Complete Families of Graphs

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ABSTRACT. A broom of order $n$ is a tree obtained by identifying an endvertex of a path of order $r$ with the central vertex of a star of order $n-r+1$. If $r$ is even (odd) then the broom is even (odd). It is shown that for every graph $G$ of order $n$, either $G$ or $\overline{G}$ contains an even (odd) broom of order $n$.

1. Introduction

A family $\mathcal{F}_n$ of graphs of order $n$ is said to be complete if for every graph $G$ of order $n$, either $G$ or $\overline{G}$ contains a member of $\mathcal{F}_n$. Bialostocki, Dierker and Voxman [2] conjectured that the family $\mathcal{B}_n$ of all brooms of order $n$ is complete, where a broom of order $n$ is a tree obtained by identifying an endvertex of a path of order $r$ with the central vertex of a star of order $n-r+1$. In [2] Burr verified their conjecture and further conjectured that, in fact, only about half of $\mathcal{B}_n$ is needed for a complete family. In Section 2 we will verify this conjecture.

Suppose that $\mathcal{F}_n$ is a complete family of connected graphs of order $n$. Then, as we shall see, $|\mathcal{F}_n| \geq \left\lceil \frac{n}{2} \right\rceil$, and the family of even brooms (odd brooms) attains this minimum. In Section 3 we will consider complete families of graphs of order $n$ in which the graphs are not necessarily connected.

2. Complete families of brooms

For integers $r$ and $n$ satisfying $1 \leq r \leq n$, let $B(r,n)$ denote the broom of order $n$ obtained by identifying an endvertex of a path of order $r$ with the central vertex of a star of order $n-r+1$. We will call the identified vertex of the path the central vertex of the broom and the endvertex of the path the top of the handle of the broom. With this notation the star of order $n$ is denoted by $B(1,n)$ or, equivalently, $B(2,n)$, and the path $P_n$ is denoted by $B(n-1,n)$ or $B(n,n)$. Thus for $n \geq 3$ there are $n-2$ distinct brooms of order $n$.

Our first result implies that there are proper complete subfamilies of $\mathcal{B}_n$. Its proof employs the classic result of Menger [3] regarding $k$-connected graphs.

Theorem 1. Let $G$ be a $k$-connected graph, $k \geq 2$. Then every $k$ vertices lie on a common cycle $C$ of $G$. Furthermore, if $k < n$, then $G$ can be chosen to contain at least $k+1$ vertices.
THEOREM 2. Let $G$ be a graph of order $n \geq 2$. Then there is a positive integer $v < n$ such that either $G$ or $\overline{G}$ contains both $B(v, n)$ and $B(v+1, n)$.

PROOF. It is straightforward to verify the theorem for small values of $n$. Also, without loss of generality we may assume that the connectivity $\kappa(G)$ of $G$ and the connectivity $\kappa(\overline{G})$ of $\overline{G}$ satisfy $\kappa(\overline{G}) \geq \kappa(G)$.

Case 1. Suppose $\kappa(G) = 0$. Then $\overline{G}$ contains the complete bipartite graph $K_{n,a}$ for integers $a \leq b$, with $a + b = n$. If $a = b$, then $\overline{G}$ contains $P_{n} = B(n-1, n) = B(n,n)$. If, on the other hand $a < b$, then $\overline{G}$ clearly contains $B(2a-1, n)$ and $B(2a, n)$.

Case 2. Suppose $\kappa(G) = 1$. Let $x$ be a cutvertex of $G$, let $A$ be the vertex set of the smallest component of $G - x$ and let $B = V(G) - A - x$. Then $\overline{G}$ contains $K_{a,b}$, where $|A| = a$ and $|B| = b$ and $a \leq b$. If $x$ is adjacent in $\overline{G}$ to a vertex of $A$, then $\overline{G}$ contains $B(2a-1, n)$, with $x$ an endvertex adjacent to the central vertex of the broom, and $B(2a, n)$, with $x$ the top of the handle of the broom. Suppose, then, that $x$ is adjacent in $\overline{G}$ to no vertex of $A$, to at least one vertex of $B$, and $a \leq b$. Then $\overline{G}$ contains $B(2a+1, n)$, with $x$ the top of the handle of the broom. If $a = b$, then $\overline{G}$ also contains the broom $B(2a, n)$. Thus we may assume that $a < b$. We consider three possibilities: (1) In $\overline{G}$, $x$ has two adjacencies $y_1$ and $y_2$ in $B$. (2) In $\overline{G}$, the subgraph induced by $B$ contains at least one edge $e$. (3) Neither (1) nor (2) occurs.

In the first situation, $\overline{G}$ contains $B(2a-2, n)$, with $x_1$ the top of the handle, followed by $y_1$ and $y_2$. In the second situation, $\overline{G}$ also contains $B(2a+2, n)$, where $x$ is the top of the handle of the broom and $e$ lies on the handle. Thus in either situation, we have the necessary pair of brooms. If neither occurs, then $x$ is adjacent in $G$ to every vertex of $A$ and to all but one vertex of $B$. Also, the subgraph induced by $B$ in $\overline{G}$ is complete. In particular, $G$ contains $B(3, n)$, $B(4, n)$, ..., $B(b+1, n)$.

Case 3. Suppose $2 \leq \kappa(G) \leq \kappa(\overline{G})$. Let $S$ be a vertex cut of $G$ of minimum order, let $A$ be the vertex set of one component of $G - S$ and let $B = V(G) - A - S$. Since $\kappa(\overline{G}) > \kappa(G) = 2$, it follows from Menger's theorem that $\overline{G}$ has a cycle $C$ containing $S$, with $|V(C)| > |S|$. Select such a cycle $C$ of maximum order. If $C$ is hamiltonian, then $\overline{G}$ contains $B(n-1, n)$ and $B(n, n)$. Otherwise, then, that $|V(C)| < n$.

Let $A^* = A - V(C)$ and let $B^* = B - V(C)$, with $a^* = |A^*|$ and $b^* = |B^*|$. Without loss of generality, assume $b^* \geq a^*$.

Subcase 1. Suppose $a^* > 0$, and that in $\overline{G}$, $C$ contains a vertex $x$ adjacent to a vertex of $A^*$ and a vertex $y$ adjacent to a vertex of $B^*$. Fix an orientation of $C$ and under this orientation, let $x^+$ and $y^+$ denote the successors of $x$ and $y$, respectively. Then $\overline{G}$ contains the brooms $B(|V(C)| - 2a^* - 1, n)$, with $x^+$ the top of the handle and $x$ the last vertex of $C$ on the handle; and the broom $B(|V(C)| + 2a^*, n)$, with $y^+$ the top of the handle and $y$ the last vertex of $C$ on the handle.

Subcase 2. Suppose $a^* = 0$. Then $b^* > 0$. Let $y^* \in A$. Then $y^*$ is on $C$, with $y^+$ and $y^{++}$ its two immediate successors. Therefore, $\overline{G}$ contains $B(|V(C)| - 1, n)$, with $y^{++}$ the top of the handle, $y$ the last vertex of $C$ on the handle; and $y^+$ the only vertex of $C$ not on the handle, as well as $B(|V(C)|, n)$, with $y^+$ the top of the handle and $y^+$ the last vertex of $C$ on the handle.
Subcase 3. Suppose \( n^* > 0 \) but Subcase 1 does not occur. Then one of two things occurs. Either no vertex of \( C \) is adjacent, in \( \bar{G} \), to a vertex of \( B^* \) so that \( A = A^* \), and \( C \) consists of the vertices of \( S \) and at least one vertex of \( B \); or no vertex of \( C \) is adjacent, in \( \bar{G} \), to a vertex of \( A^* \) so that \( B = B^* \), and \( C \) consists of the vertices of \( S \) and at least one vertex of \( A \). We consider the first case in detail, and leave the details of a similar argument for the second case to the reader.

The maximality of \( C \) implies that \( C \) contains no two consecutive vertices from \( B \); otherwise, we could extend \( C \) by a vertex from \( A \). Thus \( |B \cap C| \leq |S| \). On the other hand, let \( y \in A \). Since \( \kappa(\bar{G}) \geq |S| \), there exist \(|S|\) paths in \( \bar{G} \) from \( y \) to \( C \), disjoint except for \( y \). By the maximality of \( C \), the endvertices of these paths cannot be consecutive on \( C \), so that \( |V(C)| \geq 2|S| \). We conclude that \( |B \cap C| = |S| \) and \( |V(C)| = 2|S| \), that is, the vertices of \( C \) alternate between vertices of \( B \) and vertices of \( S \). Finally, the maximality of \( C \) implies that no two vertices of \( S \) and no pair of vertices from \( A \) and \( S \) are adjacent in \( \bar{G} \).

Let \( x \) be a vertex of \( C \) adjacent, in \( \bar{G} \), to a vertex of \( A \). Since \( n^* \leq n^* \), \( \bar{G} \) contains the broum \( B((|V(C)| + 2n^* - 1, n), \) with \( x^* \) the top of the handle and \( z \) the last vertex of \( \bar{G} \) on the handle. If there is an edge in \( \bar{G} \) between vertices of \( A = A^* \), then \( \bar{G} \) also contains the broum \( B((|V(C)| + 2n^* - 2, n), \) while an edge in \( \bar{G} \) between vertices of \( B^* \) produces the broum \( B((|V(C)| + 2n^*, n), \) Thus we may assume that \( A = A^* \) and \( B^* \) induce complete graphs in \( G \).

Thus the structure of \( G \) is completely determined except for the subgraph induced by \( B \cap C \), denoted \( <B \cap C> \). In particular, we have the following in \( G \):

(a) \( <A> \) is complete,
(b) \( <B^*> \) is complete,
(c) \( <S> \) is complete,
(d) every vertex of \( A \) is adjacent to every vertex of \( S \),
(e) every vertex of \( C \) is adjacent to every vertex of \( B^* \),
(f) no vertex of \( A \) is adjacent to a vertex of \( B \), and
(g) no vertex of \( S \) is adjacent to a vertex of \( B \cap C \).

Furthermore, since \( \kappa(G) \geq 2 \), we have that \( |B^*| \geq 2 \). But then we can construct many brooms in \( G \) by beginning in \( A \), traversing all the vertices of \( A \), followed by all the vertices in \( S \), and then completing the handle by traversing as many of the vertices of \( B^* \) as desired. The remaining vertices of \( B^* \) and those of \( B \cap C \) form the "bristles" of the brooms. In particular, \( G \) contains the brooms \( B(|A| + |S| + 1, n) \) and \( B(|A| + |S| + 2, n) \).

An even broom is a broom \( B(r.n) \), where \( r \) is even. An odd broom is defined analogously.

**Corollary 1.** For a fixed integer \( n \), the family of even brooms of order \( n \) (odd brooms of order \( n \)) forms a complete family of graphs.

Note that for a fixed \( n \), each of the collections of even brooms and odd brooms contains \( [n/2] \) graphs. Now, suppose that \( F_n \) is any complete family of connected graphs of order \( n \) and consider the \( [n/2] \) graphs \( K_{i,n-i} \) \( 1 \leq i \leq [n/2] \). Since the complement of each of these graphs is disconnected, it follows that each of these graphs must contain a member of \( F_n \). The partite sets of \( K_{i,n-i} \) have orders \( i \) and \( n-i \); so must the partite sets of the contained member of \( F_n \). The partite sets
of $K_{2, n-2}$ have orders 2 and $n-2$; so must the partite sets of the contained member of $F_n$, and so forth. Since, in a connected bipartite graph, the orders of the partite sets are unique, it follows that $|F_n| \geq \left\lceil \frac{n}{2} \right\rceil$. Thus the family of even brooms (odd brooms) is a minimal complete family of connected graphs.

For odd values of $i$ at most $n-1$, let $S_i = \{B(i, n), B(i+1, n)\}$. We have shown that a complete family of brooms may be obtained by selecting the even broom from each $S_i$, or by selecting the odd broom from each $S_i$. In fact, we believe that a much stronger result is true and we make the following conjecture.

Conjecture 1. For a fixed integer $n$ and each odd $i$ at most $n-1$, select any $F_i \in S_i$. Then $\{F_1, F_2, \ldots\}$ is complete.

3. Other complete families of graphs

We saw in Section 2 that if $F_n$ is a complete family of connected graphs of order $n$, then $|F_n| \geq \left\lceil \frac{n}{2} \right\rceil$, and the lower bound is attainable. On the other hand, if we put no restrictions on the number of components in the graphs in $F_n$, then there are complete families $F_n$ containing exactly one graph. For example, $F_n = \{n K_1\}$ is trivially a complete family of graphs of order $n$ containing exactly one graph. This graph, of course, has $n$ components. Equivalently, the Ramsey number $r(n K_1)$ $= n$. Our next result exhibits another such graph with fewer components.

Theorem 3. For positive integers $n$ and $k$ satisfying $n \geq \left\lceil \frac{n+2}{2} \right\rceil$, define $F_{k,n}$ to be the graph $K_1 \cup K_{1,1} \cup K_{1,2} \cup K_{1,3} \cup \ldots \cup K_{1,k-1} \cup \left( n - \left\lceil \frac{n+1}{2} \right\rceil \right) K_1$. Then $\{F_{k,n}\}$ is a complete family of graphs of order $n$, that is, $r(F_{k,n}) = n$.

Proof. Color the edges of $K_n$ arbitrarily red and blue. We show that there is a monochromatic $F_{k,n}$. Suppose first that there is a set $S$ of $k$ vertices, and vertices $u$ and $v$ such that all edges from $u$ to $S$ are red and all edges from $v$ to $S$ are blue. Consider $G^* = G - \{u, v\} \cup S$ of order $n^* = n - k - 1$. By induction, $G^*$ contains a monochromatic $F_{k-1,n^*}$, and so $G$ contains a monochromatic $F_{k,n}$. Assume, then, that no such vertices $u, v$ and $S$ exist.

Let $v$ be a vertex of maximum red degree $\delta$ and let $T$ be the $\delta$ vertices connected to $v$ by red edges. We count the number of red edges between $T$ and $V(K_n) - T - \{v\}$. Since every edge between $v$ and $V(K_n) - T - \{v\}$ is blue, each of the $\delta$ vertices of $T$ has at most $k-2$ red adjacencies in $V(K_n) - T - \{v\}$. Thus the number of red edges between $T$ and $V(K_n) - T - \{v\}$ is at most $\delta(k-2)$. On the other hand, since each edge from $v$ to $T$ is red, each vertex in $V(K_n) - T - \{v\}$ has at most $k-2$ blue adjacencies in $T$, and so at least $\delta - k + 2$ red adjacencies in $T$. Thus the number of red edges between $T$ and $V(K_n) - T - \{v\}$ is at least $(n - \delta - 1)(\delta - k + 2)$. We conclude that

$$\Delta(k-2) \geq (n - \delta - 1)(\delta - k + 2),$$

and so

$$\delta^2 - (n - 1)\delta + (k - 2)(n - 1) > 0.$$

If, however, $\delta = k$, then $k^2 - 2(n - 1) > 0$, which contradicts the assumption that $n \geq \left\lceil \frac{n+2}{2} \right\rceil$. We conclude that $\delta < k$, i.e., each vertex has red degree at most $k-1$. Then, of course, each vertex has blue degree at least $n-k$ and we can arbitrarily pick the $k$ blue stars to construct a blue $F_{k,n}$.
If \( F \) is the graph described in the previous theorem in the case that \( k \) is chosen as large as possible, then \( F \) has \( O(\sqrt{n}) \) components. We conjecture that \( O(\log n) \) is the smallest number of components in a complete family of graphs of order \( n \) containing exactly one graph.

**Conjecture 2.** For every \( n \) sufficiently large there exists a complete family \( \{F\} \) where \( F \) has at most \( O(\log n) \) components and \( |V(F)| = n \).

Suppose that \( \mathcal{F}_n \) is a complete family of graphs of order \( n \) and each graph in \( \mathcal{F}_n \) has at most \( k \) components. For \( k = 1 \) we know that \( |\mathcal{F}_n| \geq \lfloor n/2 \rfloor \). This count is based on considering the \( \lfloor n/2 \rfloor \) graphs \( K_{i,n-i} \), \( 1 \leq i \leq \lfloor n/2 \rfloor \). Since the complement of each of these graphs is disconnected, it follows that each of these graphs must contain a member of \( \mathcal{F}_n \). These same graphs provide us with a bound on \( |\mathcal{F}_n| \) when \( k = 2 \), that is, when each graph in \( \mathcal{F}_n \) has at most two components. If \( F \) is a member of \( \mathcal{F}_n \), then \( F \) is a spanning subgraph of at most two of the complete bipartite graphs and at most one of the complements. Thus, \( 3|\mathcal{F}_n| \geq \lfloor n/2 \rfloor \). Similarly, if each graph in \( \mathcal{F}_n \) has at most three components and \( F \) is a member of \( \mathcal{F}_n \), then \( F \) is a spanning subgraph of at most four of the complete bipartite graphs and at most three of the complements, so that \( 7|\mathcal{F}_n| \geq \lfloor n/2 \rfloor \). In general, if each graph in \( \mathcal{F}_n \) has at most \( k \) components and \( F \) is a member of \( \mathcal{F}_n \), then \( F \) is a spanning subgraph of at most \( 2^k/2 = 2^{k-1} \) of the complete bipartite graphs and at most \( (2^k-2)/2 = 2^{k-1}-1 \) of the complements. It follows, then, that \( (2^k-1)|\mathcal{F}_n| \geq \lfloor n/2 \rfloor \), giving a lower bound \( |\mathcal{F}_n| \geq \left\lfloor \frac{n}{2^k+1-2} \right\rfloor \). This bound is tight for \( k = 1 \). For \( k = 2 \) we have \( |\mathcal{F}_n| \geq \left\lfloor \frac{n}{8} \right\rfloor \). We will describe complete families \( \mathcal{F}_n \) of graphs of order \( n \) in which each graph has two components and \( |\mathcal{F}_n| = \lfloor (n+1)/3 \rfloor \). Thus the smallest such family \( \mathcal{F}_n \) satisfies \( \left\lfloor \frac{n}{8} \right\rfloor \leq |\mathcal{F}_n| \leq \lfloor (n+1)/3 \rfloor \).

For \( i = 1, 2, \ldots, n - 1 \), let \( T(i, u) \) be the forest of order \( n \) with two components, one of which is a path of order \( i \) and the other of which is a star of order \( n - i \). It is clear that \( T(i, u) \) can be obtained from either \( B(i + 1, u) \) or \( B(i + 2, u) \) by the removal of a single edge. It follows, then, from Theorem 2 that if \( \mathcal{F}_n \) contains the \( \lfloor (n+1)/3 \rfloor \) graphs \( T(1, u), T(4, u), T(7, u), \ldots \), then \( \mathcal{F}_n \) is complete.
Bibliography

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