. Then, using the fact that we assumed $m \leq k - 1$, we find $n \leq \frac{3k^2 + 3ck - (3c + 3)}{k + c - 2} = 3k + 6 - \frac{9c - 9}{k + c - 2}$. This contradicts the assumption and completes the proof. \[ \square \]
Theorem 2: Let $G$ be a 2-connected, claw-free graph of order $n \geq 51$ with $\delta(G) \geq \frac{1}{3}(n - 2)$. Then for each $k$ with $1 \leq k \leq \frac{n - 24}{3}$, $G$ has a 2-factor with exactly $k$ components.

Proof: By the assumption $n \geq 3k + 24$ and $\delta(G) \geq \frac{n - 2}{3} \geq \frac{3k + 22}{3} \geq k + 1$. Therefore, by Proposition 1, $G$ has $k$ disjoint cycles $C_1, C_2, \ldots, C_k$. Choose $C_1, \ldots, C_k$ such that $\sum_{i=1}^{k} |V(C_i)|$ is as large as possible. Let $D = \bigcup_{i=1}^{k} V(C_i)$ and assume $D \neq V(G)$. Let $H = G - D$ and let $x \in V(H)$.

Claim 1: $|V(H)| \geq 4$.

Proof: Let $h = |V(H)|$ and assume $h \leq 3$.

Since $h \leq 3$, $|D| \geq n - 3 \geq 3k + 21$. Thus, there exists some cycle, say $C_i$, such that $|V(C_i)| \geq 4$.

Let $x \in V(H)$ and let $|N_G(x) \cap V(C_i)| = t$, say $N_G(x) \cap C_i = \{a_1, \ldots, a_t\}$. We may assume $a_1, \ldots, a_t$ appear in consecutive order along some orientation of $C_i$. Let $I_j = a_j C_i a_{j+1}$ for $1 \leq j \leq t - 1$ and let $I_t = a_t C_i a_1$. If $l(I_j) = 1$ for some $1 \leq j \leq t$, then $a_{j+1} = a_j^+$. Let $C_i' = a_{j+1} C_i a_j x a_{j+1}$ and $C_j' = C_j$ for all $j \neq i$. Then $\{C_1', \ldots, C_k'\}$ is a disjoint collection of cycles of larger total order, a contradiction. Therefore, $l(I_j) \geq 2$ for each $j$, $1 \leq j \leq t$.

Since $G$ is claw-free, this implies $a_j a_j^+ \in E(G)$ for each $j$, $1 \leq j \leq t$. If $l(I_j) = 2$, then $a_j a_j^+ = a_{j+1}$. Let $C_i' = x a_{j+1} a_j a_j^+ a_j x a_{j+1}$ and $C_i' = C_j$ for all $j \neq i$. If $l(I_j) = 3$, then $a_j a_j a_j = a_{j+1}$.

Let $C_i' = x a_{j+1} a_j a_j^+ a_j x a_{j+1}$ and $C_i' = C_j$ for all $j \neq i$. In either case, the collection $\{C_i', \ldots, C_k'\}$ forms a set of independent cycles of larger order, a contradiction.

Therefore, $l(I_j) \geq 4$ for each $j$, $1 \leq j \leq t$. This implies $|V(C_i)| = \sum_{j=1}^{t} l(I_j) \geq 4t$ or $|N_G(x) \cap V(C_i)| \leq \frac{1}{4} |V(C_i)|$ for all $C_i$ such that $|V(C_i)| \geq 4$. Note that $x$ has at most one adjacency to every 3-cycle in the collection $C_1, \ldots, C_k$.

We may assume $|V(C_1)| = |V(C_2)| = \ldots = |V(C_s)| = 3$ and $|V(C_t)| \geq 4$ for $s + 1 \leq i \leq k$. Then,

$$\frac{n - 2}{3} \leq e(x, D) + \deg_H x \leq (h - 1) + s + \frac{1}{4} \sum_{i=s+1}^{k} |V(C_i)| = (h - 1) + \frac{|D| + s}{4} = \frac{n - h + s}{4},$$

which implies $n < 3s + 9h - 4$. Since $s < k$ and $h < 3$, we have $n < 3k + 23$. This contradicts the assumption. Consequently, we know $|V(H)| > 4$.

Claim 2: For each $y \in V(H) - \{x\}$, $\deg_{H-x} y \geq 2$. 


Proof: Assume \( \deg_{H-x}y \leq 1 \) for some \( y \in V(H) - \{x\} \). As in Claim 1, we count the number of edges from \( y \) to \( D \) observing that \( y \) can have at most one adjacency to a 3-cycle and \( y \) is adjacent to at most one out of every four vertices on cycles of length 4 or more.

We may assume \( |V(C_1)| = |V(C_2)| = \ldots = |V(C_s)| = 3 \) and \( |V(C_i)| \geq 4 \) for \( s+1 \leq i \leq k \). Then \( e(y, D) \leq s + \frac{1}{4} \sum_{i=s+1}^{k} |V(C_i)| = s + \frac{1}{4} (|D| - 3s) = \frac{1}{4} |D| + \frac{1}{4} s \). Therefore,

\[
\frac{n - 2}{3} \leq \deg_H y + \deg_D y \leq \deg_H y + e(y, D) \\
\leq 1 + \deg_{H-x} y + \frac{1}{4} |D| + \frac{1}{4} s \\
\leq 1 + \deg_{H-x} y + \frac{1}{4} (n - 4) + \frac{1}{4} k \\
\leq 1 + \deg_{H-x} y + \frac{1}{4} (n - 4) + \frac{n - 24}{12} .
\]

Thus, \( \deg_{H-x} y \geq 2 \). \( \Box \)

By Claims 1 and 2, we know that for every \( x \in V(H) \), \( H - x \) contains a cycle, call it \( C_x \).

Claim 3: For every \( x \in H \), the set \( N_D^+(x) \) is independent.

Proof: Assume, to the contrary, \( a_1^+ a_2^+ \in E(G) \) for some \( a_1, a_2 \in N_D(x) \). If \( a_1 \) and \( a_2 \) lie in the same cycle of \( D \), say \( C_i \). Then we increase the total order of \( D \) by replacing \( C_i \) by \( C_i' = a_1^+ C_i^+ a_2^+ a_1^+ C_i a_2^+ a_1^+ \). If \( a_1 \) and \( a_2 \) lie in different cycles of \( D \), we may assume without loss of generality \( a_i \in V(C_i), i = 1, 2 \). Then let \( C_1' = C_x \), \( C_2' = xa_1^+ C_i^+ a_2^+ a_1^+ C_i a_2^+ a_1^+ \) and for \( j \neq 1, 2 \) let \( C_j' = C_j \). Then the collection \( \{C_1', \ldots, C_k'\} \) forms a set of \( k \) disjoint cycles of larger total order, a contradiction. \( \Box \)

Claim 4: Since \( G \) is a claw-free graph of order \( n \), then \( \alpha(G) \leq \frac{2n}{\delta(G) + 2} \).

Proof: Let \( S \) be a largest independent set in \( G \). For each \( x \in V(G) - S \), we have \( e_G(x, S) \leq 2 \) since \( G \) is claw-free. Therefore, \( e_G(S, V(G) - S) \leq 2 |V(G) - S| = 2(n - \alpha(G)) \). On the other hand, since \( S \) is independent, we know \( e_G(S, V(G) - S) = \sum_{x \in S} \deg_G x \geq \delta(G) |S| = \delta(G) \alpha(G) \).

Therefore, we have \( 2(n - \alpha(G)) \geq \delta(G) \alpha(G) \). Solving this inequality for the independence number and we get \( \alpha(G) \leq \frac{2n}{\delta(G) + 2} \). \( \Box \)
By Claims 3 and 4, for each \( x \in V(H) \) we have that
\[
|N_D[x]| = |N_D^+(x) \cup \{x\}| \leq \alpha(G) \leq \frac{2n}{\delta(G) + 2} \leq \frac{2n}{\frac{n-2}{3} + 2} < 6.
\]
Therefore, \( |N_D(x)| \leq 4 \) and we have \( \deg_H x \geq \frac{n-14}{3} \).

Let \( P \) be a longest path in \( H \) and let \( x \) be one of its end vertices. Then \( N_H(x) \subseteq V(P) \) or a longer path is possible. Therefore, if we choose \( y \in N_H(x) \) so that \( x \rightarrow P \rightarrow y \) is as long as possible, we form a cycle \( C = x \rightarrow P \rightarrow y \rightarrow x \) with \( N_H(x) \subseteq V(C) \). This implies \( |V(C)| \geq \deg_H x + 1 \geq \frac{n-14}{3} + 1 = \frac{n-11}{3} \).
Then by the maximality of \( D \), we know \( |V(C_i)| \geq \frac{n-11}{3} \), for all \( 1 \leq i \leq k \).

**Claim 5**: The number of independent cycles, \( k \), is 2.

**Proof**: Assume \( k \geq 3 \). Then \( n = |V(G)| \geq |V(C)| + |V(C_1)| + |V(C_2)| + |V(C_3)| \geq 4\left(\frac{n-11}{3}\right) \). This forces \( n \leq 44 \), a contradiction. \( \Box \)

Since \( C_1 \) and \( C_2 \) each have at least \( \frac{n-11}{3} \) vertices, we know \( |V(H)| \leq n - |V(C_1)| - |V(C_2)| \leq \frac{n+22}{3} \).

**Claim 6**: The subgraph \( H \) is hamiltonian connected.

**Proof**: If \( H \) is not hamiltonian-connected by [6],
\[
\frac{n-14}{3} \leq \delta(H) \leq \frac{1}{2}|V(H)| \leq \frac{n+22}{6}.
\]
This forces \( n \leq 50 \), a contradiction. \( \Box \)

In particular, \( H \) has a hamiltonian cycle, say \( C_0 \). By the maximality of \( D \), we know \( |V(C_0)| \leq |V(C_i)| \) for \( i = 1, 2 \). Thus, \( |V(C_0)| \leq \frac{1}{3}n \).

Since \( G \) is 2-connected, there exist at least two independent edges between \( C_0 \) and \( C_1 \cup C_2 \).

**Claim 7**: There do not exist two independent edges from \( C_0 \) to \( C_i \), for \( i = 1, 2 \).
Proof: Without loss of generality, let $i = 1$. Assume there are two independent edges, say $a_1b_1$ and $a_2b_2$ between $C_0$ and $C_1$ (where $a_1, a_2 \in C_0$, $b_1, b_2 \in C_1$). Without loss of generality, we may assume $l(b_1 \overrightarrow{C_1} b_2) \geq \frac{1}{3}|V(C_1)|$. Since $\{a_2 \overrightarrow{P} a_1b_1 \overrightarrow{C_1} b_2a_2, C_2\}$ forms a set of disjoint cycles where $P$ is a hamiltonian $a_1, a_2$-path in $H$, we know $l(b_2 \overrightarrow{C_1} b_1) \geq |V(C_0)| + 1 \geq \delta(H) + 2 \geq \frac{n-8}{3}$. Then $|V(C_1)| \geq 2l(b_2 \overrightarrow{C_1} b_1) \geq \frac{2n-16}{3}$. Therefore,

$$n = |V(C_0)| + |V(C_1)| + |V(C_2)| \geq 2 \left(\frac{n-11}{3}\right) + \frac{2n-16}{3} = \frac{4n-38}{3}.$$ 

This forces $n \leq 38$ which is a contradiction. \qed

Therefore we may assume $a_1b_1, a_2b_2 \in E(G)$ where $a_1, a_2 \in V(C_0)$, $a_1 \neq a_2$, $b_1 \in V(C_1)$, and $b_2 \in V(C_2)$. As a consequence of Claim 7 and 2-connectivity, we know there exists an edge $d_1d_2 \in E(G)$ such that $d_1 \in V(C_1) - b_1$ and $d_2 \in V(C_2)$.

Let $x \in H - \{a_1, a_2\}$. (Since $|V(H)| = |V(C_0)| \geq \frac{n-11}{3}$ we know such an $x$ exists.) Then by Claim 6, $NC_1 \cup C_2(x) \subset \{b_1, b_2\}$. Therefore, $\deg_H x \geq \frac{n-2}{3} - 2 = \frac{n-8}{3}$, and hence $|V(C_0)| \geq \frac{n-5}{3}$.

Claim 8: The graph $H - \{a_1, a_2\}$ has a triangle $T$ and $H - V(T)$ is hamiltonian-connected.

Proof: Let $H' = H - \{a_1, a_2\}$ and assume $\delta(H') \leq \frac{|V(H')|}{2}$. Since $\delta(H') \geq \delta(H) - 2 \geq \frac{n-8}{3} - 2 \geq \frac{n-14}{3}$ and $|V(H')| \leq \frac{n}{3} - 2 = \frac{n-6}{3}$, we get $\frac{n-14}{3} \leq \frac{1}{2} \left(\frac{n-6}{3}\right)$. This forces $n \leq 18$, a contradiction.

Thus $\delta(H') \geq \frac{|V(H')|+1}{2}$ and $|V(H')| \geq \frac{n}{3} - 2 \geq 3$, which implies by [1] that $H'$ is pancyclic. Thus $H'$ has a triangle $T$. Let $H'' = H - V(T)$. Then $|V(H'')| = |V(H)| - 3 \geq \frac{n}{3} - 3 = \frac{n-9}{3}$ and $\delta(H'') \geq \frac{n-9}{3} - 3 \geq \frac{n-22}{3}$. Therefore, since $n \geq 40$, $\delta(H'') > \frac{1}{2}|V(H'')|$. Hence, by [6] $H''$ is hamiltonian connected. \qed

First, suppose $d_2 \neq b_2$. We may assume $l(d_1 \overrightarrow{C_1} b_1) \leq \frac{1}{2}(|V(C_1)|)$ and $l(b_2 \overrightarrow{C_2} d_2) \leq \frac{1}{2}(|V(C_2)|)$. By the maximality of $C_1$ and $C_2$ and the fact that $G$ is claw-free, $b_1^+b_1^+b_2^+b_2^+ \in E(G)$. Let $C'' = a_1b_1b_1^+b_1^+C_1^+d_1d_2C_2^+b_2^+b_2^+b_2a_2P_1$, where $P$ is a hamiltonian $a_1a_2$-path in $H - T$. Since $C''$ and $T$ are disjoint cycles, $l(d_1 \overrightarrow{C_1b_1}^{-}) + l(b_2^+ \overrightarrow{C_2d_2}) \geq 2 \geq |V(H)|$. Thus $\frac{|V(C_1)|+|V(C_2)|}{2} - 4 \geq |V(H)| \geq \frac{n-5}{3}$, which implies that $|V(C_1)| + |V(C_2)| \geq \frac{2n+14}{3}$. Since $|V(H)| = |V(C_0)| \geq \frac{n-5}{3}$, we have $n = |V(H)| + |V(C_1)| + |V(C_2)| \geq \frac{3n+10}{3} = n + 3$, a contradiction. Therefore, we know $d_2 = b_2$.
which implies that there cannot be three independent edges between the cycles $C, C_1,$ and $C_2$.

Since $G$ is 2-connected, there exists an edge $b'_2u$ from $C_2 - \{b_2\}$ to $C_0 \cup C_1$

**Case 1:** We consider the case where $u \in C_0$. If $u \neq a_1$ the three edges $a_1b_1, a_1b_2,$ and $b'_2u$ are independent, a contradiction. Thus, $u = a_1$. But now the two edges $a_2b_2$ and $a_1b'_2$ between $C_0$ and $C_2$ are independent. This contradicts Claim 7.

**Case 2:** We consider the case where $u \in C_1$. If $u \neq b_1$, then the three edges $a_1b_1, ub'_2$, and $a_2b_2$ are independent, a contradiction. If $u = b_1$, consider $b_1$ and $\{a_1, b_1^+, b'_2\}$. We know $b'_2b_1^+ \notin E(G)$ because $u = b_1$. By Claim 7, $a_1b'_2 \notin E(G)$. If $a_1b_1^+ \in E(G)$, then the three edges $a_1b_1^+, b_1b'_2$ and $a_2b_2$ are independent, a contradiction. Thus, $(b_1, b_1^+, a_1, b'_2)_G$ is a claw, a contradiction.

Hence, in all cases we reach a contradiction, and the result is proved.  

□

References


