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Two-factors with few cycles in claw-free graphs

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Abstract

Let G be a graph of order n. Define $f_k(G)$ ($F_k(G)$) to be the minimum (maximum) number of components in a k-factor of G. For convenience, we will say that $f_k(G)=0$ if G does not contain a k-factor. It is known that if G is a claw-free graph with sufficiently high minimum degree and proper order parity, then G contains a k-factor. In this paper we show that $f_2(G)\leqslant n/\delta$ for n and δ sufficiently large and G claw-free. In addition, we consider $F_2(G)$ for claw-free graphs and look at the potential range for the number of cycles in a 2-factor. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The study of k-factors, i.e. k-regular spanning subgraphs, has long been fundamental in graph theory. Especially well studied are 2-factors, the disjoint union of cycles that span the vertex set. Historically, two questions have been at the forefront of this study. Under what conditions will a 2-factor exist? Is this 2-factor a single cycle (the hamiltonian problem)? However, harder questions about the actual structure of general 2-factors have also been considered. For example, Corrádi and Hajnal [5] showed that if a graph G has order $n=3t$ and minimum degree $\delta(G)\geqslant 2t$ then G has a 2-factor composed of triangles. In [2] it was shown that the classic hamiltonian condition of Dirac [6] (G satisfies $\delta(G)\geqslant |V(G)|/2$) not only implies the graph is hamiltonian, but in fact, G must contain 2-factors with t cycles, for each integer t satisfying $1\leqslant t\leqslant |V(G)|/4$. The complete bipartite graph $K_{n/2,n/2}$ shows this result is best possible.

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The class of claw-free graphs (no induced $K_{1,3}$) has played a major role in a number of different studies. This broad class admits many interesting graph properties, often under somewhat weaker conditions than those for arbitrary graphs. For example, Matthews and Sumner [10] showed that if G is a 2-connected claw-free graph of order n with $\delta(G) \geqslant (n-2)/3$, then G is hamiltonian. The graph of Fig. 1 shows this result is best possible. This result was extended in [3] when the same conditions were shown to imply the existence of a 2-factor with t cycles for each t in the range $1 \leqslant t \leqslant (n-24)/3$. Acree and Leist [1] studied the number of cycles in 2-factors for several classes of graphs obtained by forbidding the claw and another graph.

Independently, results of Egawa and Ota [7] and Choudum and Paulraj [4] imply the following.

Theorem 1. *A connected claw-free graph with minimum degree at least* 4 *contains a* 2-*factor*.

Thus, 2-factors exist in claw-free graphs under very weak conditions. Since a hamiltonian cycle is only guaranteed if G is 2-connected and $\delta(G) \geqslant (n-2)/3$, it is natural to ask what is the minimum number of cycles in a 2-factor of a claw-free graph G of order n with $\delta(G) \geqslant 4$? Hence, we define $f_k(G)$ ($F_k(G)$) to be the minimum (maximum) number of components in a k-factor of G. For convenience, we will say that $f_k(G) = 0$ if G does not contain a k-factor. Faudree et al. [8] investigated the question and showed the following.

Theorem 2. *If G is a connected claw-free graph of order n and minimum degree $\delta(G)$ then $f_2(G) \leqslant 6n/(\delta(G)+2) - 1$.*

In this paper we prove the following result which improves the last result from roughly $6n/\delta(G)$ to $n/\delta(G)$.

Theorem 3. *Let $k \geqslant 2$ be a fixed positive integer. If G is a claw-free graph of order $n \geqslant 16k^3$ and $\delta(G) \geqslant n/k$, then G has a 2-factor with at most k cycles.*

Let H be a 2-factor of a graph G. Let $s(H,G)$ denote the number of cycles in H and $S_2(G) = \bigcup_{H \subset G} \{s(H,G) | H \text{ is a 2-factor of } G\}$ be the set of values assumed by the number of cycles in a 2-factor of G. The purpose of this paper is to improve the Faudree, Flandrin, Liu bound when $\delta(G)$ is large and develop more information about the set $S_2(G)$ and the function $f_2(G)$.

In what follows, all graphs are finite with no loops or multiple edges. We let $V(G)$ denote the vertex set of G and $\alpha(G)$ denote the independence number of G, that is, the maximum cardinality of an independent set of vertices. Given a cycle C and a vertex $x \in V(C)$, we let x^+ and x^- denote the successor and predecessor of x under some orientation of C. We use the notation $C[a,b]$ to denote a segment of the cycle C from the vertex a to the vertex b following the orientation of C. Let $C^-[a,b]$ denote the segment traversing the vertices of C under the reverse of the orientation of C. Also, C^- will denote traversing C in the reverse direction.

2. Proof of the main result

In this section we prove Theorem 3 and to do this we need the following consequence of a result in [9].

Theorem 4. *If G is a claw-free graph of order n, then $\alpha(G) \leqslant 2n/(\delta(G)+2)$.*

Proof of Theorem 3. Clearly by Theorem 1, G contains a 2-factor. Suppose the result fails to hold, then G contains a 2-factor with at least $k+1$ components. Now suppose over all 2-factors with the minimum number of components, we choose one with a smallest cycle C_1. Further, note by Theorem 4 that $\alpha(G) \leqslant 2n/(\delta(G)+2) < 2k$.

Claim 1. *The cycle C_1 is K_3.*

Proof. Suppose not, say that $|V(C_1)| \geqslant 4$. Since $|V(C_1)| \leqslant n/(k+1)$, we see that any vertex $x \in V(C_1)$ must send at least $n/(k^2+k)$ edges to $V(G) - V(C_1)$. Further, $n/(k^2+k) \geqslant 8k$ since $n \geqslant 16k^3$.

We now consider the structure of adjacencies from $x \in V(C_1)$ to vertices on the other cycles $C_2, C_3, \ldots, C_t$, $(t \geqslant k+1)$. In order to complete the proof of Claim 1, we make the following claim.

Claim 2. *The set of successors of neighbors of x on $C_2, \ldots, C_t$ form an independent set.*

Proof. Suppose $x \in V(C_1)$ is adjacent to vertex $x_2 \in V(C_2)$ and $x_3 \in V(C_3)$. Further, suppose that x_2^+ and x_3^+ are the successors of x_2 and x_3 under some orientation of the cycles C_2 and C_3, respectively. Suppose that x_2^+ and x_3^+ are adjacent. Then by considering the claw centered at x with x_2, x_3 and $x^- \in V(C_1)$, we see that either x_2 is adjacent to x_3 or x^- is adjacent to one of x_2 or x_3. However, if x_2 is adjacent to x_3, then cycles C_2 and C_3 can easily be combined into one cycle, contradicting our assumption that our cycle system had the least number of cycles. Now without loss of generality, suppose that x^- is adjacent to x_2. Then $x^-, x_2, C_2^-, x_2^+, x_3^+, C_3^-, x_3, x, C_1, x^-$ is a cycle that combines all three of C_1, C_2, and C_3, contradicting our assumptions again. Thus, we conclude that x_2^+ and x_3^+ are nonadjacent.

Next we suppose that x_2 and x_3 are both on the same cycle, say C_2. Then again suppose that x_2^+ and x_3^+ are adjacent. Now note that on C_1, the vertices x^- and x^+ are not adjacent, for otherwise, since $|V(C_1)| \geqslant 4$ we could remove x from C_1 leaving a cycle C_1^* and we could incorporate the vertex x into C_2 forming the cycle C_2^* as $x, x_2, \ldots, x_3^+, x_2^+, \ldots, x_3, x$. However, this produces a cycle system with the same number of cycles and a cycle smaller that C_1, contradicting our assumptions. Now the claw centered at x with x^+, x^- and x_2 implies that (without loss of generality) $x^-x_2 \in E(G)$. Then $x^-, x_2, x_2^-, \ldots, x_3^+, x_2^+, \ldots, x_3, x, C_1, x^-$ is a cycle incorporating $V(C_1)$ and $V(C_2)$ again producing a 2-factor with fewer cycles, contradicting our assumptions. This proves Claim 2.

But, x has at least $2k$ neighbors on $C_2, \ldots, C_t$ whose successors, by Claim 2, form an independent set, while $\alpha(G)$ is less than $2k$, a contradiction. This completes the proof of the Claim 1.

Thus, C_1 must be K_3 and let $V(C_1) = \{u_1, u_2, u_3\}$.

Claim 3. *The number of different cycles in $C_2, \ldots, C_t$ containing neighbors of $V(C_1) = \{u_1, u_2, u_3\}$ is less than $2k$.*

Proof. Suppose the claim fails to hold so that $V(C_1)$ has neighbors on at least $2k$ other cycles. Again using $\alpha(G) < 2k$, we know that the set of successors of neighbors of $\{u_1, u_2, u_3\}$ cannot be an independent set. Thus, either for one vertex of C_1, say u_1 the set of successors of neighbors on $C_2, \ldots, C_t$ are not independent, or for two vertices of C_1, without loss of generality say u_1 and u_2, the set of successors of neighbors on $C_2, \ldots, C_t$ are not independent.

In the first case, a method of proof similar to that used in Claim 2 may be applied to produce a smaller cycle system, contradicting our assumptions. In the second case, suppose that u_1 is adjacent to $x_1 \in V(C_i)$ and u_2 is adjacent to $x_2 \in V(C_j)$ $(i \neq j)$. Then if x_1^+ and x_2^+ are adjacent, we see that $u_1, x_1, C_i^-, x_1^+, x_2^+, C_j, x_2, u_2, u_3, u_1$ is a cycle that combines all the vertices of C_1, C_2 and C_3, contradicting our assumptions. Thus, in either case, the vertices of C_1 have adjacencies to at most $2k-1$ other cycles as claimed.

Now, we note that each vertex of C_1 must have at least $n/k-2$ adjacencies to vertices off of C_1. Thus each vertex of C_1 has $n/2k^2$ neighbors on some one cycle other than C_1. Say that u_i has these adjacencies to cycle C_{j_i}, $i = 1, 2, 3$. As $n/2k^2 \geqslant 8k > 4\alpha(G)$, the set of all successors of neighbors of u_i cannot form an independent set. If the cycles C_{j_i}, $i = 1, 2, 3$, are all distinct, then each of the vertices u_{j_i} can be absorbed into C_{j_i}, and a 2-factor with fewer cycles results. Thus, at least two of the vertices of C_1 have their $n/2k^2$ adjacencies to the same cycle, say C_j. Without loss of generality, say that u_1 and u_2 are these two vertices.

Now over all possible pairs of neighbors of either u_1, or u_2 we select a closest pair along C_j with the property that their successors along C_j are adjacent. Without loss of generality, say that $x_1, x_2 \in N(u_1) \cap V(C_j)$ is such a pair. Let $S_1 = C[x_1, x_2]$. Note that u_2 can have at most $2k$ neighbors in S_1 or we could find a pair closer along C_j than x_1 and x_2 with adjacent successors, contradicting our choice. Thus, u_2 has at least $6k$ neighbors to C_j outside S_1. Among these neighbors select a pair y_1, y_2 such that $y_1^+ y_2^+ \in E(G)$. Thus, we can find $x_1, x_2 \in N(u_1) \cap V(C_j)$ with $x_1^+ x_2^+ \in E(G)$ and $y_1, y_2 \in N(u_2) \cap V(C_j)$ with $y_1^+ y_2^+ \in E(G)$ and such that $C[x_1, x_2] \cap C[y_1, y_2] = \emptyset$. Then the cycle u_1, $x_2, \ldots, x_1^+$, $x_2^+, \ldots, y_1$, u_2, $y_2, \ldots, y_1^+$, $y_2^+, \ldots, x_1$, u_1 incorporates both u_1

and u_2 into C_j. The vertex u_3 may then be incorporated into C_{j_3} and we will have a 2-factor with fewer cycles, a contradiction.

Finally, we consider the case when each u_i, $(i=1,2,3)$ has all of its $n/2k^2$ neighbors on the same cycle, say C_j. As before over all possible pairs of neighbors of either u_1, u_2 or u_3 we select a closest pair along C_j with the property that their successors along C_j are adjacent. Without loss of generality, let $x_1, x_2 \in N(u_1) \cap V(C_j)$ be such a pair. Let $S_1 = C[x_1, x_2]$. Again, note that u_2 and u_3 each have at most $2k$ neighbors in S_1 or we could find a pair closer along C_j than x_1 and x_2 with adjacent successors, contradicting our choice. Thus, u_2 and u_3 each have at least $6k$ neighbors to C_j outside S_1. Now repeat the above argument on these neighbors of u_2 and u_3. Without loss of generality, suppose that $y_1, y_2 \in N(u_2) \cap V(C_j) - S_1$ are a closest pair with the property that $y_1^+ y_2^+ \in E(G)$. Let $S_2 = C[y_1, y_2]$. Now the deletion of S_1 and S_2 from C_j partitions the remaining vertices of C_j into at most two segments. The vertex u_3 has at most $2k$ neighbors into either S_1 or S_2. Thus, it has at least $4k$ neighbors into the remaining vertices, and hence at least $2k$ neighbors into one of these segments. Thus, in this segment we may select a pair $z_1,\ z_2 \in N(u_3)$ such that $z_1^+ z_2^+ \in E(G)$. Let $S_3 = C[z_1,\ z_2]$. Now it is clear that $S_i \cap S_j = \emptyset$ for $i, j \in \{1,2,3\}$ and $i \neq j$. Hence, each of u_1, u_2 and u_3 can be incorporated into C_j. Once again we have a 2-factor with fewer cycles and a contradiction. This completes the proof. □

3. Examples

We now turn our attention to several examples that are key to our investigation. These examples illustrate the behavior of $f_2(G)$ as well as that of $S_2(G)$.

Example 1. Sharpness of Sumner's result.

The graph H contains three copies of $K_{n/3}$ with distinct vertices x_i and y_i $(i=1,2,3)$ in each copy joined by an edge to the corresponding vertices in the other two copies (Fig. 1). That is, x_1 is joined to x_2 and x_3 and similarly for y_1. The graph H has many 2-factors, but $f_2(H) = 2$.

Example 2. Increasing values for $f_2(G)$.

Consider the graph R obtained by replacing the vertices of a P_t with copies of K_{d+1}, where there is exactly one edge between consecutive copies of K_{d+1} (see Fig. 2). Clearly, R has order $n = t(d+1)$ and $\delta(R) = d$. Finally, it is easy to see that $f_2(R) = t$. Thus, for fixed n as $\delta(G)$ decreases, clearly $f_2(G)$ must increase.

Example 3. The sharpness of the bound on $f_2(G)$.

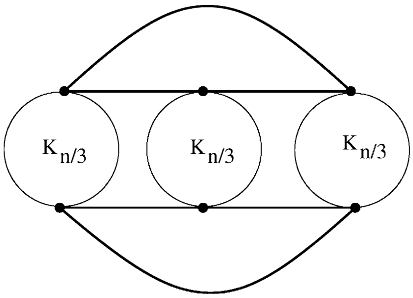

Fig. 1.

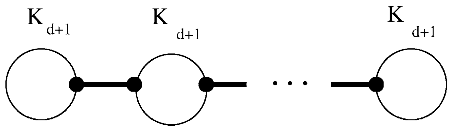

Fig. 2.

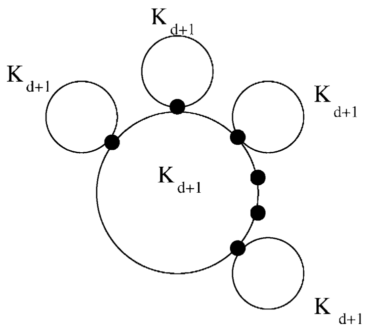

Fig. 3.

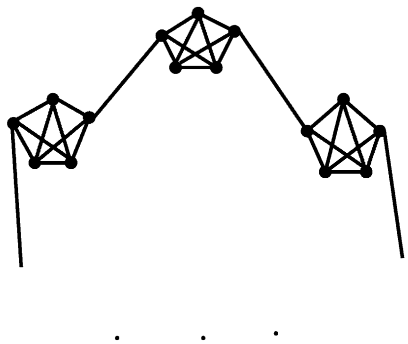

Fig. 4.

Consider the graph W composed of one central copy of K_{d+1} and $d-1$ other copies of K_{d+1} where one vertex from each of the $d-1$ copies of K_{d+1} is identified with a distinct vertex of the central K_{d+1}. Note that two vertices of the central K_{d+1} are unused in this process (see Fig. 3). Then W has order $n=(d-1)(d+1)+2=d^2+1$ and minimum degree d. Further, $f_2(W)=d$. Also note that $\lfloor n/\delta(W)\rfloor = d$.

Example 4. A graph where $S_2(G)$ does not assume consecutive values.

Finally, consider the graph M composed of k copies of the graph $L_i = K_5 - e$ ($e = x_i y_i$, $i=0,\ldots,k-1$) where the graphs L_i are connected by placing an edge between x_i and y_{i+1}, (subscripts mod k). (See Fig. 4.) This graph has order $n=5k$ and $\delta(M)=4$. Further, M is hamiltonian and $F_2(M)=k$, but there are no other 2-factors of M. Hence, $S_2(M)=\{1,k\}$ and is not a set of consecutive integers.

4. Conclusions and problems

For claw-free graphs we have established a new bound on $f_2(G)$. However, we wonder about the values of $f_2(G)$, especially as $\delta(G)$ decreases.

As we have seen, when the minimum degree of a claw-free graph is sufficiently high, there is a wide range of 2-factors. In fact, as shown by the result in [3] mentioned earlier, $S_2(G) = \{1, 2, \ldots, (n-24)/3\}$. This set of consecutive integers is nearly best possible. But the interesting feature is that the set $S_2(G)$ is a set of consecutive integers. We wonder if $S_2(G)$ is a set of consecutive integers whenever G is claw-free and $\delta(G) \geqslant n/k$, for some integer k? Recall the graph of Fig. 4 shows that this need not be the case for small values of $\delta(G)$. What is the maximum $\delta(G)$ such that $S_2(G)$ (G claw-free) is not a set of consecutive integers?

Finally, we note the case when $\delta(G) \geqslant (n-2)/3$ but G has connectivity one can be considered. A straightforward but tedious analysis of the structure of G based on the number of cut vertices in G, the values of the orders of the blocks of G mod 3 and applications of the result from [3] to these blocks shows that large G will have 2-factors with t cycles for $3 \leqslant t \leqslant n/3 - 17$.

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