Partitioning Vertices of a Tournament into Independent Cycles

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Let $k$ be a positive integer. A strong digraph $G$ is termed $k$-connected if the removal of any set of fewer than $k$ vertices results in a strongly connected digraph. The purpose of this paper is to show that every $k$-connected tournament with at least $8k$ vertices contains $k$ vertex-disjoint directed cycles spanning the vertex set. This result answers a question posed by Bollabás.

This article will generally follow the notation and terminology defined in [1]. A digraph is called strongly connected or strong if for every pari of vertices $u$ and $v$ there exists a directed path from $u$ to $v$ and a directed path from $v$ to $u$. Let $k$ be a positive integer. A digraph $G$ is $k$-connected if the removal of any set of fewer than $k$ vertices results in a strong digraph. A tournament with $n$ vertices will also be called an $n$-tournament.

It is well-known that every tournament contains a hamiltonian path and every strong tournament contains a hamiltonian cycle. Reid [2] proved that if $T$ is a $2$-connected $n$-tournament, $n \geq 6$, that is, $T$ is not the $7$-tournament that contains no transitive subtournament with $4$ vertices (i.e., the quadratic residue $7$-tournament), then $T$ contains two vertex-disjoint cycles

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spanning $V(T)$. In fact, he showed that one cycle can be taken to be a triangle. This result established an affirmative answer (for $r = s = 1$) to the following problem asked by Thomassen (see [3]): If $r$ and $s$ are positive integers, does there exist a (least) positive integer $m = m(r, s)$ so that all but a finite number of $m$-connected tournaments can be partitioned into an $r$-connected subtournament and an $s$-connected subtournament? Song [4] was able to show that if $T$ is a 2-connected $n$-tournament with $n \geq 6$ then the vertices of $T$ can be partitioned into two cycles of lengths $s$ and $n - s$ for any integer $s$ with $3 \leq s \leq n - 3$, unless $T$ is the 7-tournament described above. The following problem was posed by Bollobás (see [2]) for tournaments.

**Problem 1.** If $k$ is a positive integer, what is the least integer $g(k)$ so that all but a finite number of $g(k)$-connected tournaments contain $k$ vertex-disjoint cycles that span $V(T)$?

Reid observed that $g(k)$ exists and $g(k) \leq 3k - 4$ for $k \geq 2$ as follows: Recall that $g(1) = 1$ and $g(2) = m(1, 1) = 2$. If $T$ is $(g(k - 1) + 3)$-connected, then the removal of a triangle leaves a $g(k - 1)$-connected tournament that can be expressed as $k - 1$ nontrivial vertex-disjoint cycles; that is, $g(k) \leq g(k - 1) + 3$. Thus, $g(3) \leq 5$, and, in general, $g(k) \leq 3k - 4$. The following example shows that $g(k) \geq k$.

Let $n \geq 3k$. Let $T$ be an $n$-tournament with $V(T) = \{v_1, v_2, \ldots, v_n\}$, where $v_i$ dominates $v_j$ for all $1 \leq i < j \leq n$ except when $1 \leq i \leq k$ and $n - k + 1 \leq j \leq n$ (in which case $v_j$ dominates $v_i$). If $S$ is any set of fewer than $k$ vertices, then $T - S$ is strongly connected; that is, $T$ is $k$-connected. Clearly, any nontrivial cycle in $T$ must use an arc $v_i v_j$ for some $1 \leq i < k$ and some $n - k + 1 \leq j \leq n$, so that $T$ contains at most $k$ vertex-disjoint cycles.

The main result of this article, stated below, shows that $g(k) = k$.

**Theorem 1.** Every $k$-connected $n$-tournament $T$ with $n \geq 8k$ contains $k$ vertex-disjoint cycles that span $V(T)$.

In [4], Song posed the following problem.

**Problem 2.** If $k$ is a positive integer, what is the least integer $f(k)$ so that all but a finite number of $f(k)$-connected tournaments contain $k$ vertex-disjoint cycles of lengths $n_1, n_2, \ldots, n_k$ where $n = n_1 + n_2 + \cdots + n_k$ and $n_i \geq 3$ for all $i = 1, 2, \ldots, k$?

Clearly, $f(1) = g(1) = 1$. Song showed that $f(2) = g(2) = 2$. Clearly, $f(k) \geq g(k)$ holds for every $k$. Song conjectured that $f(k) = g(k)$. 214 CHEN, GOULD, AND LI
Let $T$ be a tournament. The arc set of $T$ will be denoted by $E(T)$. If $u \rightarrow v$ is an arc in $T$, then $u$ dominates $v$ and $v$ is dominated by $u$. A set $A \subseteq V(T)$ dominates a set $B \subseteq V(T)$ if every vertex of $A$ dominates every vertex of $B$. If $A = \{x\}$, we say that $x$ dominates $B$. For any $X \subseteq V(T)$, let $T[X]$ denote the subtournament induced by $X$.

Let $T$ be a tournament and let $C$ be a cycle in $T$. For every vertex $v \in V(C)$, let $v^+_C$ denote the successor of $v$ on $C$ and let $v^-_C$ denote the predecessor of $v$ on $C$. If no confusion arises, $v^+$ and $v^-$ will be used to denote $v^+_C$ and $v^-_C$, respectively. Let $X$ be a cycle or a path of $T$ and let $u$ and $v$ be two vertices on $X$ ($u$, $v$ are in that order along $X$ if $X$ is a path).

We define $X[u, v]$ as the subpath of $X$ from $u$ to $v$. For any $u \not\in V(C)$, if $u$ is dominated by a vertex $x \in V(C)$ and $u$ dominates $x^+$, then $ux^+ C[x^+, x] xu$ is a cycle longer than $C$. In this case, we say that $u$ can be inserted into $C$. So, if $u$ cannot be inserted into a cycle $C$, then either $u$ dominates $V(C)$ or $V(C)$ dominates $u$. In the case, we call $C$ an out-cycle of $u$ while in the second case we call $C$ an in-cycle of $u$. The following lemma will be used in the proof of Theorem 1.

**Lemma 1.** Every $k$-connected tournament with $n \geq 5k - 3$ vertices and $k \geq 2$ contains $k$ vertex-disjoint cycles.

**Proof.** To the contrary, let $k \geq 2$ be the smallest positive integer such that there is a $k$-connected tournament $T$ with $n \geq 5k - 3$ vertices, which does not contain $k$ vertex-disjoint cycles. By the minimality of $k$ and the fact that every strong tournament has a cycle, $T$ contains $k - 1$ vertex-disjoint cycles. Since every cycle of length at least 4 contains a chord, $T$ contains $k - 1$ vertex-disjoint triangles, say, $T_1$, $T_2$, ..., $T_{k-1}$. Let $H = T - \bigcup_{i=1}^{k-1} V(T_i)$. Since $H$ does not contain a cycle, $H$ is a transitive tournament. Let $P = v_1 v_2 \cdots v_m$ be the unique hamiltonian path in $H$. Since $H$ is transitive, then $v_i v_j \in E(T)$ for any $1 \leq i < j \leq m$.

Let $F = \{v_1, v_2, ..., v_k\}$ and $B = \{v_{m-k+1}, v_{m-k+2}, ..., v_m\}$. Since $m \geq (5k - 3) - 3(k - 1) = 2k$, then $F \cap B = \emptyset$. Since $T$ is $k$-connected, there exist $k$ vertex-disjoint paths $P_1$, $P_2$, ..., $P_k$ from $B$ to $F$. Clearly, these paths plus the appropriate arcs from $F$ to $B$ form $k$ vertex-disjoint cycles.

**Proof of Theorem 1.** Let $T$ be a $k$-connected tournament with $n \geq 8k$ vertices. Since $8k \geq 5k - 3$, $T$ contains $k$ vertex-disjoint cycles by Lemma 1. Let $C_1$, $C_2$, ..., $C_k$ be $k$ vertex-disjoint cycles of $T$ such that $\sum_{i=1}^{k} |V(C_i)|$ is maximum. Let $\mathcal{C} = \{C_1, C_2, ..., C_k\}$. To the contrary, then, we may assume that $\sum_{i=1}^{k} |V(C_i)| < n$. Let $H = T - \bigcup_{i=1}^{k} V(C_i)$. Since $H$ is a tournament, $H$ has a hamiltonian path. Let $P = v_1 v_2 \cdots v_m$ be a hamiltonian path in $H$. The linear order of $v_1, v_2, ..., v_m$ will play a role in our proof.
For each \( v_i \in V(H) \) (1 \( \leq i \leq m \)) and each \( C_r \in \mathcal{C} \) (1 \( \leq r \leq k \)), since \( v_i \) cannot be inserted into \( C_r \), \( C_r \) is either an in-cycle of \( v_i \) or an out-cycle of \( v_i \).

We partition \( \mathcal{C} \) into two sets \( \mathcal{F}_i \) and \( \mathcal{E}_i \) for each \( i = 1, 2, \ldots, m \) as

\[
\mathcal{F}_i = \{ C_r \mid C_r \text{ is an in-cycle of } v_i \},
\]

\[
\mathcal{E}_i = \{ C_r \mid C_r \text{ is an out-cycle of } v_i \}.
\]

For any two vertices \( v_i, v_j \in V(H) \) and a cycle \( C_r \in \mathcal{C} \), if \( i < j \) and \( C_r \) is an out-cycle of \( v_i \), then \( C_r \) is also an out-cycle of \( v_j \); otherwise, let \( x \) and \( x^+ \) be two consecutive vertices on \( C_r \). The cycle \( P[ v_i, v_j ] C_r[ x^+, x ] v_i \) is longer than \( C_r \) which leads to a contradiction of the maximality of \( \sum_{r=1}^{k} |V(C_r)| \). Thus, \( \mathcal{E}_i \subseteq \mathcal{E}_j \). As a consequence,

\[
\mathcal{E}_m \subseteq \mathcal{E}_{m-1} \subseteq \cdots \subseteq \mathcal{E}_1 \quad \text{and} \quad \mathcal{F}_m \supseteq \mathcal{F}_{m-1} \supseteq \cdots \supseteq \mathcal{F}_1.
\]

**Claim 1.** If \( S \) is a strong subtournament of \( H \), then \( \mathcal{F}_i = \mathcal{F}_j \) and \( \mathcal{E}_i = \mathcal{E}_j \) for any two vertices \( v_i \) and \( v_j \in V(S) \).

**Proof.** Suppose, to the contrary, that there is a cycle \( C_r \in \mathcal{C} \) such that \( C_r \in \mathcal{F}_i \) and \( C_r \in \mathcal{E}_j \). Let \( P[ v_i, v_j ] \) be a path in \( S \) connecting \( v_i \) and \( v_j \) and let \( x \) be an arbitrary vertex on \( C_r \). Then, the cycle \( P[ v_i, v_j ] C_r[ x^+, x ] v_i \) is longer than \( C_r \), a contradiction.

We will show that there exist \( k \) vertex disjoint cycles which contain all vertices of \( \bigcup_{r=1}^{k} V(C_r) \) and \( v_m \), which produces a contradiction. For convenience, let \( \mathcal{F} = \mathcal{F}_m \), \( \mathcal{E} = \mathcal{E}_m \), and \( H^* = H - v_m \).

**Claim 2.** \( \sum_{C_r \in \mathcal{F}} |V(C_r)| \geq k \) and \( \sum_{C_r \in \mathcal{E}} |V(C_r)| \geq k \).

**Proof.** Let \( S \) be the strong component containing \( v_m \) in \( H \). (Note that \( S \) could be \( \{ v_i \} \).) Since \( P = v_1 v_2 \cdots v_m \) is a hamiltonian path in \( H \), \( V(H) - V(S) \) dominates \( V(S) \). By Claim 1, \( \bigcup_{C_r \in \mathcal{F}} V(C_r) \) dominates \( V(S) \).

Also, \( S \) is the strong component of \( v_m \) in \( T[H] \cup \bigcup_{C_r \in \mathcal{F}} V(C_r) \). Thus, \( \sum_{C_r \in \mathcal{F}} |V(C_r)| \geq k \) is \( k \)-connected.

Since \( v_m \) dominates \( \bigcup_{C_r \in \mathcal{E}} V(C_r) \), \( V(H) \) dominates \( \bigcup_{C_r \in \mathcal{E}} V(C_r) \). As \( S \) is the strong component of \( v_m \) in \( T[H] \cup \bigcup_{C_r \in \mathcal{E}} V(C_r) \), we see that \( \sum_{C_r \in \mathcal{E}} |V(C_r)| \geq k \).

Without loss of generality, we may assume that \( \sum_{C_r \in \mathcal{F}} |V(C_r)| \geq \sum_{C_r \in \mathcal{E}} |V(C_r)| \). Otherwise, we may reverse the directions of all arcs of \( T \) and exchange the roles of \( v_1 \) and \( v_m \) and consider \( \mathcal{E}_1 \). Since \( \mathcal{E}_1 \supseteq \mathcal{E}_m \), \( \sum_{C_r \in \mathcal{E}_1} |V(C_r)| \geq \sum_{C_r \in \mathcal{E}_1} |V(C_r)| \).
Since \(|V(T)| = n \geq 8k\), we have that
\[
\sum_{C \in \mathcal{I}} |V(C)| + |V(H^*)| \geq 4k.
\]

Define
\[
R = \left\{ y \in \bigcup_{C \in \mathcal{I}} V(C) : xy \in E(T) \text{ for some } x \in \bigcup_{C \in \mathcal{I}} V(C) \right\}
\]
and
\[
U = \bigcup_{C \in \mathcal{I}} V(C) - R.
\]

That is, any \(y \in R\) is dominated by some vertices in \(\bigcup_{C \in \mathcal{I}} V(C)\) and any \(u \in U\) dominates all vertices in \(\bigcup_{C \in \mathcal{I}} V(C)\) for all \(u \in U\).

**Claim 3.** For each \(C_i \in \mathcal{I}\), \(|V(C_i) \cap R| \leq 3\) and equality holds only when \(C_i\) is a triangle.

**Proof.** Let \(x \in C_j \in \mathcal{I}\) and \(y \in V(C_i) \cap R\) such that \(xy \in E(T)\). If \(y = z \in E(T)\) for some \(z \in V(C_i) - \{y, y^-\}\), the cycles \(v_m C_i[z^+, y] C_j[y, z^-] v_m\) and \(C_i[z, y^-] y\) plus the remaining \(k - 2\) cycles of \(\mathcal{I}\) contradict the maximality of \(\sum_{C \in \mathcal{I}} |V(C)|\). Hence, \(V(C_i) - \{y, y^-\}\) dominates \(y^-\). Suppose \(w\) is another vertex in \(R \cap V(C_i)\). Similarly, we have that \(V(C_i) - \{w, w^-\}\) dominates \(w^-\). If \(w\) and \(y\) are not two consecutive vertices on \(C_i\), then \(w^-\) and \(y^-\) dominate each other, a contradiction. Thus, every two vertices in \(R \cap V(C_i)\) must be consecutive vertices on \(C_i\). Consequently, \(|R \cap V(C_i)| \leq 3\) and the equality holds only when \(C_i\) is a triangle.

Since \(\sum_{C \in \mathcal{I}} |V(C)| + |H^*| \geq 4k\) and \(|R \cap V(C_i)| \leq 3\) for each \(C_i \in \mathcal{I}\), then \(|U \cup H^*| = |U| + |V(H^*)| \geq k\) follows. Since \(T\) is \(k\)-connected and
\[
\ell = \ell_m \leq \ell_{m-1} \cdots \leq \ell_1,
\]
there exist \(k\) vertex-disjoint paths, \(P_i[x_i, y_i] (i = 1, 2, ..., k)\), such that \(x_i\) is in some cycle in \(\ell\) and \(y_i \in U \cup V(H^*)\) and all internal vertices of the path are in \(R \cup \{u\}\). Furthermore, we can assume that all internal vertices of the path \(P_i[x_i, y_i]\) are in \(R\). Otherwise, suppose that \(v_m \in V(P_i[x_i, y_i])\) for some \(i = 1, ..., k\). Let \(u\) be the predecessor of \(v_m\) on \(P_i[x_i, y_i]\) and \(w\) be the successor of \(v_m\) on \(P_i[x_i, y_i]\). We can suppose that \(u\) is in \(\mathcal{I}\) and \(b\) is in \(H^*\). So the arc \(uw\) belongs to \(T\), and thus \(v_m\) can be omitted in the path \(P_i[x_i, y_i]\).
For each $P_i[x_i, y_i]$, we define a *hop* to be two consecutive vertices $u$ and $u^*$ on $P_i[x_i, y_i]$ such that $u$ and $u^*$ are not consecutive vertices on the same cycle of $\mathcal{C}$. Let $h_i$ be the number of hops on $P_i[x_i, y_i]$. We choose $k$ vertex-disjoint paths $P_i[x_i, y_i]$, $P_2[x_2, y_2], \ldots, P_k[x_k, y_k]$ such that:

1. For each $i$, $x_i \in \bigcup_{C \in \mathcal{C}} V(C)$, $y_i \in U \cup V(H^\ast)$, and all internal vertices are in $R$.
2. Under Condition 1, $\sum_{i=1}^k h_i$ is minimum.
3. Under Conditions 1 and 2, $\sum_{i=1}^k |V(P_i[x_i, y_i])|$ is maximum.

A cycle $C_i \in \mathcal{C}$ is called a *used in-cycle* with respect to $P_i[x_i, y_i], \ldots, P_k[x_k, y_k]$ if $C_i$ contains some vertices in $\bigcup_{j=1}^k V(P_j[x_j, y_j])$, otherwise it is called an *unused in-cycle*. Similarly, a cycle $C_i \in \mathcal{C}$ is called a *used out-cycle* if it contains some vertices in $\{x_1, x_2, \ldots, x_k\}$, otherwise it is called an *unused out-cycle*. All used in-cycles and out-cycles are called *used cycles* and all unused in-cycles and out-cycles are called *unused cycles*.

**Claim 4.** For each used in-cycle $C_j$, $V(C_j) - \bigcup_{i=1}^k V(P_i[x_i, y_i]) \subseteq R$.

**Proof.** Suppose, to the contrary, that there is a vertex $u \in U \cap (V(C_j) - \bigcup_{i=1}^k V(P_i[x_i, y_i]))$. Let $u^*$ be the first vertex in $\bigcup_{i=1}^k V(P_i[x_i, y_i])$ along $C_j$ in the reverse direction from $u$. Suppose that $u^* \in V(P_i[x_i, y_i])$. Let $P_i^* = P_i[x_i, u^*] C_j[u^*, u]$. If $u^* \neq y_i$, the number of hops on $P_i^*$ is less than $h_i$, a contradiction to the minimality of $\sum_{i=1}^k h_i$. If $u^* = y_i$, the number of hops on $P_i^* = h_i$, but $P_i^*$ is longer than $P_i[x_i, y_i]$, a contradiction to the maximality of $\sum_{i=1}^k |V(P_i[x_i, y_i])|$.

For each $i = 1, \ldots, k$, let $C^\ast_i$ be the cycle in $\mathcal{C}$ containing $y_i$ and $C_i^\ast$ be the cycle in $\mathcal{C}$ containing $x_i$. Starting from $x_i$, let $x_i^\ast$ be the first vertex along cycle $C_i^\ast$ in the reverse direction from $x_i$ such that $(x_i^\ast)^- \in \{x_1, x_2, \ldots, x_k\}$. For each $i = 1, 2, \ldots, k$, let

$$Q_i = C_i^\ast(x_i^\ast, x_i) P_i[x_i, y_i].$$

Clearly, all vertices in used out-cycles are in $\bigcup_{i=1}^k V(Q_i[x_i^\ast, y_i])$. By Claim 4, we choose $k$ vertex-disjoint paths $Q_1[x_1, y_1], Q_2[x_2, y_2], \ldots, Q_k[x_k, y_k]$ such that:

1. For each $i = 1, 2, \ldots, k$, $x_i \in \bigcup_{C \in \mathcal{C}} V(C)$, $y_i \in U \cup V(H^\ast)$, and all internal vertices are in $\bigcup_{C \in \mathcal{C}} V(C)$.
2. For each used in-cycle $C_j$, $V(C_j) - \bigcup_{i=1}^k V(Q_i[x_i, y_i]) \subseteq R$.
3. For each used out-cycle $C_j$, $V(C_j) \subseteq \bigcup_{i=1}^k V(Q_i[x_i, y_i])$.
4. Under the above three conditions, $\sum_{i=1}^k |V(Q_i[x_i, y_i])|$ is maximum.
Let $r$ be the number of unused cycles with respect to $Q_1[x_1, y_1]$, $Q_2[x_2, y_2], \ldots, Q_k[x_k, y_k]$. Let $S$ be the set of vertices in used cycles but not in $\bigcup_{i=1}^k V(Q_i[x_i, y_i])$. Then, from Statements 2 and 3 above, $S \subseteq R$.

Note that \( \{y_1, y_2, \ldots, y_k\} \subseteq U \cup V(H^*) \) dominates $\bigcup_{C \in \mathcal{E}} V(C_i)$. In particular, we have $y_i \in V(T)$ for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k$.

If $S = \emptyset$, let

$$C_1^* = Q_1[x_1, y_1] y_1 v_m x_1,$$
$$C_i^* = Q_i[x_i, y_i] \quad \text{for} \quad i = 2, \ldots, k - r - 1,$$

and

$$C_{k-r}^* = Q_{k-r}[x_{k-r}, y_{k-r}, y_{k-r+1}] Q_{k-r+1}[x_{k-r+1}, y_{k-r+1}];$$
$$\cdots$$
$$C_k^* = Q_{k-r}[x_{k-r}, y_{k-r}, y_{k-r+1}] Q_k[x_k, y_k] x_{k-r}.$$

Let $\mathcal{C}^*$ be the set containing the above cycles and all unused cycles. Clearly, $\mathcal{C}^*$ contains exactly $k$ vertex-disjoint cycles, and the union of the vertex sets of these cycles contains all vertices in $\bigcup_{i=1}^k V(C_i)$ and $v_m$, a contradiction to the maximality of $\sum_{i=1}^k |V(C_i)|$.

Thus, we conclude that $S \neq \emptyset$. Let $Q[w_1, w_2] = w_1 w_2 \cdots w_q$ be a hamiltonian path in $T[S]$.

**Claim 5.** $w_1$ dominates $\{y_1, y_2, \ldots, y_k\}$.

**Proof.** Suppose, to the contrary and without loss of generality, that $y_i \not\in E(T)$. Let

$$C_1^* = Q_1^*[x_1, y_1] Q[w_1, w_q] v_m x_1,$$
$$C_i^* = Q_i[x_i, y_i] x_i, \quad \text{for} \quad i = 2, \ldots, k - r - 1,$$

and

$$C_{k-r}^* = Q_{k-r}[x_{k-r}, y_{k-r}, y_{k-r+1}] Q_{k-r+1}[x_{k-r+1}, y_{k-r+1}];$$
$$\cdots$$
$$C_k^* = Q_{k-r}[x_{k-r}, y_{k-r}, y_{k-r+1}] Q_k[x_k, y_k] x_{k-r}.$$

In the same manner as before, these cycles lead to a contradiction of the maximality of $\sum_{i=1}^k |V(C_i)|$.

**Claim 6.** $w_1$ dominates $\bigcup_{i=1}^k V(Q_i[x_i, y_i])$. 
Proof. Suppose, to the contrary, that there is a vertex \( u \in V(Q[x_i, y_i]) \) such that \( uw_1 \in E(T) \). Since \( w_1, y_i \notin E(T) \), there are two consecutive vertices \( u_i \) and \( u_i^+ \) on \( Q[x_i, y_i] \) such that \( uw_1 \in E(T) \) and \( w_1u_i \in E(T) \). Path \( Q[x_i, u_i] w_1 Q[u_i^+, y_i] \) plus the other \( k-1 \) paths contradict the maximality of \( \sum_{i=1}^{k} |V(Q[x_i, y_i])| \).

Since \( w_1 \in R \), there is a vertex \( x \in \bigcup_{C \in S} V(C) \) which dominates \( w_1 \). From Claim 6, \( x \) must be on an unused out-cycle \( C_s \) since \( \bigcup_{i=1}^{k} V(Q[x_i, y_i]) \) contains all vertices in all used out-cycles. Let \( x^+ \) be the successor of \( x \) on \( C_s \). We construct \( k-r+1 \) cycles as follows.

\[
C^*_1 = Q_1[x_1, y_1] C_i[x^+, x] Q[w_1, w_q] v_m v_1, \\
C^* = Q_i[x_i, y_i] x_i, \text{ for } i = 2, ..., k-r, \\
C^*_{k-r+1} = Q_{k-r+1}[x_{k-r+1}, y_{k-r+1}] Q_{k-r+2}[x_{k-r+2}, y_{k-r+2}] \cdots Q_k[x_k, y_k] x_{k-r+1}.
\]

These \( k-r+1 \) cycles and \( r-1 \) remaining unused cycles lead to a contradiction of the maximality of \( \sum_{i=1}^{k} |V(C_i)| \), which completes the proof of Theorem 1.

REFERENCES