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# PATH SPECTRA AND FORBIDDEN FAMILIES

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ABSTRACT. The *path spectrum*, $sp(G)$, of a graph $G$ is the set of all lengths of maximal paths in $G$. The path spectrum is *continuous* if $sp(G) = \{\ell, \ell+1, \ldots, m\}$ for some $\ell \leq m$. A graph whose path spectrum consists of a single element is called *scenic* and is by definition continuous. In this paper, we determine when a $\{K_{1,3}, S\}$-free graph has a continuous path spectrum where $S$ is one of $C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, N, B$, or $W$.

## 1. INTRODUCTION

All graphs considered in this paper are simple graphs, no loops or multiple edges are allowed. For terms not defined here, see [4]. A graph $G$ is *hamiltonian* if $G$ contains a cycle spanning the vertex set of $G$. A path $P$ in $G$ is *maximal* if it cannot be extended to a longer path by adding an edge and a vertex to one of the end vertices of $P$. A graph $G$ is $\{H_1, H_2, \ldots, H_k\}$*-free* ($k \geq 1$) if $G$ contains no induced subgraph isomorphic to an $H_i$, $1 \leq i \leq k$

The *path spectrum* of a connected graph $G$, $sp(G)$, is the set of lengths of all maximal paths in $G$. The path spectra of graphs have been studied in [5] and [2]. In [5] and [2], the focus of the work is on determining whether a given set of integers is in the path spectrum of some graph. Also, in [5], Jacobson *et al.* asked about the complexity of computing the path spectrum of a given graph $G$. They considered the related question of whether there is a maximal path of length $k$. This question is NP-hard since if $k$ is one less than the order of $G$, the problem asks whether the graph has a hamiltonian path. Hence, the path spectrum question for an arbitrary graph was determined to be NP-complete.

However, Bedrossian in [1] proved the following (see Figure 1 for drawings of some of the graphs).

**Theorem 1.** Let $R$ and $S$ be connected graphs with $R, S \neq P_3$, and let $G$ be a 2-connected graph that is not a cycle. Then $G$ being $\{R, S\}$-free implies $G$ is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$, or $W$.

Research partially supported by O.N.R. Grant N00014-97-1-0499.

Faudree and Gould in [3] improved on the work of Bedrossian to get the following theorem.

**Theorem 2.** Let $R$ and $S$ be connected graphs with $R, S \neq P_3$, and let $G$ be a 2-connected graph of order $n \geq 10$. Then $G$ being $\{R, S\}$-free implies $G$ is hamiltonian if and only if $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, Z_3, B$, $N$, or $W$.

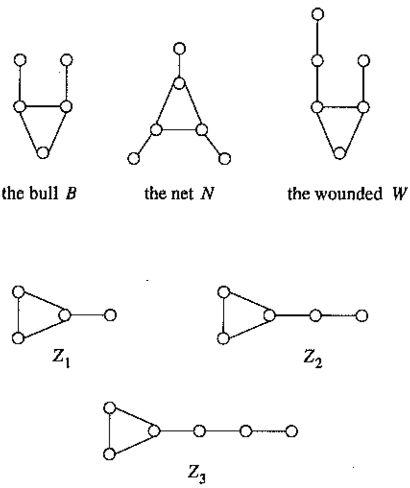


FIGURE 1. The graphs $B, N, W, Z_1, Z_2, Z_3$.

Since we know that these 2-connected, $\{R, S\}$-free graphs are hamiltonian (and hence have a hamiltonian path), we ask what can be said about the path spectrum of such graphs. In particular, are they continuous? By a *continuous path spectrum*, we mean that $sp(G) = \{\ell, \ell+1, \ldots, m\}$ where $\ell$ is the length of the shortest maximal path in $G$ and $m$ is the length of the longest maximal path in $G$. Note a path spectrum consisting of only one element is continuous. A graph with such a path spectrum is called *scenic*. Thomassen characterized when a traceable graph is scenic in [7]. Jacobson, Kézdy, and Lehel also studied scenic graphs in [6].

We need some notation to state Thomassen's result. A matching of $t$ edges will be denoted by $tK_2$. A graph that is a complete graph minus a matching with $1 \leq t \leq n/2$ will be denoted by $K_n - tK_2$. A complete bipartite graph plus (resp. minus) an edge is denoted by $K_{p,p} + K_2$ (resp. $K_{p,p} - K_2$). The graph obtained by adding an edge to each partite set of $K_{p,p}$ is denoted by $K_{p,p} + 2K_2$. If $H \in \{K_3, 2K_2, K_{1,q}\}$, the graph $K_{p,p+1} + H$ denotes the graph formed by adding all the edges of $H$ to the largest partite set of $K_{p,p+1}$. The *cube* is the graph $K_{4,4} - 4K_2$ and the *prism* is the graph formed from $K_6$ by removing the edges of a six-cycle. The following result is Thomassen's characterization of traceable, scenic graphs.

**Theorem 3.** [7] A traceable graph is scenic if and only if it belongs to one of the following families:

$$\begin{aligned}
\Phi[K_n] &= \{K_n, K_n - tK_2 \ (1 \le t \le n/2)\}, \\
\Phi[K_{p,p}] &= \{K_{p,p}, K_{p,p} - K_2, K_{p,p} + K_2, K_{p,p} + 2K_2\}, \\
\Phi[K_{p,p+1}] &= \{K_{p,p+1}, K_{p,p+1} + K_3, K_{p,p+1} + 2K_2, \\
&\quad K_{p,p+1} + K_{1,q} \ (1 \le q \le p)\}, \\
\Psi &= \{P_n, C_n, \mathit{prism}, \mathit{cube}\}.
\end{aligned}$$

We answer the question concerning the 2-connected, $\{R, S\}$-free graphs in Theorem 2 that have continuous path spectra in the following result.

**Theorem 4.** Let $G$ be a 2-connected, $\{K_{1,3}, S\}$-free graph of order $n \ge 10$ where $S$ is one of $C_3, P_4, P_5,\ P_6, Z_1, Z_2, Z_3, N, B,$ or $W$. Then $G$ has a continuous path spectrum if and only if $S$ is one of the graphs $C_3, P_4, Z_1$ or $Z_2$. Furthermore, $G$ is scenic if and only if $G$ is one of $K_n, K_n - tK_2$ or $C_n$.

The proof of this theorem is in Section 3. Two preparatory propositions are in the next section.

## 2. Two Results

**Proposition 2.1.** Let $G$ be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n$. Let $P$ be a maximal $u$–$v$ path of order $m < n$. Then $P$ can be extended to a maximal $u$–$v$ path of order $m + 1$.

*Proof.* Let $P$ be $u = x_1, x_2, \ldots, x_m = v$. Since $G$ is connected, $m < n$, and $P$ is maximal, there is a vertex $w$ in $V(G) - V(P)$ such that $w$ is adjacent to a vertex $x_j$, $1 < j < n$, on $P$. Also, since $G$ is 2-connected, there is at least one other path from $w$ to $P$. Consider the collection $C$ of these paths that have the shortest length. Let $Q$ be the path from $C$ that hits closest to $x_j$. Suppose that $Q$ hits $P$ at $x_k$ and with out loss of generality that $j < k < n$. Let $Q$ be $x_k = z_1, z_2, \ldots, z_\ell = w$.

**CASE 1:** Suppose that $k \ge j+3$. We first note that $\langle\{x_{j-1}, x_j, x_{j+1}, w\}\rangle$ is a claw centered at $x_j$ and that $\langle\{x_{k-1}, x_k, x_{k+1}, z_2\}\rangle$ forms a claw centered at $x_k$. Observe that $w$ cannot be adjacent to $x_{j-1}$ or $x_{j+1}$ otherwise $Q$ would not be the closest path to $x_j$ from $w$ to $P$. Also, $z_2$ is not adjacent to $x_{k-1}$ or else $Q$ would not hit closest to $x_j$. Now if $z_2$ is adjacent to $x_{k+1}$, then $P$ can be easily extended as follows:

$$u = x_1, x_2, \dots, x_k, z_2, x_{k+1}, \dots, x_m = v.$$

Thus, suppose that $z_2$ is not adjacent to $x_{k-1}$ or $x_{k+1}$. Consequently, $x_{j-1}x_{j+1}$ and $x_{k-1}x_{k+1}$ must be edges of $G$.

Now, if $Q$ has 3 or more vertices, then $\langle\{x_{k-1}, x_k, x_{k+1}, z_2, z_3\}\rangle$ forms a $Z_2$. Thus, either $x_{k-1}z_2$, $x_{k+1}z_2$, $x_{k-1}z_3$, $x_{k+1}z_3$, or $x_kz_3$ is an edge of $G$. Since $Q$ is the shortest path from $w$ to $P$, $x_{k-1}z_3$, $x_{k+1}z_3$, and $x_kz_3$ cannot be edges in $G$. Since $Q$ hits closest to $x_j$, $x_{k-1}z_2$ cannot be an edge in $G$. If $x_{k+1}$ is adjacent to $z_2$, then $P$ can be extended as above. Therefore, we assume that $Q$ has only two vertices; that is, $w$ is adjacent to $x_k$.

Next, we note that $\{x_{j-1}, x_j, x_{j+1}, w, x_k\}$ induces a $Z_2$. Hence at least one of the following edges is in $G$: $x_{j-1}x_k$, $x_{j+1}x_k$, or $x_jx_k$. (The pairs $wx_{j-1}$ and $wx_{j+1}$ were eliminated since $Q$ hits closest to $x_j$.) If $x_{j-1}x_k$ (or similarly $x_{j+1}x_k$) is an edge, then $P$ can be extended as follows:

$$u = x_1, x_2, \dots, x_{j-1}, x_k, w, x_j, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_m = v.$$

Therefore, assume that $x_jx_k$ is an edge in $G$.

Note that by a symmetric argument on $x_k$, the edges $wx_{k+1}$, $x_{k-1}x_j$, and $x_{k+1}x_j$ can shown to extend the path $P$.

Before proceeding, we make the following notational convention and two observations. We will denote the subpath $\{x_a, x_{a+1}, \dots, x_b\}$ of $P$ as $[x_a, x_b]$. Now we observe that if $x_{j-1}$ and $x_{k+1}$ are adjacent to adjacent vertices of $[x_{j+1}, x_{k-1}]$, then $P$ can be extended. To see why, suppose that $x_{j-1}$ is adjacent to $x_i$ and that $x_{k+1}$ is adjacent to $x_{i+1}$. Then a path of order $m+1$ can be formed as follows:

$$u = x_1, x_2, \dots, x_{j-1}, x_i, x_{i-1}, \dots, x_j, w, x_k, x_{k-1} \dots, \\ x_{i+1}, x_{k+1}, \dots, x_m = v.$$

Secondly, we note that if $x_j$ and $x_k$ are adjacent to adjacent vertices of $[x_{j+1}, x_{k-1}]$, then $P$ can be extended to a path of order $m+1$. To see this, suppose without loss of generality that $x_j$ is adjacent to $x_i$ and $x_k$ is adjacent to $x_{i+1}$. Then a path of order $m+1$ is formed as follows:

$$u = x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_i, x_j, w, x_k, x_{i+1}, \dots, \\ x_{k-1}, x_{k+1}, \dots, x_m = v.$$

Now, notice that $\langle\{w, x_j, x_k, x_{k-1}, x_{k-2}\}\rangle$ and $\langle\{w, x_j, x_k, x_{j+1}, x_{j+2}\}\rangle$ each forms a $Z_2$. We will only consider the $Z_2$ induced by $\{w, x_j, x_k, x_{k-1}, x_{k-2}\}$ in detail since the $Z_2$ $\langle\{w, x_j, x_k, x_{j+1}, x_{j+2}\}\rangle$ is symmetric. We see that at least one of the following pairs is an edge of $G$: $x_jx_{k-2}$, $x_jx_{k-1}$, or $x_{k-2}x_k$. If either of the edges $x_jx_{k-2}$ or $x_jx_{k-1}$ is an edge of $G$, then $x_{j-1}$ and $x_{k+1}$ are adjacent to adjacent vertices in $[x_{j+1}, x_{k-1}]$, and thus $P$ can be extended. Hence, we assume that $x_jx_{k-2}$ and $x_jx_{k-1}$ are not edges in $G$ but that $x_{k-2}x_k$ is an edge in $G$. By symmetry, we assume that $x_kx_{j+2}$ and $x_kx_{j+1}$ are not edges in $G$ but that $x_jx_{j+2}$ is an edge in $G$. If $j+2=k-2$, then $x_j$ and $x_k$ are adjacent to adjacent vertices in $[x_{j+1}, x_{k-1}]$ and $P$ can be extended. If $j+2 \neq k-2$, we apply arguments similar to the preceding arguments to the following $Z_2$'s: $\langle\{w, x_j, x_k, x_{k-2}, x_{k-3}\}\rangle$ and $\langle\{w, x_j, x_k, x_{j+2}, x_{j+3}\}\rangle$. We see that the only edges that do not immediately lead to a path of length $m+1$ are $x_jx_{j+3}$ and $x_k, x_{k-3}$. We continue the process until the path extends or we reach a point where $x_j$ and $x_k$ are adjacent to adjacent vertices in $[x_{j+1}, x_{k-1}]$ which also implies $P$ can be extended.

Thus, we see that if $k \geq j+3$, $P$ can be extended to a path of length $m+1$.

**CASE 2:** Suppose that $k=j+2$. Then, by the arguments of Case 1, we may assume that the edges $x_{j-1}x_{j+1}$ and $x_{k-1}x_{k+1}$ and that $w$ is adjacent to $x_k$. Thus $P$ can be extended as follows:

$$u = x_1, \ldots, x_{j-1}, x_{j+1}, x_j, w, x_k, x_{k+1}, \ldots, x_m = v.$$

**CASE 3:** Suppose that $k=j+1$. Note that if $w$ is adjacent to both $x_j$ and $x_k$, $P$ can easily be extended by exactly one vertex. Thus, we assume that $w$ is not adjacent to $x_k$; that is, $Q$ has at least three vertices. Thus, $\langle\{x_j, x_k, x_{k+1}, z_2\}\rangle$ forms a $K_{1,3}$ centered at $x_k$. If either $x_jz_2$ or $x_{k+1}z_2$ is an edge in $G$, $P$ is easily seen to be extendable. Hence, we suppose that $x_jx_{k+1}$ is an edge of $G$. Since $\langle\{x_j, x_k, x_{j-1}, w\}\rangle$ forms a claw centered at $x_j$, we assume by symmetry that $x_{j-1}x_k \in E(G)$.

Now, we note that if $Q$ has more than three vertices, a $Z_2$ is formed by $\langle\{x_j, x_k, x_{k+1}, z_2, z_3\}\rangle$. Observe that $x_{k+1}z_3$ cannot be an edge of $G$ or $Q$ would not be the shortest path from $w$ to $P$. If $z_2$ is adjacent to $x_{k+1}$, $P$ is easily seen to be extendable to a path of length $m+1$. Thus, we suppose that $x_jz_3$ is an edge of $G$. Observe that this is really the case when $Q$ has exactly three vertices as $z_3$ assumes the role of $w$.

Thus, suppose that $Q$ has three vertices, say $w$, $z$, and $x_k$. Note that $\langle\{w, x_{j-1}, x_j, x_{k+1}\}\rangle$ forms a claw centered at $x_j$. Since $Q$ is the shortest path from $w$ to $P$ (except for $wx_j$),$x_{j-1}x_{k-1}$ must be an edge in $G$. However, we see that $\{x_{j-1}, x_{k+1}, x_k, z, w\}$ induces a $Z_2$. Since $Q$ is the shortest path, the only possible edges that can exist are $x_{j-1}z$ and $x_{k+1}z$. Clearly if $x_{k+1}z$ is an edge of $G$, $P$ can be extended. Now, if $x_{j-1}z \in E(G)$, $P$ can be extended as follows:

$$u = x_1, \ldots, x_{j-1}, z, x_k, x_j, x_{k+1}, \ldots, x_m = v.$$

Thus, when $k=j+1$, $P$ can be extended to a path of length $m+1$. □

**Proposition 2.2.** Let $G$ be a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n$. Let $P$ be a maximal $u$–$v$ path of order $m < n$. Then $P$ can be extended to a maximal $u$–$v$ path of order $m+1$.

*Proof.* Let $P$ be $u = x_1, x_2, \ldots, x_m = v$. Since $G$ is connected and $m < n$, there is a vertex $w$ in $V(G) - V(P)$ such that $w$ is adjacent to a vertex $x_j$ on $P$. Also, since $G$ is 2-connected, there is at least one other path from $w$ to $P$. Consider the collection $C$ of these paths that have the shortest length. Among this collection let $Q$ be the path that hits closest to $x_j$. Suppose that $Q$ hits $P$ at $x_k$ and with out loss of generality that $j < k < n$. Let $Q$ be $x_k = z_1, z_2, \ldots, z_\ell = w$. Observe that since $G$ is $P_4$-free, $\ell \leq 3$.

**CASE 1:** Suppose that $k \geq j + 2$. First, we note that since $G$ is claw-free, $x_{j-1}x_{j+1}$ and $x_{k-1}x_{k+1}$ are edges in $G$. Next, we observe that $\ell = 2$. To see why, suppose $\ell = 3$. Then $wz_2x_kx_{k-1}$ forms a $P_4$. Note that the addition of any edge to this $P_4$ contradicts the choice of $Q$. Hence, $\ell = 2$; that is, $w$ is adjacent to $x_k$.

Now, we see that if $k = j + 2$, $P$ can be extended as follows:

$$u = x_1, x_2, \ldots, x_j, w, x_k, x_{k-1}, x_{k+1}, \ldots, x_m = v.$$

Thus, we assume that $k > j + 2$ and observe that $\langle\{x_{j-1}, x_j, w, x_k\}\rangle$ forms a $P_4$. The vertex $w$ cannot be adjacent to $x_{j-1}$ (contradicts the choice of $Q$). If $x_{j-1}$ is adjacent to $x_k$, then $P$ can be extended as follows:

$$u = x_1, x_2, \ldots, x_{j-1}, x_k, w, x_j, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m = v.$$

Thus, we suppose that $x_j$ is adjacent to $x_k$.

Now, we see that $\langle\{x_{j+1}, x_j, x_k, x_{k+1}\}\rangle$ forms a $P_4$. If $x_{j+1}$ is adjacent to $x_{k+1}$, then $P$ can be extended as follows:

$$u = x_1, x_2, \ldots, x_j, w, x_k, x_{k-1}, x_{k-2}, \ldots, x_{j+1}, x_{k+1}, \ldots, x_m = v.$$

If $x_{j+1}$ is adjacent to $x_k$, then $P$ can be extended as follows:

$$u = x_1, x_2, \ldots, x_j, w, x_k, x_{j+1}, x_{j+2}, \ldots x_{k-1}, x_{k+1} \ldots, x_m = v.$$

Finally, if $x_j$ is adjacent to $x_{k+1}$, then $P$ can be extended as follows:

$$u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots, x_k, w, x_j, x_{k+1}, \ldots, x_m = v.$$

**CASE 2:** Suppose that $k = j + 1$. If $\ell = 2$, then $P$ is easily extendable. Hence, we assume that $\ell = 3$. Then $\langle\{w, z_2, x_k, x_{k+1}\}\rangle$ forms a $P_4$. The edges $wx_k$ and $wx_{k+1}$ cannot be in $G$ by the choice of $Q$. Thus, $x_{k+1}z_2$ must be an edge in $G$. Consequently, $P$ is easily seen to be extendable. □

## 3. Proof of Theorem 4

*Proof.* First we note that if $G$ is a 2-connected, $\{K_{1,3}, C_3\}$-free graph, then $G$ is a cycle, $C_n, n \geq 10$. Also note that a 2-connected, $\{K_{1,3}, Z_1\}$-free graph is either a cycle or a complete graph minus a matching. By Theorem 3, the only 2-connected, $\{K_{1,3}, S\}$-free scenic graphs of order $n \geq 10$ are $K_n, K_n - tK_2$, and $C_n$. Thus, these graphs have continuous path spectra.

Now suppose $G$ is a nonscenic, 2-connected, $\{K_{1,3}, S\}$-free graph of order $n \geq 10$ where $S$ is $P_4$ (or $Z_2$). Then by choosing the shortest maximal path in $G$ and repeatedly applying Proposition 2.2 (or Proposition 2.1), we see that the path spectrum of $G$ is continuous.

Finally, suppose that $G$ is a nonscenic, 2-connected $\{K_{1,3}, S\}$-free graph of order $n \geq 10$ where $S$ is one of $B, N, W, P_5, P_6$, or $Z_3$. We consider the graph $H$ in Figure 2. The path spectrum of $H$ is easily seen to be

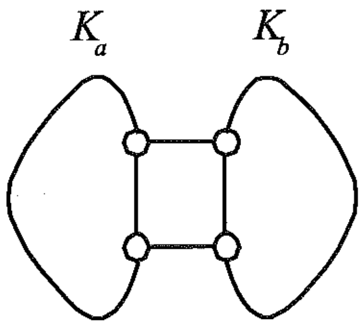


FIGURE 2. The graph $H$ with $b > a + 1$, $a \geq 4$.

$sp(H) = \{a-1, a+1, a+2, \ldots, a+b-1\}$. The graph $H$ is also free of claws, $B$'s, $N$'s, $W$'s, $P_5$'s, $P_6$'s, and $Z_3$'s. □

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