On Graph Irregularity Strength

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Received May 1, 2001; Revised April 29, 2002

DOI 10.1002/jgt.10056

Abstract: An assignment of positive integer weights to the edges of a simple graph G is called irregular, if the weighted degrees of the vertices are all different. The irregularity strength, s(G), is the maximal weight, minimized over all irregular assignments. In this study, we show that s(G) ≤ c₁ n/δ, for graphs with maximum degree Δ ≤ n₁/₂ and minimum

Contract grant sponsor: NSF; Contract grant number: CCR-9818411 (to A.F.)
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Let $d$, and $s(G) \leq c_2 (\log n)n/d$, for graphs with $\Delta > n^{1/2}$, where $c_1$ and $c_2$ are explicit constants. To prove the result, we are using a combination of deterministic and probabilistic techniques.

**1. INTRODUCTION**

Perhaps, the second oldest "fact" in graph theory is that in a simple graph, two vertices must have the same degree. This fact no longer holds for multigraphs. By an irregular multigraph, we mean one in which each vertex has a different degree. Hence, a natural question would be: What is the least number of edges we would need to add to a graph in order to convert a simple graph into an irregular multigraph?

Another way to view this question is through an assignment of integer weights to the edges of the graph. Given a simple graph $G$ of order $n$, an assignment $f : E(G) \to \{1, \ldots, w\} = [w]$ of positive integers weights to the edges of $G$ is called irregular if the weighted degrees, $w(G) = \sum_{u \in N(v)} w(uv)$ of the vertices are all different. The irregularity strength, $s(G)$, is the maximal weight $w$, minimized over all irregular weight assignments, and is set to $\infty$, if no such assignment is possible. Clearly, $s(G) < \infty$ if and only if $G$ contains no isolated edges and at most one isolated vertex.

The irregularity strength was introduced by Chartrand et al. [3]. The irregularity strength of regular graphs was considered by Faudree and Lehel [4]. They showed that if $G$ is a $d$-regular graph of order $n$, $d \geq 2$, then $s(G) \leq \left\lceil \frac{n}{2} \right\rceil + 9$, and they conjectured that $s(G) = \left\lceil \frac{n+d-1}{d} \right\rceil + c$ for some constant $c$. This conjecture comes from the lower bound $s(G) \geq \left\lceil \frac{n+d-1}{d} \right\rceil$. For general graphs with finite irregularity strength, Aigner and Triesch [1] showed that $s(G) \leq n - 1$, if $G$ is connected and $s(G) \leq n + 1$ otherwise. Nierhoff [8] refined their method to show $s(G) \leq n - 1$ holds for all graphs with finite irregularity strength, except for $K_3$. We will provide an improvement of both the Faudree–Lehel bound and the Aigner–Triesch–Nierhoff bound in this study.

For a review of other results and open problems in this area, we refer the reader to a survey by Lehel [7].

In this study, all graphs are simple of order $n$. The degree of a vertex $v$ is denoted by $d_v$ or $\deg(v)$, we shall denote the minimum degree of $G$ by $\delta$ and the maximum degree by $\Delta$. For terms not found here, see [2] or [6]. Our upper bounds on $s(G)$ involve a function of $n$ and $\delta$ or both $\delta$ and $\Delta$, and are stated in the next theorem.

**Theorem 1.** Let $G$ be a graph with no isolated vertices or edges.

(a) If $\Delta \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 7n(\frac{1}{2} + \frac{1}{\Delta})$.

(b) If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq \Delta \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 60n/\delta$.

(c) If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$, $\delta \geq 6\log n$ then $s(G) \leq 336(\log n)n/\delta$.

For regular graphs, we get the following theorem with improved constants.
Theorem 2. Let $G$ be a $d$-regular graph with no isolated vertices or edges.

(a) If $d \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 10n/d + 1$.
(b) If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq d \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 48n/d + 1$.
(c) If $d \geq n^{1/2} + 1$, then $s(G) \leq 240(\log n)n/d + 1$.

Observe that both (a) and (b) give bounds of the correct order of magnitude. If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$ and $\delta < 6\ln n$, Theorem 1 does not apply, but we can still make the following statement.

Theorem 3. Let $G$ be a graph with no isolated vertices or edges. If $n$ is sufficiently large, then $s(G) \leq 14n/\delta^{1/2}$.

To explain the main technique used to prove all results, let us define

$$m_g = \max_{X \subseteq V(G)} \{|X| : g(v) = g(u) \text{ for all } v, u \in X\},$$

where $g$ is defined as a weight assignment, i.e., $g : E(G) \rightarrow \{1, 2, \ldots, w\} = [w]$, for some integer $w$. In the deterministic part of our proof (see Lemma 4), we show that $s(G) \leq 3(w + 1)m_g$. Next, we use probabilistic tools to establish bounds on $m_g$. Here the idea is to assign weights to edges from the set $\{1, 2\}$ or $\{1, 2, 3\}$, and show that for such weightings, there exist assignments with $m_g$ of the order $n/\delta$ or $n \log n/\delta$ (see Lemma 7, 8 and 9).

2. DETERMINISTIC LEMMAS

The next two lemmas will be fundamental to our results. Their proofs follow below.

Lemma 4. Let $G$ be a graph without isolated vertices or isolated edges. Let $g : E(G) \rightarrow [w]$ be a weight assignment. Then, there exists an irregular assignment $f : E(G) \rightarrow \{2m_g, \ldots, (3w + 1)m_g\}$.

Lemma 5. Let $G$ be a $d$-regular graph without isolated vertices or isolated edges. Let $g : E(G) \rightarrow [w]$ be a weight assignment. Then, there exists an irregular assignment $f : E(G) \rightarrow [(3w - 1)m_g + 1]$.

We begin with a lemma needed to prove Lemma 4. We will call a tree with at most one vertex of degree greater than two, and $k$ vertices of degree one, a generalized $k$-star.

Lemma 6. Let $G$ be a graph without isolated vertices or isolated edges. Then, $G$ has a factor consisting of generalized stars of order at least three.

Proof. Let $T$ be a spanning tree of a component of $G$. Note that $|V(T)| \geq 3$ by our hypothesis. We show that $T$ can be broken into disjoint generalized stars
that together span \( V(T) \). Then repeating this argument on each component produces the result.

To do this, we induct on \(|U|\), where \( U = \{ u \in V(T) \mid \deg_T(u) \geq 3 \} \). If \(|U| \leq 1\), we are done, as \( T \) is itself a generalized star. Now assume the result holds on any tree \( T\) with \(|U| = l \geq 1\) and suppose \( T\) is a tree with \(|U| = l + 1\). Now root \( T\), at \( u \) and select any vertex \( v \in U\), such that the distance in \( T\) between \( u\) and \( v\) is maximum over all vertices of \( U\). Let \( T_v\) be the subtree of \( T\) rooted at \( v\) and consider \( T' = T \setminus T_v\). This tree has \(|U| = l\) and by the induction hypothesis, we can find generalized stars in \( T'\) that span \( V(T')\). Further, the tree \( T_v\) is, by our choice of \( v\), a generalized star of order at least three. This star, together with the collection of stars that spans \( T'\), spans \( T\), completing the proof. 

\[ \blacksquare \]

**Proof of Lemma 4.** Denote the weight class of a vertex \( v \in V(G)\) as

\[
C_v = \{ u \in V(G) : g(u) = g(v) \}.
\]

Define a new weight function \( \hat{f} : E \rightarrow [3m_gw] \) by \( \hat{f}(e) = 3m_gg(e)\). Note that the weight classes are unchanged under this function. Let \( S\) be a generalized star factor of \( G\), guaranteed by Lemma 6. We select one generalized star \( S\) from \( S\). Let \( u\) be a vertex of maximum degree in \( S\) and suppose that \( S\) consists of \( t\) paths rooted at \( u\). Let \( u_1, u_2, \ldots, u_t\) be the neighbors of \( u\) in \( S\). Consider the first branch (path) of \( S\), say \( v_1, v_2, \ldots, v_r\), where \( v_1 = u_1\) and \( r \geq 2\) (if such a branch of \( S\) exists). Now begin with the last edge \( v_r,v_{r-1}\). We change the weight of this edge as follows. Put \( f(v_r,v_{r-1}) = \hat{f}(v_r,v_{r-1}) + x\), where \( x\) is selected from the set \( L = \{0, -1, \ldots, -(m_g - 1)\} \) in such a way that \( f(v_r)\), its new weighted degree, is different from the current weighted degrees of any vertex from \( C_{v_r} \setminus \{ v_r \}\). Since \(|C_v| \leq m_g\), it is always possible to select an appropriate \( x\). We now repeat this process to the edges \( v_{r-1}, v_{r-2}, v_{r-3}, \ldots, v_2v_1\), thus making \( f'(v_{r-1})\), \( f'(v_{r-2})\), \ldots, \( f'(v_2)\) unique also. To complete the first phase, repeat the procedure on the paths emanating from \( u_2, u_3, \ldots, u_t\), in this order.

It remains to adjust the weights of the star centered at \( u\). So, we change the weights of the edges \( uu_1, uu_2, \ldots, uu_{t-1}\), one by one, starting at \( uu_1\). Let \( f(uu_i) = \hat{f}(uu_i) + y_i\), where \( y_i\) is chosen from the set \( L' = \{ -m_g, -(m_g - 1), \ldots, m_g - 1, m_g\} \) in such a way that \( f(u_i)\), \( i = 1, 2, \ldots, t - 1\), the new weighted degree of \( u_i\), is different from the current weighted degrees of any vertex from \( C_{u_i} \setminus \{ u_i \}\) and, additionally, such that \( \sum_{k=1}^{t} y_i\) belongs to the set \( (L' \cup \{-m_g\}) \setminus \{ f(u_i, v) - f(u_i, v) \} \), where \( v\) is the second vertex of the path starting in \( u_i\)(if no such vertex \( v\) exists, use instead \( L' \cup \{-m_g\} \setminus \{0\})\). Now, we are left with \( uu_i\). Observe that \( u\) and \( u_i\) have different weighted degrees at this time. Now let \( f(uu_i) = \hat{f}(uu_i) + x\), where \( x\in L' \setminus \{-m_g\}\), such that both \( f(u)\) and \( f(u_i)\) are unique in their respective classes. This is possible, since there are \( 2m_g\) options, and \( C_u \) and \( C_w\) can only block \( 2(m_g - 1)\) of these. Finally, repeat the process for all remaining stars \( S\in S\).

Now, for every weight class \( C_u\), all vertices have different weighted degrees under \( f\). The weighted degrees were altered from \( \hat{f}\) by total values from the range \( \{-2m_g + 1, \ldots, m_g\}\), the different classes were at least \( 3m_g\) apart from each
Proof of Lemma 5. Use Lemma 4 to get an irregular weight assignment $f': E(G) \to \{2m_g, 2m_g + 1, \ldots, 3m_gw + m_g\}$. Now define $f: E(G) \to [(3w - 1) m_g + 1]$ by $f'(e) = f'(e) - 2m_g + 1$. This assignment is irregular, since the weighted degree of every vertex is reduced by $d(2m_g - 1)$.

3. PROBABILISTIC LEMMAS

The following two lemmas will be used to get bounds on the irregularity strength of graphs with maximal degree $\Delta \leq n^{1/2}$. Again, the proofs follow below.

Lemma 7. Let $G$ be a graph. If $\Delta \leq (n/\ln n)^{1/4}$, then $\exists g: E(G) \to \{1, 2\}$, such that $m_g \leq \frac{n}{\Delta} + \frac{n}{\Delta^2}$.

Lemma 8. Let $G$ be a graph. If $\Delta \leq n^{1/2}$, then $\exists g: E(G) \to \{1, 2, 3\}$, such that, $m_g \leq 6n/\delta$.

The next lemma is used for graphs with $\Delta > n^{1/2}$.

Lemma 9. Let $G$ be a graph. If $n \geq 10$ and $\delta \geq 10 \log n$, then $\exists g: E(G) \to \{1, 2\}$, such that $m_g \leq 48(\log n)n/\delta$.

Finally, we state the lemma which provides bounds on $m_g$, without any restrictions on vertex degrees of a graph $G$, but for sufficiently large $n$ only.

Lemma 10. Let $G$ be a graph. If $n$ is sufficiently large, then $\exists g: E(G) \to \{1, 2\}$, such that $m_g \leq 2n/\delta^{3/2}$.

Since the proofs of both Lemma 7 and 9 use the same model of assigning weights to the edges, at random, we will present their proof together.

Proof of Lemma 7 and 9. Let $X_v, v \in V$ be independent random variables with uniform distribution over the interval $[0, 1]$, and then for $e = uv \in E$, let

$$g(e) = \begin{cases} 2 & \text{if } X_u + X_v \geq 1, \\ 1 & \text{if } X_u + X_v < 1. \end{cases}$$

For the non-negative integer $y \in \{0, 1, \ldots, d_v\}$,

$$\Pr(g(v) = d_v + y) = \int_{x=0}^{1} \binom{d_v}{y} x^y (1-x)^{d_v-y} dx = \frac{1}{d_v + 1} \leq \frac{1}{\delta + 1}. \quad (1)$$

It follows for every $y$ with $\delta \leq y \leq 2\Delta$ and $Z_y = |\{v \in V: g(v) = y\}|$ that

$$\mathbb{E}(Z_y) \leq \frac{n}{\delta + 1}. \quad (2)$$
To prove Lemma 7, we assume that $G$ is a graph with maximum degree $\Delta \leq (n/\log n)^{1/4}$.

We apply the Hoeffding–Azuma inequality, see, for example, Janson, Łuczak, and Ruciniski [6]. Changing the value of an $X_v$ can only change the value of $Z_y$ by at most $1$. It follows that for $t > 0$,

$$\Pr(Z_y \geq E(Z_y) + t) \leq \exp\left\{-\frac{t^2}{2n(\Delta + 1)^2}\right\}. \tag{3}$$

Putting $t = \frac{n}{\Delta + 1}$ and using (2) we see that

$$\Pr(Z_y \geq E(Z_y) + t) < \frac{1}{2\Delta},$$

and thus

$$\Pr(\exists y: Z_y \geq \frac{n}{\beta} + \frac{n}{\Delta}) < 1,$$

and Lemma 7 follows.  

We now prove Lemma 9. We use the Markov inequality for $t, k > 0$ and any event $\mathcal{E}$, to obtain

$$\Pr(Z_y > t | \mathcal{E}) \leq \frac{E(Z_y k | \mathcal{E})}{\binom{i}{k}}. \tag{4}$$

But

$$E\left(\binom{Z_y}{k} | \mathcal{E}\right) = \sum_{|S| = k} \Pr(g(v) = y, v \in S | \mathcal{E}). \tag{5}$$

Now fix $S = \{v_1, v_2, \ldots, v_k\}$ in Equation (5). For $v \in S$, let $N_S(v) = N(v) \setminus S$, and let $\mu(v) = |N_S(v)|$. Note that $d_v - \mu(v) \leq k - 1$. For $v \in S$, let $\xi_1 < \xi_2 < \cdots < \xi_{d_v}$ be the values of $X_u, u \in N(v)$, sorted in increasing order and let $\eta_1 < \eta_2 < \cdots < \eta_{\mu(v)}$ be the values of $X_u, u \in N_S(v)$, also sorted in increasing order.

Note that, in general, if $\xi_1 < \xi_2 < \cdots < \xi_{d_v}$ is the sequence of order statistics from the uniform distribution over $[0,1]$, then $\xi_i$ has the same distribution as $(Y_1 + Y_2 + \cdots + Y_{i})/(Y_1 + Y_2 + \cdots + Y_{s+1})$ where $Y_1, Y_2, \ldots, Y_{s+1}$ is a sequence of independent random variables, each having exponential distribution with mean one, see for example Ross, Theorem 2.3.1 [9].

To prove the lemma we need to show the following general statement.
Lemma 11. Let $Y_1, Y_2, \ldots, Y_s$ be a sequence of independent random variables, each having exponential distribution with mean one. Then for any real $a > 0$, $0 < b < 1$, we have

\[
\Pr(Y_1 + \cdots + Y_s \geq (1 + a)s) \leq ((1 + a)e^{-a})^s,
\]
\[
\Pr(Y_1 + \cdots + Y_s \leq (1 - b)s) \leq ((1 - b)e^b)^s.
\]

**Proof.**

\[
\Pr(Y_1 + \cdots + Y_s \geq t) \leq \Pr(e^{\lambda(Y_1 + \cdots + Y_s - t)} \geq 1) \\
\leq e^{-\lambda t} \mathbb{E}(e^\lambda(Y_1 + \cdots + Y_s)) \\
= \frac{e^{-\lambda t}}{(1 - \lambda)^s},
\]

provided $\lambda \in (0, 1)$. So putting $t = (1 + a)s$, we see that

\[
\Pr(Y_1 + \cdots + Y_s \geq (1 + a)s) \leq \left(\frac{e^{-\lambda(1+a)}}{1 - \lambda}\right)^s = ((1 + a)e^{-a})^s,
\]
on putting $\lambda = a/(1 + a)$.

A similar argument shows that

\[
\Pr(Y_1 + \cdots + Y_s \leq (1 - b)s) \leq ((1 - b)e^b)^s,
\]
completing the proof of Lemma 11.

Let $k = \lfloor \log n \rfloor$ and

\[
\mathcal{E} = (\Theta < (16 \log n)/\delta),
\]

where

\[
\Theta = \max_{v \in V} \Theta_v, \text{ and } \Theta_v = \max_{0 \leq i \leq d_v - 2k + 1} \xi_{i+2k} - \xi_i.
\]

Here, by default, we take $\xi_0 = 0$ and $\xi_{d_v+1} = 1$.

Now, observe that $g(v) = y$ implies

\[
1 - X_v \in [\xi_{2d_v-y}, \xi_{2d_v-y+1}] \subset [\eta_{2d_v-y-k+1}, \eta_{2d_v-y+1}] \subseteq [\xi_{2d_v-y-k+1}, \xi_{2d_v-y+k}].
\]

In the above formula, we take $\xi_0 = \eta_0 = 0$ for $j \leq 0$, and $\xi_{d_v+j} = \eta_{\mu(v)+j} = 1$ for $j \geq 1$. 

126 JOURNAL OF GRAPH THEORY
Applying Lemma 11 to the order statistics defining $\Theta$, we see that
\[
\Pr(\neg \mathcal{E}) = \Pr\left(\exists v \in V : \Theta_v \geq \frac{16 \log n}{\delta}\right)
\]
\[
\leq n \Pr\left(\exists 0 \leq i \leq \Delta - 2k + 1 : \frac{Y_i + \cdots + Y_{i+2k-1}}{Y\_{i+1} + \cdots + Y_{\delta+1}} \geq \frac{16 \log n}{\delta}\right)
\]
\[
\leq n \Pr(Y_1 + \cdots + Y_{\delta+1} \leq \delta/2) + n^2 \Pr(Y_1 + \cdots + Y_{2k} \geq 8k)
\]
\[
\leq n(e^{1/2}/2)^{\delta+1} + n^2(4e^{-3})^{2k}
\]
\[
\leq 1/10.
\]
Further,
\[
\Pr(g(v) = y, v \in S| \mathcal{E}) \leq \Pr(1 - X_{v_i} \in [\eta_{2d_{v_i} - y - k+1}, \eta_{2d_{v_i} - y +1}], i = 1, 2, \ldots, k| \mathcal{E})
\]
\[
\leq 2 \Pr\left(1 - X_{v_i} \in \left[\eta_{2d_{v_i} - y - k+1}, \eta_{2d_{v_i} - y - k+1} + \frac{16 \log n}{\delta}\right], i = 1, 2, \ldots, k\right)
\]
\[
\leq 2\left(\frac{16 \log n}{\delta}\right)^k.
\]
From Equation (4) and (5) we obtain
\[
\Pr(\exists y : Z_y > t| \mathcal{E}) \leq 2n\binom{t}{k}^{-1}n\binom{16 \log n}{\delta}^k.
\]
Putting $t = 48(\log n)n\delta^{-1}$ together with Equation (6) establishes
\[
\Pr(\exists y : Z_y > t) \leq \Pr(\exists y : Z_y > t| \mathcal{E}) + \Pr(\neg \mathcal{E}) < 1,
\]
proving Lemma 9. \hfill \blacksquare

**Proof of Lemma 8.** For every vertex $v$ independently assign a number $W_v$ from $\{0, \ldots, d_v\}$ uniformly at random. Now pick a random subset $N \subseteq N(v)$ of size $W_v$, and for every $u \in N$, set $v_u = 1$, and for every $u \in N(v) \setminus N$, set $v_u = 0$.

Let $g : E \to [3]$ as follows: For $uv \in E$, let $g(uv) = 1 + v_u + u_v$. For a vertex $v$, let $g(v) = \sum_{u \in N(v)} g(uv)$. For some integer $y$ with $\delta \leq y \leq 3\Delta$, let $Z_y = |\{v \in V : g(v) = y\}|$. Then
\[
E(Z_y) \leq \frac{n}{\delta},
\]
(7)
since
\[
\Pr(g(v) = y) = \Pr\left(W_v = y - d - \sum_{u \in N(v)} u_v\right) \leq \frac{1}{d_v + 1}.
\]
By the symmetry of the construction, we know that $\forall x \in V, v, u \in N(x)$:

\[
\begin{align*}
\Pr(x_v = 1) &= 1/2, \\
\Pr(x_v = x_u = 1) &= \Pr(x_v = x_u = 0) = 1/3, \\
\Pr(x_v = 1, x_u = 0) &= \Pr(x_v = 0, x_u = 1) = 1/6.
\end{align*}
\]

To use Chebyshev’s inequality, we have to bound the variance of $Z_y$:

\[
\Var(Z_y) = \sum_{v \in V} \sum_{u \in V} (\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y)).
\]

Fix a $v \in V$, and consider

\[
S_v = \sum_{u \in V} (\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y)).
\]

Divide $V$ into three classes $V_1, V_2, V_3$, and consider the partial sums

\[
S_i = \sum_{u \in V_i} (\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y)).
\]

**Class 1**

$V_1 = \{ v \}$.

\[
S_1 \leq \Pr(g(v) = y) \leq \frac{1}{d_v} \leq \frac{\Delta}{\delta^2}.
\]

**Class 2**

$V_2 = N(v)$.

\[
S_2 \leq d_v \Pr(g(v) = g(u) = y)
\]

\[
\leq d_v \Pr \left( W_v = y - d_v - \sum_{x \in N(v)} x_v | g(u) = y \right) \Pr \left( W_u = y - d_u - \sum_{x \in N(u)} x_u \right)
\]

\[
\leq d_v \frac{2}{(d_v + 1)(d_u + 1)} \leq \frac{2\Delta}{\delta^2}.
\]

**Class 3**

$V_3 = V \setminus (\{v\} \cup N(v))$.

Let $u \in V_3$, and let $c = |N(v) \cap N(u)|$. For the sake of the analysis, pick a random subset $A$ from $\{x \in N(u) \cap N(v) : x_u = x_v\}$, by choosing each
vertex with probability 1/2. So, using Equation (8), for every vertex \( x \in N(u) \cap N(v) \),

\[
\Pr(x_u = x_v = 1 \land x \in \mathcal{A}) = \Pr(x_u = x_v = 1 \land x \notin \mathcal{A})
\]

\[
= \Pr(x_u = x_v = 0 \land x \in \mathcal{A}) = \Pr(x_u = x_v = 0 \land x \notin \mathcal{A})
\]

\[
= \Pr(x_u = 0 \land x_v = 1) = \Pr(x_u = 1 \land x_v = 0) = 1/6,
\]

and

\[
\Pr(x \in \mathcal{A}) = 1/3.
\]

Let \( A \subseteq N(u) \cap N(v) \), and let \( a = |A| \).

Then, for every vertex \( x \in N(u) \cap N(v) \),

\[
\Pr(x_u = x_v = 1 \mid A = A \land x \notin A) = \frac{\Pr(x_u = x_v = 1 \land A = A \mid x \notin A)}{\Pr(A = A \mid x \notin A)}
\]

\[
= \frac{(1/6)(1/3)^a(2/3)^{c-a-1}}{1/3)^a(2/3)^{c-a}} = \frac{1}{4}.
\]

By symmetry, we get

\[
\Pr(x_u = x_v = 0 \mid A = A) = \Pr(x_u = 0, x_v = 1 \mid A = A)
\]

\[
= \Pr(x_u = 1, x_v = 0 \mid A = A) = 1/4.
\]

Thus, given \( x \notin A \) and \( A = A \), the events \((x_u = 1)\) and \((x_v = 1)\) are independent.

For \( x \in A \), we get

\[
\Pr(x_u = x_v = 1 \mid A = A \land x \in A) = \Pr(x_u = x_v = 0 \mid A = A \land x \in A) = 1/2.
\]

We introduce the following notation:

\[
P_A = \Pr(g(v) = g(w) = y \mid A = A) - \Pr(g(v) = y \mid A = A)\Pr(g(w) = y \mid A = A)
\]

\[
= \Pr(g(v) = g(w) = y \mid A = A) - \Pr(g(v) = y)\Pr(g(w) = y),
\]

since \( \Pr(g(v) = y) \) is independent from the choice of \( A \). In particular,

\[
P_\emptyset = \Pr(g(v) = g(w) = y \mid A = \emptyset) - \Pr(g(v) = y)\Pr(g(w) = y) = 0. \quad (11)
\]

For \( A \neq \emptyset \), pick any \( x \in A \). We want to bound the difference \( P_A - P_{A \setminus x} \). Let

\[
b_v = d_v + \sum_{z \in N(v) \setminus x} z_v, \quad b_u = d_u + \sum_{z \in N(u) \setminus x} z_u.
\]
Now consider the difference between \( P_A \) and \( P_{A \setminus x} \), given that \( b_v = l \) and \( b_u = r \), and denote it by

\[
P_{p}^{l,r} - P_{p}^{l,r} = \]

\[
= \Pr(g(v) = g(w) = y | A = A \land b_v = l \land b_u = r) - \Pr(g(v) = g(w) = y | A \setminus x = A \setminus x \land b_v = l \land b_u = r)
\]

\[
= [\Pr(x_u = x_v = 1 | A = A) - \Pr(x_u = x_v = 0 | A = A \setminus x)] \times \Pr(W_v = y - l - 1) \Pr(W_u = y - r - 1)
\]

\[
+ [\Pr(x_u = x_v = 0 | A = A) - \Pr(x_u = x_v = 0 | A = A \setminus x)] \times \Pr(W_v = y - l) \Pr(W_u = y - r)
\]

\[
+ [\Pr(x_u = 1 \land x_v = 0 | A = A) - \Pr(x_u = 1 \land x_v = 0 | A = A \setminus x)] \times \Pr(W_v = y - l) \Pr(W_u = y - r - 1)
\]

\[
+ [\Pr(x_u = 0 \land x_v = 1 | A = A) - \Pr(x_u = 0 \land x_v = 1 | A = A \setminus x)] \times \Pr(W_v = y - l - 1) \Pr(W_u = y - r)
\]

\[
= \frac{1}{4} [\Pr(W_v = y - l - 1) \Pr(W_u = y - r - 1) + \Pr(W_v = y - l) \Pr(W_u = y - r)
\]

\[
- \Pr(W_v = y - l) \Pr(W_u = y - r - 1) - \Pr(W_v = y - l - 1) \Pr(W_u = y - r)].
\]

Therefore,

\[
P_{p}^{l,r} - P_{p}^{l,r} = \begin{cases} 
1/[4(d_v + 1)(d_u + 1)] & \text{if } (r = y - d_u - 1 \land l = y - d_v - 1), \\
-1/[4(d_v + 1)(d_u + 1)] & \text{if } (r = y - d_u - 1 \land l = y), \\
0 & \text{otherwise.}
\end{cases}
\]

Thus, summing over all possible values of \( l, r \) and \( t = |\{z \in A \setminus x : z_u = z_v = 1\}|\),

\[
P_A - P_{A \setminus x} \leq \frac{1}{4(d_v + 1)(d_u + 1)}
\]

\[
\times [\Pr(b_u = y - d_u - 1 \land b_v = y - d_v - 1) + \Pr(b_u = y \land b_v = y)]
\]

\[
\leq \frac{1}{4(d_v + 1)(d_u + 1)}
\]

\[
\times \left[ \sum_{t=0}^{a-1} \left( \frac{a - 1}{t} \right) 2^{-a+1} \left( \frac{d_u - a}{y - 2d_u - 1 - t} \right) \left( \frac{d_v - a}{y - 2d_v - 1 - t} \right) 2^{-d_u - d_v + 2a} \\
\right.
\]

\[
\left. + \sum_{t=0}^{a-1} \left( \frac{a - 1}{t} \right) 2^{-a+1} \left( \frac{d_u - a}{y - d_u - t} \right) \left( \frac{d_v - a}{y - d_v - t} \right) 2^{-d_u - d_v + 2a} \right]
\]

\[
\leq \frac{1}{(d_v + 1)(d_u + 1)} \left( \frac{d_u - a}{(d_u - a)/2} \right) \left( \frac{d_v - a}{(d_v - a)/2} \right) 2^{-d_u - d_v + a} \sum_{t=0}^{a-1} \left( \frac{a - 1}{t} \right).
\]
Suppose first that $1 \leq a \leq \delta/3$. Then,

$$P_A - P_A \leq \frac{2^{d_v-a+1}}{(d_v+1)(d_u+1)} \left( \frac{2^{d_v-a+1}}{(d_v-a)^{1/2}} \right) \left( \frac{2^{d_v-a+1}}{(d_u-a)^{1/2}} \right) \leq \frac{3}{d_v\delta^2}.$$ 

Hence,

$$P_A \leq \frac{3a}{d_v\delta^2} \leq \frac{3c}{d_v\delta^2}. \tag{12}$$

Note that for all $A$,

$$\Pr(g(v) = g(u) = y | A = A) \leq \frac{1}{(d_v+1)(d_u+1)},$$

hence, for $a > \delta/3$,

$$P_A \leq \Pr(g(v) = g(u) = y | A = A) \leq \frac{3a}{d_v\delta^2} \leq \frac{3c}{d_v\delta^2}. \tag{13}$$

Therefore, combining (11), (12) and (13),

$$\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y) \leq \sum_{A \subseteq N(u) \cap N(v)} (3c/d_v\delta^2)\Pr(A = A) = \frac{3|N(v) \cap N(u)|}{d_v\delta^2}. $$

Now notice that $\sum_{u \in V} |N(v) \cap N(u)|$ counts the number of walks of length two starting in $v$, thus $\sum_{u \in V} |N(v) \cap N(u)| \leq d_v\Delta$, and therefore,

$$S_3 \leq \sum_{u \in V} \frac{3|N(v) \cap N(u)|}{d_v\delta^2} \leq \frac{3\Delta}{\delta^2}. \tag{14}$$

Altogether, we get from (9), (10), and (14),

$$S_v = S_1 + S_2 + S_3 \leq \frac{6\Delta}{\delta^2},$$

and thus,

$$\text{Var}(Z_y) = \sum_{v \in V} S_v \leq \frac{6n\Delta}{\delta^2}.$$
By Chebyshev’s inequality and (7) we get
\[ \Pr(Z_y > 6n/\delta) \leq \frac{\text{Var}(Z_y)}{(6n/\delta)^2} < \frac{1}{3\Delta}, \]
and thus,
\[ \Pr(\exists y : Z_y > 6n/\delta) < 1, \]
finishing the proof.

**Proof of Lemma 10.** Choose \( g \) randomly from \( \{1, 2\}^E \). Observe that \( g(v) - d_v \) has the binomial distribution \( \text{Bi}(d_v, 1/2) \). For a non-negative integer \( y \) let
\[ V_y = \left\{ v : \frac{y}{2} \leq d_v \leq (2d_v \log n)^{1/2} \right\}. \]
The Chernoff bounds for the tails of the binomial (see, e.g., [6]) imply that for any \( t > 0 \),
\[ \Pr(|g(v) - \frac{3}{2}d_v| \geq t) \leq e^{-2t^2/d_v}. \]
Hence,
\[ \Pr(g(v) = y) \leq \frac{1}{n^4} \quad \text{if} \quad v \notin V_y. \tag{15} \]
Now consider \( v \in V_y \). Clearly,
\[ \Pr(g(v) = y) = 0 \quad \text{if} \quad d_v < y/2. \tag{16} \]

**Case 1** \( y \geq n^{1/4} \)

If \( d_v \geq y/2 \geq n^{1/4}/2 \), then we can use Stirling’s inequality or apply Feller [5], Chapter VII (2.7) to get
\[ \Pr(g(v) = y) = \frac{1}{2d_v} \left( \frac{d_v}{y} \right) \approx \sqrt{\frac{2}{\pi d_v}} e^{-z^2/2}, \tag{17} \]
where \( z = 2(y - \frac{3}{2}d_v)/d_v^{1/2} \).

Let \( Z_y = |\{v : g(v) = y\}| \). It follows from (15), (16), and (17) that
\[ E(Z_y) \leq \frac{|V_y|}{\delta^{1/2}}. \tag{18} \]
Let
\[ Z_1^y = |\{ v \in V_y : g(v) = y \}| \quad \text{and} \quad Z_2^y = |\{ v \notin V_y : g(v) = y \}|. \]

It follows from (15) that
\[ \Pr\left( Z_2^y \neq 0 \right) \leq \frac{1}{n^3}. \] 

(19)

Note also that \( v \in V_y \) implies that
\[ y = \frac{3}{2} d_v + O\left( (d_v \log n)^{1/2} \right). \] 

(20)

Now for \( t > 0 \) and \( k = (\log n)^2 \), we use the Markov’s inequality to obtain
\[ \Pr(Z_1^y > t) \leq \frac{\mathbb{E}\left( \left( Z_1^y \right)^k \right)}{t^k}. \] 

(21)

But
\[ \mathbb{E}\left( \left( Z_1^y \right)^k \right) = \sum_{S \subseteq V_y, |S| = k} \Pr(g(v) = y, v \in S) \]
\[ = \sum_{S \subseteq V_y, |S| = k} \sum_{\xi \in \{1,2\}^S} \Pr(g(v) = y, v \in S \mid g(E_S) = \xi) \Pr(g(E_S) = \xi) \]
\[ \] 

(22)

where \( E_S = \{ e \in E : e \subseteq S \} \).

Now fix \( S \) in Inequality (22). For \( v \in S \), let
\[ A_v = \{ e = uv \in E : u \notin S \} \text{ and } B_v = \{ e = uv \in E : u \in S \}. \]

Then, if \( |g(B_v)| \) denotes \( \sum_{u \in B_v} g(u) \),
\[ \Pr(g(v) = y \mid g(E_S) = \xi) = \Pr(|g(A_v)| = y - |g(B_v)|) \]
\[ = 2^{-|A_v|} \left( y - |g(B_v)| - |A_v| \right). \] 

(23)
Therefore,

\[
\frac{\Pr(|g(A_v)| = y - |g(B_v)|)}{\Pr(g(v) = y)} = 2^{2d_v}(y - |g(B_v)| - |A_v|) \left( \frac{|A_v|}{d_v} \right)
\]

\[
= 2^{2d_v}\frac{|A_v|(|A_v| - 1) \cdots (2|A_v| - y - 1)}{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)}
\]

\[
= \frac{1 \times 2 \times \cdots \times (y - d_v)}{d_v(d_v - 1) \cdots (2d_v - y + 1)}
\]

\[
\Pr(g(v) = y) = 1 + O\left(k\left(\frac{\log n}{d_v}\right)^{1/2}\right)
\]

(24)

Now we use

\[|A_v| + |B_v| = d_v \quad \text{and} \quad |B_v| \leq |g(B_v)| \leq 2|B_v| \leq 2k\]

and Equation (20) to verify that

\[
\frac{1 \times 2 \times \cdots \times (y - d_v)}{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)}
\]

\[
= (y - d_v)(y - d_v - 1) \cdots (y - |g(B_v)| - |A_v| + 1)
\]

\[
= \left(\frac{1}{2d_v}\right)^{|g(B_v)|-|B_v|} \left(1 + O\left(k\left(\frac{\log n}{d_v}\right)^{1/2}\right)\right)
\]

(25)

and

\[
\frac{|A_v|(|A_v| - 1) \cdots (2|A_v| + |g(|B_v|)| - y + 1)}{d_v(d_v - 1) \cdots (2d_v - y + 1)}
\]

\[
= (2d_v - y)(2d_v - y - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)
\]

\[
= d_v^{d_v - |g(B_v)|} \times 2^{2|g(B_v)| - 2|B_v|} \left(1 + O\left(k\left(\frac{\log n}{d_v}\right)^{1/2}\right)\right).
\]

(26)

Plugging Equation (25) and (26) into (24), we see that

\[
\frac{\Pr(|g(A_v)| = y - |g(B_v)|)}{\Pr(g(v) = y)} = 1 + O\left(k\left(\frac{\log n}{d_v}\right)^{1/2}\right).
\]
So from Equation (22) and (23) we see that

\[
\mathbb{E}\left(\left(\frac{Z^1_Y}{k}\right)\right)
\leq \sum_{s \subseteq V_y, |s| = k} \sum_{\xi \in \{1, 2\}^E_s} \prod_{v \in s} \left(1 + O\left(k \left(\frac{\log n}{d_v}\right)^{1/2}\right)\right) \Pr(g(v) = y) \Pr(g(E_s) = \xi)
\leq \left(1 + O\left(k^2 \left(\frac{\log n}{n^{1/8}}\right)^{1/2}\right)\right) \sum_{s \subseteq V_y, |s| = k} \prod_{v \in s} \Pr(g(v) = y)
\leq (1 + o(1)) \frac{\mathbb{E}(Z^1_Y)^k}{k!}
= (1 + o(1)) \frac{(2n/\delta^{1/2})^k}{k!}.
\]

So Inequality (18), (21) imply

\[
\Pr\left(Z^1_Y > 2 \frac{n}{\delta^{1/2}}\right) \leq (1 + o(1)) \frac{\mathbb{E}(Z^1_Y)^k}{(2n/\delta^{1/2})^k} \leq (1 + o(1))2^{-k}
\]

and then together with Inequality (19) we get

\[
\Pr\left(\exists y: Z_y > 2 \frac{n}{\delta^{1/2}}\right) \leq 2n((1 + o(1))2^{-k} + n^{-3}) = o(1).
\]

**Case 2** \(y \leq n^{1/4}\).

Assume that \(V_y \neq \emptyset\). We apply the Hoeffding–Azuma inequality. Changing the value of \(g\) on a single edge can only change the value of \(Z^1_Y\) by at most 2. Also, \(Z^1_Y\) is determined by the outcome of at most

\[
\sum_{v \in V_y} d_v \leq |V_y|(y + (\log n)^2)
\]

random choices. It follows that for \(t > 0\),

\[
\Pr\left(Z^1_Y \geq \mathbb{E}(Z^1_Y) + t\right) \leq \exp\left\{-\frac{t^2}{2|V_y|(y + (\log n)^2)}\right\}.
\]

(28)
Putting $t = n/\delta^{1/2}$ and observing that $V_y \neq \emptyset$ implies $\delta \leq n^{1/4}$ and $y\delta \leq n^{1/2}$, and applying Inequality (18), (19), (28), we see that

$$\Pr \left( Z^1_y > \frac{n}{\delta^{1/2}} \right) \leq e^{-n^{1/2}/3}. \quad (29)$$

The lemma follows from Inequality (19), (27), and (29).

\section*{4. PROOFS OF THEOREMS}

We are now able to prove the Theorems.

\textbf{Proof of Theorem 1.} Let $\Delta \leq n^{1/2}$. By Lemma 8, there exists a weight assignment \( g : E \to [w] \) with $m_g \leq 6n/\delta$ and $w = 3$. Now by Lemma 4, $s(G) \leq 3m_g + m_g \leq 60n/\delta$, proving (b). Similar arguments, using Lemma 7 and Lemma 9 in place of Lemma 8, provide part (a) and (c).

\textbf{Proof of Theorem 2.} The proof is similar to the proof of Theorem 1, just use Lemma 5 in place of Lemma 4.

\textbf{Proof of Theorem 3.} The proof is similar to the proof of Theorem 1, just use Lemma 4 and Lemma 10.

\section*{ACKNOWLEDGMENT}

This research was supported in part by NSF grant CCR-9818411 (to A.F.).

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