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Complete Families of Graphs

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ABSTRACT. A broom of order n is a tree obtained by identifying an endvertex of a path of order r with the central vertex of a star of order n-r+1. If r is even (odd) then the broom is even (odd). It is shown that for every graph G of order n, either G or $\overline{G}$ contains an even (odd) broom of order n.

1. Introduction

A family $\mathcal{F}_n$ of graphs of order n is said to be *complete* if for every graph G of order n, either G or $\overline{G}$ contains a member of $\mathcal{F}_n$. Bialostocki, Dierker and Voxman [2] conjectured that the family $\mathcal{B}_n$ of all brooms of order n is complete, where a *broom* of order n is a tree obtained by identifying an endvertex of a path of order r with the central vertex of a star of order n - r + 1. In [2] Burr verified their conjecture and further conjectured that, in fact, only about half of $\mathcal{B}_n$ is needed for a complete family. In Section 2 we will verify this conjecture.

Suppose that $\mathcal{F}_n$ is a complete family of connected graphs of order n. Then, as we shall see, $|\mathcal{F}_n| \geq \left\lfloor \frac{n}{2} \right\rfloor$, and the family of even brooms (odd brooms) attains this minimum. In Section 3 we will consider complete families of graphs of order n in which the graphs are not necessarily connected.

2. Complete families of brooms

For integers r and n satisfying $1 \leq r \leq n$, let B(r,n) denote the broom of order n obtained by identifying an endvertex of a path of order r with the central vertex of a star of order n - r + 1. We will call the identified vertex of the path the *central vertex of the broom* and the endvertex of the path the *top of the handle of the broom*. With this notation the star of order n is denoted by B(1, n) or , equivalently, B(2, n), and the path P_n is denoted by B(n-1,n) or B(n,n). Thus for $n \geq 3$ there are n-2 distinct brooms of order n.

Our first result implies that there are proper complete subfamilies of $\mathcal{B}_n$. Its proof employs the classic result of Menger [3] regarding k-connected graphs.

THEOREM 1. *Let G be a k-connected graph, $k \geq 2$. Then every k vertices lie on a common cycle C of G. Furthermore, if $k < n$, then C can be chosen to contain at least k+1 vertices.*

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THEOREM 2. *Let G be a graph of order $n \geq 2$. Then there is a positive integer $r < n$ such that either G or $\overline{G}$ contains both $B(r, n)$ and $B(r+1, n)$.*

PROOF. It is straightforward to verify the theorem for small values of n. Also, without loss of generality we may assume that the connectivity $\kappa(G)$ of G and the connectivity $\kappa(\overline{G})$ of $\overline{G}$ satisfy $\kappa(\overline{G}) \geq \kappa(G)$.

Case 1. Suppose $\kappa(G) = 0$. Then $\overline{G}$ contains the complete bipartite graph $K_{a,b}$ for integers $a \leq b$, with $a + b = n$. If $a = b$, then $\overline{G}$ contains P_n = B(n-1,n) = B(n,n). If, on the other hand $a < b$, then $\overline{G}$ clearly contains B(2a-1, n) and B(2a, n).

Case 2. Suppose $\kappa(G) = 1$. Let x be a cutvertex of G, let A be the vertex set of the smallest component of G - x and let B = V(G) - A - x. Then $\overline{G}$ contains $K_{a,b}$, where $|A| = a$ and $|B| = b$ and $a \leq b$. If x is adjacent in $\overline{G}$ to a vertex of A, then $\overline{G}$ contains B(2a-1, n), with x an endvertex adjacent to the central vertex of the broom, and B(2a, n), with x the top of the handle of the broom. Suppose, then, that x is adjacent in $\overline{G}$ to no vertex of A, to at least one vertex of B, and $a \leq b$. Then $\overline{G}$ contains B(2a+1, n), with x the top of the handle of the broom. If $a = b$, then G also contains the broom B(2a, n). Thus we may assume that $a < b$. We consider three possibilities. (1) In $\overline{G}$, x has two adjacencies y_1 and y_2 in B. (2) In $\overline{G}$, the subgraph induced by B contains at least one edge e. (3) Neither (1) nor (2) occurs.

In the first situation, $\overline{G}$ contains B(2a+2, n), with y_1 the top of the handle, followed by x and y_2. In the second situation, $\overline{G}$ also contains B(2a+2, n), where x is the top of the handle of the broom and e lies on the handle. Thus in either situation, we have the necessary pair of brooms. If neither occurs, then x is adjacent in G to every vertex of A and to all but one vertex of B. Also, the subgraph induced by B in G is complete. In particular, G contains B(3, n), B(4, n), ... , B(b+1, n).

Case 3. Suppose $2 \leq \kappa(G) \leq \kappa(\overline{G})$. Let S be a vertex cut of G of minimum order, let A be the vertex set of one component of G - S and let B = V(G) - A - S. Since $\kappa(\overline{G}) \geq \kappa(G) \geq 2$, it follows from Menger's theorem that $\overline{G}$ has a cycle C containing S, with $|V(C)| > |S|$. Select such a cycle C of maximum order. If C is hamiltonian, then $\overline{G}$ contains B(n-1,n) and B(n,n). Assume, then, that $|V(C)| < n$.

Let $A^* = A - V(C)$ and let $B^* = B - V(C)$, with $a^* = |A^*|$ and $b^* = |B^*|$. Without loss of generality, assume $b^* \geq a^*$.

Subcase 1. Suppose $a^* > 0$, and that in $\overline{G}$, C contains a vertex x adjacent to a vertex of A^* and a vertex y adjacent to a vertex of B^*. Fix an orientation of C and under this orientation, let x^+ and y^+ denote the successors of x and y, respectively. Then $\overline{G}$ contains the brooms $B(|V(C)| + 2a^* - 1, n)$, with x^+ the top of the handle and x the last vertex of C on the handle, and the broom $B(|V(C)| + 2a^*, n)$, with y^+ the top of the handle and y the last vertex of C on the handle.

Subcase 2. Suppose $a^* = 0$. Then $b^* \neq 0$. Let $y \in A$. Then y is on C, with y^+ and y^{++} its two immediate successors. Therefore, $\overline{G}$ contains $B(|V(C)| - 1, n)$, with y^{++} the top of the handle, y the last vertex of C on the handle and y^+ the only vertex of C not on the handle, as well as $B(|V(C)|, n)$, with y^+ the top of the handle and y the last vertex of C on the handle.

Subcase 3. Suppose $a^* > 0$ but Subcase 1 does not occur. Then one of two things occurs. Either no vertex of C is adjacent, in $\overline{G}$, to a vertex of B^* so that A = A^*, and C consists of the vertices of S and at least one vertex of B; or no vertex of C is adjacent, in $\overline{G}$, to a vertex of A^* so that B = B^*, and C consists of the vertices of S and at least one vertex of A. We consider the first case in detail, and leave the details of a similar agument for the second case to the reader.

The maximality of C implies that C contains no two consecutive vertices from B; otherwise, we could extend C by a vetex from A. Thus $|B \cap C| \leq |S|$. On the other hand, let $y \in A$. Since $\kappa(\overline{G}) \geq |S|$, there exist |S| paths in $\overline{G}$ from y to C, disjoint except for y. By the maximality of C, the endvertices of these paths cannot be consecutive on C, so that $|V(C)| \geq 2|S|$. We conclude that $|B \cap C| = |S|$ and $|V(C)| = 2|S|$, that is, the vertices of C alternate between vertices of B and vertices of S. Finally, the maximality of C implies that no two vertices of S and no pair of vertices from A and S are adjacent in $\overline{G}$.

Let x be a vertex of C adjacent, in $\overline{G}$, to a vertex of A. Since $a^* \leq b^*$, $\overline{G}$ contains the broom $B(|V(C)| + 2a^* - 1, n)$, with x^+ the top of the handle and x the last vertex of on the handle. If there is an edge in $\overline{G}$ between vertices of A = A^*, then $\overline{G}$ also contains the broom $B(|V(C)| + 2a^* - 2, n)$, while an edge in $\overline{G}$ between vertices of B^* produces the broom $B(|V(C)| + 2a^*, n)$. Thus we may assume that A = A^* and B^* induce complete graphs in G.

Thus the structure of G is completely determined except for the subgraph induced by $B \cap C$, denoted $\prec B \cap C \succ$. In particular, we have the following in G :

(a) $\prec A \succ$ is complete,
(b) $\prec B^* \succ$ is complete,
(c) $\prec S \succ$ is complete,
(d) every vertex of A is adjacent to every vertex of S,
(e) every vertex of C is adjacent to every vertex of B^*,
(f) no vertex of A is adjacent to a vertex of B, and
(g) no vertex of S is adjacent to a vertex of $B \cap C$.

Furthermore, since $\kappa(G) \geq 2$, we have that $|B^*| \geq 2$. But then we can construct many brooms in G by beginning in A, traversing all the vertices of A, followed by all of the vetices in S, and then completeing the handle by traversing as many of the vertices of B^* as desired. The remaining vertices of B^* and those of $B \cap C$ form the "bristles" of the brooms. In particular, G contains the brooms $B(|A| + |S| + 1, n)$ and $B(|A| + |S| + 2, n)$. ∎

An *even broom* is a broom B(r,n), where r is even. An *odd broom* is defined analogously.

COROLLARY 1. *For a fixed integer n, the family of even brooms of order n (odd brooms of order n) forms a complete family of graphs.*

Note that for a fixed n, each of the collections of even brooms and odd brooms contains $\lfloor n/2 \rfloor$ graphs. Now, suppose that $\mathcal{F}_n$ is any complete family of connected graphs of order n and consider the $\lfloor n/2 \rfloor$ graphs $K_{i,n-i}$, $1 \leq i \leq \lfloor n/2 \rfloor$. Since the complement of each of these graphs is disconnected, it follows that each of these graphs must contain a member of $\mathcal{F}_n$. The partite sets of $K_{1,n-1}$ have orders 1 and n-1; so must the partite sets of the contained member of $\mathcal{F}_n$. The partite sets of $K_{2,n-2}$ have orders 2 and n-2; so must the partite sets of the contained member of $\mathcal{F}_n$, and so forth. Since, in a connected bipartite graph, the orders of the partite sets are unique, it follows that $|\mathcal{F}_n| \geq \lfloor n/2 \rfloor$. Thus the family of even brooms (odd brooms) is a minimal complete family of connected graphs.

For odd values of i at most n-1, let S_i = {B(i, n), B(i+1, n)}. We have shown that a complete family of brooms may be obtained by selecting the even broom from each S_i, or by selecting the odd broom from each S_i. In fact, we believe that a much stronger result is true and we make the following conjecture.

CONJECTURE 1. *For a fixed integer n and each odd i at most n-1, select any $F_i \in S_i$. Then $\{F_1, F_3, \ldots\}$ is complete.*

3. Other complete families of graphs

We saw in Section 2 that if $\mathcal{F}_n$ is a complete family of connected graphs of order n, then $|\mathcal{F}_n| \geq \lfloor n/2 \rfloor$, and the lower bound is attainable. On the other hand, if we put no restrictions on the number of components in the graphs in $\mathcal{F}_n$, then there are complete families $\mathcal{F}_n$ containing exactly one graph. For example, $\mathcal{F}_n = \{nK_1\}$ is trivially a complete family of graphs of order n containing exactly one graph. This graph, of course, has n components. Equivalently, the ramsey number $r(nK_1)$ = n. Our next result exhibits another such graph with fewer components.

THEOREM 3. *For positive intgers n and k satisfying $n \geq \binom{k+2}{2}$, define $F_{k,n}$ to be the graph $K_1 \cup K_{1,1} \cup K_{1,2} \cup K_{1,3} \cup \ldots \cup K_{1,k-1} \cup (n - \binom{k+1}{2})K_1$. Then $\{F_{k,n}\}$ is a complete family of graphs of order n, that is, $r(F_{k,n}) = n$.*

PROOF. Color the edges of K_n arbitrarily red and blue. We show that there is a monochromatic $F_{k,n}$. Suppose first that there is a set S of k-1 vertices, and vertices u and v such that all edges from u to S are red and all edges from v to S are blue. Consider G* = G - ({u,v}∪ S} of order n* = n - k - 1. By induction, G* contains a monochromatic $F_{k-1,n*}$, and so G contains a monochromatic $F_{k,n}$. Assume, then, that no such vertices u,v and S exist.

Let v be a vertex of maximum red degree δ and let T be the δ vertices connected to v by red edges. We count the number of red edges between T and $V(K_n) - T$ - {v}. Since every edge between v and $V(K_n) - T$ - {v} is blue, each of the δ vertices of T has at most k-2 red adjacencies in $V(K_n) - T$ - {v}. Thus the number of red edges between T and $V(K_n) - T$ - {v} is at most δ(k-2). On the other hand, since each edge from v to T is red, each vertex in $V(K_n) - T$ - {v} has at most k-2 blue adjacencies in T, and so at least δ - k + 2 red adjacencies in T. Thus the number of red edges between T and $V(K_n) - T$ - {v} is at least (n - δ - 1)(δ - k + 2). We conclude that

$$\delta(k-2) \geq (n - \delta - 1)(\delta - k + 2),$$

and so

$$\delta^2 - (n - 1)\delta + (k - 2)(n - 1) > 0.$$

If, however, δ = k, then $k^2 - 2(n - 1) \geq 0$, which contradicts the assumption that $n > \binom{k+2}{2}$. We conclude that $\delta < k$, i.e., each vertex has red degree at most k-1. Then, of course, each vertex has blue degree at least n-k and we can arbitrarily pick the k blue stars to construct a blue $F_{k,n}$. ∎

If F is the graph described in the previous theorem in the case that k is chosen as large as possible, then F has $O(\sqrt{n})$ components. We conjecture that O(log n) is the smallest number of components in a complete family of graphs of order n containing exactly one graph.

CONJECTURE 2. *For every n sufficiently large there exists a complete family* $\{F\}$ *where F has at most O(log n) components and* $|V(F)| = n$.

Suppose that $\mathcal{F}_n$ is a complete family of graphs of order n and each graph in $\mathcal{F}_n$ has at most k components. For k = 1 we know that $|\mathcal{F}_n| \geq \lfloor n/2 \rfloor$. This count is based on considering the $\lfloor n/2 \rfloor$ graphs $K_{i,n-i}$, $1 \leq i \leq \lfloor n/2 \rfloor$. Since the complement of each of these graphs is disconnected, it follows that each of these graphs must contain a member of $\mathcal{F}_n$. These same graphs provide us with a bound on $|\mathcal{F}_n|$ when k = 2, that is, when each graph in $\mathcal{F}_n$ has at most two components. If F is a member of $\mathcal{F}_n$, then F is a spanning subgraph of at most two of the complete bipartite graphs and at most one of the complements. Thus, $3|\mathcal{F}_n| \geq \lfloor n/2 \rfloor$. Similarly, if each graph in $\mathcal{F}_n$ has at most three components and F is a member of $\mathcal{F}_n$, then F is a spanning subgraph of at most four of the complete bipartite graphs and at most three of the complements, so that $7|\mathcal{F}_n| \geq \lfloor n/2 \rfloor$. In general, if each graph in $\mathcal{F}_n$ has at most k components and F is a member of $\mathcal{F}_n$, then F is a spanning subgraph of at most $2^k/2 = 2^{k-1}$ of the complete bipartite graphs and at most $(2^k-2)/2 = 2^{k-1}-1$ of the complements. It follows, then, that $(2^k - 1)|\mathcal{F}_n| \geq \lfloor n/2 \rfloor$, giving a lower bound $|\mathcal{F}_n| \geq \left\lfloor \frac{n}{2^{k+1}-2} \right\rfloor$. This bound is tight for k = 1. For k = 2 we have $|\mathcal{F}_n| \geq \left\lfloor \frac{n}{6} \right\rfloor$. We will describe complete families $\mathcal{F}_n$ of graphs of order n in which each graph has two components and $|\mathcal{F}_n| = \lfloor (n+1)/3 \rfloor$. Thus the smallest such family $\mathcal{F}_n$ satisfies $\left\lfloor \frac{n}{6} \right\rfloor \leq |\mathcal{F}_n| \leq \lfloor (n+1)/3 \rfloor$.

For i = 1, 2, ..., n - 1, let T(i, n) be the forest of order n with two components, one of which is a path of order i and the other of which is a star of order n - i. It is clear that T(i, n) can be obtained from either B(i + 1, n) or B(i + 2, n) by the removal of a single edge. It follows, then, from Theorem 2 that if $\mathcal{F}_n$ contains the $\lfloor (n+1)/3 \rfloor$ graphs T(1, n), T(4, n), T(7, n), ..., then $\mathcal{F}_n$ is complete.

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