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Pancyclicity in claw-free graphs

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Abstract

In this paper, we present several conditions for $K_{1,3}$-free graphs, which guarantee the graph is subpancyclic. In particular, we show that every $K_{1,3}$-free graph with a minimum degree sum $\delta_2 > 2\sqrt{3n+1} - 4$; every $\{K_{1,3}, P_7\}$-free graph with $\delta_2 \geqslant 9$; every $\{K_{1,3}, Z_4\}$-free graph with $\delta_2 \geqslant 9$; and every $K_{1,3}$-free graph with maximum degree Δ, $\mathrm{diam}(G) < (\Delta + 6)/4$ and $\delta_2 \geqslant 9$ is subpancyclic.

Keywords: Claw-free; Pancyclicity; Forbidden subgraphs

1. Introduction

If not specified otherwise, we will use notation from [1]. We consider finite simple graphs only. A graph on n vertices is called *subpancyclic* if it contains cycles of every length l with $3 \leqslant l \leqslant c(G)$, where $c(G)$ denotes the circumference of G. If G is subpancyclic and hamiltonian, it is called *pancyclic*.

We will always denote the edge set of the graph G by E, and V will denote its vertex set. For some graph H, a graph is said to be *H-free*, if it does not contain an induced copy of H. The complete bipartite graph $K_{1,3}$ is also called the *claw*. The graph Z_4 is a triangle with a path of length four attached to one of its vertices, the graph P_7 is the path on seven vertices.

The degree of a vertex v is denoted by $d(v)$. We will write $\Delta(G)$ or (if no confusion arises) Δ for the maximum degree in G, and $\delta(G)$ or δ for the minimum degree in G. By $\delta_2(G)$ or δ_2, we will denote the minimum of $\{d(u) + d(v) \mid u, v \in V, uv \notin E\}$.

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Let C be a cycle in G, and assign some orientation to C. For two vertices $x, y \in V(C)$, the notation xCy will stand for the path from x to y along C following the orientation of C. An xy-path P in G is called a *shortening path of* C, if $V(P) \cap V(C) = \{x, y\}$ and $|P| < \min\{|xCy|, |yCx|\}$. An edge $xy \notin E(C)$ with $x, y \in V(C)$ is called a *chord of* C.

We will start by proving the following lemma.

Lemma 1. *Let G be a claw-free graph with $\delta_2(G) \geqslant 9$. Suppose, for some $m > 3$, G has an m-cycle C, but no $(m-1)$-cycle. Then there is no shortening path of C.*

As we will see, Lemma 1 has several interesting consequences.

Theorem 2. *Let G be a claw-free graph with maximum degree Δ and $\delta_2(G) \geqslant 9$. If* $\mathrm{diam}(G) < (\Delta + 6)/4$, *then G is subpancyclic.*

Theorem 3. *Let G be a claw-free graph with minimum degree δ and $\delta_2(G) \geqslant 9$. If G is not a line graph, and* $\mathrm{diam}(G) < (\delta + 3)/2$, *then G is subpancyclic.*

Theorem 4. *Let G be a $\{K_{1,3}, Z_4\}$-free graph with $\delta_2 \geqslant 9$. Then G is subpancyclic. If G is 2-connected, then G is pancyclic.*

Theorem 5. *Let G be a $\{K_{1,3}, P_7\}$-free graph with $\delta_2 \geqslant 9$. Then G is subpancyclic.*

Theorem 6. *Let G be a claw-free graph on $n \geqslant 5$ vertices with $\delta_2 > 2\sqrt{3n+1} - 4$. Then G is subpancyclic.*

From Theorem 6, we obtain as a corollary the following theorem of Trommel et al. [4]:

Theorem 7. *Let G be a claw-free graph on $n \geqslant 5$ vertices. If the minimum degree δ is $\delta > \sqrt{3n+1} - 2$, then G is subpancyclic.*

In the proofs of Theorems 2–6, we will frequently use the following theorem from Flandrin et al. [3], and its corollaries:

Theorem 8. *Let G be a claw-free graph. Then the graph $\langle N \rangle$ induced by the neighborhood N of any vertex x falls in one of the three cases:*

1. *$\langle N \rangle$ is hamiltonian.*
2. *$\langle N \rangle$ consists of two complete subgraphs G_1 and G_2, connected with some edges, all of them having a common vertex in G_1.*
3. *$\langle N \rangle$ consists of two complete subgraphs with no edges in between.*

Corollary 9. *Let G be a claw-free graph with maximum degree Δ. Then G contains cycles of length l for all l with $3 \leqslant l \leqslant \lceil \Delta/2 \rceil + 1$.*

Proof. The proof is obvious. □

Corollary 10. *Let G be a claw-free graph with minimum degree δ. If G is not a line graph, then G contains cycles of length l for all l with* $3 \leqslant l \leqslant \delta + 1$.

Proof. Observe that G is a line graph if the neighborhoods of all vertices are in the third class of Theorem 8. Therefore, there is a vertex x with $\langle N(x) \rangle$ in the first or second class of Theorem 8. In either case, $\langle N(x) \rangle$ is traceable, implying $\langle N(x) \cup \{x\} \rangle$ is pancyclic. □

2. Proof of Lemma 1

Suppose instead P is the shortest shortening path. We will distinguish two cases.

Case 1: Suppose P is a chord ($P = xy$).

Pick two chords u_1u_2 and v_1v_2, such that $u_1, u_2 \in xCy, v_1, v_2 \in yCx$, where both chords are minimal in the sense that there is no other chord uv with $u, v \in u_1Cu_2$ or $u, v \in v_1Cv_2$. This does not exclude the possibility of one or both of these chords being identical with xy.

Let $K := \{v \in V(C) \mid \exists u \in V(C)\colon\ uv \text{ is a chord}\}$, $L := V(C) - K$. If there is a shortening path of C with length exactly two with both its endvertices in u_1Cu_2 (v_1Cv_2), pick such a shortening path $s_1s_2s_3$ ($t_1t_2t_3$), such that s_1Cs_3 (t_1Ct_3) is as short as possible, else set $s_1 = s_2 = u_1, s_3 = u_2$ ($t_1 = t_2 = v_1, t_3 = v_2$).

Let $a_1, a_2, \ldots, a_r$ be the vertices of $s_1^+Cs_3^- \cap L$ (in order), let $b_1, b_2, \ldots, b_l$ be the vertices of $t_1^+Ct_3^- \cap L$. Without loss of generality, by symmetry, we may assume that $l \geqslant r$. Further, if $l = r$ we may assume that $d(b_i) \geqslant 5$ for all $1 \leqslant i \leqslant l$ (since they belong to L, there are no edges between the a_i and the b_j, so $\delta_2(G) \geqslant 9$ guarantees the statement).

Now we will construct a cycle $C' \subset \langle C \cup s_2 \rangle$ with $m - r - 1 \leqslant |C'| \leqslant m - 1$, which we will then extend to a C_{m-1} to get a contradiction.

Start with the cycle $s_1s_2s_3Cs_1$. Note that $c = |s_1s_2s_3Cs_1| \leqslant m - 1$. If $c \geqslant m - r - 1$, this cycle is the desired C'. Otherwise, $s_1^+Cs_3^- \cap K \neq \emptyset$ and we can pick a vertex $u \in s_1^+Cs_3^- \cap K$. Then u has an edge to some vertex $v \in s_3^+Cs_1^-$. There cannot be an edge v^-v^+, else there is a C_{m-1}. There is no claw centered at v, so $v^+u \in E$ or $v^{-u} \in E$. Therefore, u can be inserted in the cycle between v and one of its neighbors to extend the cycle. If two vertices $u, w \in s_1^+Cs_3^- \cap K$ share the same neighbors $v, v^+ \in s_3^+Cs_1^-$, then all of uCw (or wCu) can be inserted between v and v^+ to extend the cycle. Thus, any number of vertices in $s_1^+Cs_3^- \cap K$ can be inserted (we do not have control about the number of vertices of $s_1^+Cs_3^- \cap L$ inserted in the process). With this process, we insert $m - r - 1 - c$ vertices out of $s_1^+Cs_3^- \cap K$. The resulting cycle C' is of the desired length, since at most r vertices out of L were inserted.

To extend C', consider $b_1, b_4, b_7, \ldots, b_{3\lceil l/3 \rceil - 2}$. Since t_1Ct_3 is the shortest such segment possible, these vertices have pairwise disjoint neighborhoods. Further, none of them is a neighbor of s_2, else there is a claw at s_2. By Theorem 8, C' can be extended through the neighborhoods of these vertices by any number of vertices up to $d(b_i) - 2$ for each b_i, $i = 1, 4, \ldots$.

If $l = r$, then $d(b_i) \geqslant 5$, so this extends C' by up to $3\lceil l/3 \rceil \geqslant r$ vertices, resulting in a C_{m-1}.

If $3 \leqslant r < l$, let $d := \min\{4, d(b_1), d(b_4), \ldots\}$. Then C' is extendable by $\sum_{i=0}^{\lceil l/3 \rceil - 1}(d(b_{1+3i}) - 2) \geqslant d - 2 + (\lceil l/3 \rceil - 1)(7 - d) \geqslant 3\lceil l/3 \rceil - 1$ vertices, yielding a C_{m-1}.

If $1 \leqslant r < l = 3$, consider b_1 and b_3. One of them has degree at least 5, so we can extend by up to 3 vertices, which is again enough.

If $r = 1, l = 2$, the only problem would be if $d(b_1) = d(b_2) = 2$, else we could extend by one, which is enough. But then, $d(a_1) \geqslant 7$, and by a symmetric argument we can find a cycle $C'' \subset \langle C \cup t_2 \rangle$ which includes $a_1^- a_1 a_1^+$, and $m - 3 \leqslant |C''| \leqslant m - 1$. This cycle can now be extended around a_1 to a C_{m-1}.

Finally, if $r = 0$, C' is already a C_{m-1}. This contradiction concludes the argument, hence C has no chords.

Case 2: Suppose P has length $\geqslant 2$ ($P = z_0 z_1 z_2 \cdots z_l$, with $x = z_0, y = z_l$).

Assume, that P is chosen such that $k = |xCy|$ is minimal. Observe, that $k > l - 1$ (else $k = l - 1$, and a C_{m-1} is easily found). Let $v_0 = y, v_1 = y^+, \ldots, v_{m-k+1} = x$. Since k is minimal, $x^+ z_1 \notin E$. Since C is chordless, $x^+ v_{m-k} \notin E$. Thus, $v_{m-k} z_1 \in E$ to prevent a claw at x. A symmetric argument shows that $v_1 z_{l-1} \in E$. Now $m - k \geqslant k$, else k would not have been minimal.

Consider $C' = xPyCx$. We know that $m - k + l + 1 = |C'| \leqslant m - 2$. We will now extend C' to a C_{m-1} to get the contradiction. None of the edges $v_i z_j, 2 \leqslant i \leqslant m - k - 1, 0 \leqslant j \leqslant l$ exists, else let j be minimal, such that for some $2 \leqslant i \leqslant m - k - 1$, there is an edge $v_i z_j$ ($j \geqslant 1$, else chord). To prevent a claw at z_j, $z_{j+1} v_i \in E$ is necessary. But now, consider the paths $P' = v_i z_{j+1} P y$ and $P'' = v_i z_j P x$. Both of them are shorter than P. Since P is the shortest shortening path, P' and P'' cannot be shortening paths, thus $1 + l - j = |P'| \geqslant |yCv_i| = i + 1$, and $j + 2 = |P''| \geqslant |v_i C x| = m - k - i + 2$. But this implies that $l \geqslant m - k \geqslant k$, a contradiction to P being a shortening path.

Now, note that none of the neighborhoods of $v_2, v_5, \ldots, v_{3\lfloor k/3 \rfloor - 1}$ intersect, else k was not minimal.

If $k \geqslant 6$, let $d := \min\{4, d(v_2), d(v_5), \ldots, d(v_{3\lfloor k/3 \rfloor - 1})\}$. We can extend C' around $v_2, v_5, \ldots, v_{3\lfloor k/3 \rfloor - 1}$ by up to $\sum_{i=0}^{\lfloor k/3 \rfloor - 1}(d(v_{2+3i}) - 2) \geqslant d - 2 + (\lfloor l/3 \rfloor - 1)(7 - d) \geqslant 3\lfloor k/3 \rfloor - 1 \geqslant k - 3$ vertices to get a C_{m-1}, just like in the first case.

If $k = 5$, $|C'| = m + l - 4 \geqslant m - 2$. As either $d(v_2) \geqslant 5$ or $d(v_4) \geqslant 5$, we can extend it around by one vertex, and we have our contradiction.

If $k \leqslant 4$, then $m - 2 \geqslant |C'| \geqslant m - k + l + 1 \geqslant m - 4 + 2 + 1 = m - 1$, a contradiction.

3. Proofs of the theorems

Proof of Theorem 2. Suppose G is not subpancyclic. Then for some m, G has a C_m, but no C_{m-1}. By Corollary 9, $m \geqslant (\Delta + 6)/2$. But now, the diameter condition guarantees a shortening path, which is impossible by Lemma 1. □

Proof of Theorem 3. Suppose G is not subpancyclic. Then for some m, G has a C_m, but no C_{m-1}. By Corollary 10, $m \geqslant \delta + 3$. But now, the diameter condition guarantees a shortening path, which is impossible by Lemma 1. □

Proof of Theorem 4. Suppose G is not subpancyclic. Then for some m, G has a C_m, but no C_{m-1}. By Corollary 9, $m \geqslant 6$. By Lemma 1, $C := C_m$ has no chords. By the degree condition, there is a vertex $v \in V(C)$ with $d(v) \geqslant 5$.

If $m = 6$, the neighborhood of v is split into two complete subgraphs, not connected by edges. Else there was a 5-cycle in $N(v) \cup v$ by Theorem 8 (take a P_4 in $N(v)$, and connect both its ends with v). Without loss of generality, let $x, y \in N(v) \cap N(v^+)$ (so $xv^-, yv^- \notin E, xy \in E$). Observe that $xv^{++}, yv^{++} \notin E$, else there is a 5-cycle. Further, x and y cannot be adjacent to any other vertex of C, else there is a shortening path of C, which is not possible by Lemma 1. Now yxv^+Cv^- form a Z_4, a contradiction.

If $m \geqslant 7$, there exist $z \in V - V(C), y \in V(C)$, such that $z \in (N(y) \cap N(y^+)) - N(y^{++})$. But then, zyy^+Cy^{5+} form a Z_4 (Again, z cannot be adjacent to any of $y^{3+}, y^{4+}y^{5+}$, else there is a shortening path of C).

If G is 2-connected, then it is hamiltonian by a result of Brousek et al. [2], thus G is pancyclic. □

Proof of Theorem 5. Suppose G is not subpancyclic. Then for some m, G has a C_m, but no C_{m-1}. By Corollary 9, $m \geqslant 6$. By Lemma 1, $C := C_m$ has no chords.

If $m = 6$, let $v \in V(C)$ be the vertex on C with the largest degree. The degree condition guarantees $d(v) \geqslant 5$. By Theorem 8, the neighborhood of v is split into two complete subgraphs of size at most 3, not connected by edges, else there is a C_5 in $N(v) \cup v$. Hence, $|N(v) \cap N(v^-)|, |N(v) \cap N(v^+)| \in \{1, 2\}$, with one of them being 1 only in the case that $d(v) = 5$. Let $x \in N(v) \cap N(v^+), y \in N(v) \cap N(v^-)$. Clearly $xy \notin E$ or a C_5 is immediate. Now neither of the two edges xv^{++}, yv^{--} can exist by the following argument: If $|N(v) \cap N(v^+)| = 2$, then xv^{++} completes a 5-cycle. If $|N(v) \cap N(v^+)| = 1$, then $d(v) = 5$, and therefore $d(z) \geqslant 4$ for all $z \in V(C)$ (a z with a smaller degree would guarantee a vertex of degree $\geqslant 6$ in the chordless C, contradicting $d(v)$'s maximality). In particular $d(v^+) \geqslant 4$. Let $x' \in N(v^+) - \{v, v^{++}, x\}$. Then $x'v^{++} \in E$ to prevent a claw at v^+. If $xv^{++} \in E$, then $xv^{++}x'v^+vx$ is a C_5. Hence, in either case $xv^{++} \notin E$. The argument against $yv^{--} \in E$ is symmetric.

Further, x, y cannot have any other adjacencies on C, else a shortening path of C exists, a contradiction to Lemma 1. Now xv^+Cv^-y is a P_7, a contradiction.

If $m = 7$, observe that there are at most two vertices on C with degree $\leqslant 4$. Thus, there is a vertex $v \in V(C)$ with $d(v), d(v^+), d(v^-) \geqslant 5$. By Theorem 8, the neighborhood of v is split into two complete subgraphs, not connected by edges, else there is a C_6 in $N(v) \cup v$. Without loss of generality, let $x, y \in N(v) \cap N(v^+)$. Then $xv^{++} \notin E$, else there is a C_6 in $v^+ \cup N(v^+)$, using v, v^+, v^{++}, x, y and one other neighbor of v^+. Further, x has no other neighbors on C, else a shortening path of C exists. But now xv^+Cv^- is a P_7, a contradiction.

If $m \geqslant 8$, vCv^{6+} forms a P_7 for any $v \in V(C)$, a contradiction.

Hence, G is subpancyclic. □

Proof of Theorem 6. Suppose G is not subpancyclic. Then for some m, G has a C_m, but no C_{m-1}. By Corollary 9, $m \geqslant \Delta/2 + 3 \geqslant \delta_2/4 + 3$.

Case 1: *Suppose* $n<12$.

Note that the degree sum condition implies the following bounds on δ_2:

$$\begin{aligned}
n &\in \{6,7,8\} \Rightarrow \delta_2 \geqslant n-1,\\
n &\in \{9,10\} \Rightarrow \delta_2 \geqslant n-2,\\
n &= 11 \Rightarrow \delta_2 \geqslant n-3.
\end{aligned}$$

Consider all the possible values for m. Since $n\geqslant 5$, $m\geqslant \Delta/2+3>\sqrt{3n+1}/2+3\geqslant 5$. Say, $C=C_m=v_1\ldots v_m v_1$.

If $m=6$, then the only chords C could have are of the form $v_i v_{i+3}$. But then claw freeness forces either $v_i v_{i+2}$ or $v_i v_{i+4}$, which leads to a C_5. So C has no chords.

For $n\leqslant 8$, there are at least

$$\sum_{i=1}^{6}(d(v_i)-2)\geqslant 3\delta_2-12\geqslant 3(n-1)-12=3n-15$$

edges from C to $V-C$, but at most $3(n-6)=3n-18$ edges from $V-C$ to C, since no vertex in $V-C$ can have more than three neighbors on C without producing a C_5. Thus, $n\geqslant 9$.

For $9\leqslant n\leqslant 10$, the same count shows that there are exactly $3n-18$ edges from C to $V-C$, hence every vertex of $V-C$ has exactly three neighbors on C. To avoid a claw and a C_5, all the three have to be in a row. If two of the vertices $u,w\in V-C$ are adjacent, a C_5 can easily be found. But now $d(u)+d(w)=6<\delta_2$, a contradiction.

For $n=11$, there are at least $3\delta_2-12\geqslant 12$ edges from C to $V-C$, so out of the five vertices in $V-C$, at least two vertices $u,w\in V-C$ have three neighbors on the cycle, and two more vertices $x,y\in V-C$ have at least two neighbors on the cycle. If any of the edges uw,ux,uy,wx,wy exists, a C_5 can easily be found. Since $\delta_2\geqslant 8$, both u and w must be adjacent to the remaining vertex z. But now again, a C_5 can be found.

If $m=7$, the only possible chords are of the form $v_i v_{(i+3)\bmod 7}$. To avoid claws, all chords of this form have to exist if one exists. But now $v_1v_2v_5v_6v_7v_4v_1$ is a C_6. Therefore, C has no chords. This yields immediately $n\geqslant 8$. Observe, that for $n<12$, the degree sum condition ensures that $\delta_2\geqslant n-3$. Now a similar count as in the last case gives at least

$$\sum_{i=1}^{7}(d(v_i)-2)\geqslant \frac{7}{2}\delta_2-14\geqslant \frac{7}{2}(n-3)-14=3n-21+\frac{n-7}{2}$$

edges going out of C, with at most $3(n-7)$ going in, a contradiction.

If $m=8$ and C has a chord, then C has exactly the chords (after a cyclic renumbering of the vertices) $v_1v_5, v_1v_6, v_2v_5, v_2v_6$. If any of those are missing, there is a claw, if there are any more than those, there is a C_7. But now the degree sum condition forces v_3 or v_8 to have a neighbor outside the cycle, say $v_3x\in E$. To avoid a claw, $v_2x\in E$ or $v_4x\in E$. But this again yields a C_7. So C has no chords, and a similar count as before yields

$$\sum_{i=1}^{8}(d(v_i)-2)\geqslant 4\delta_2-16\geqslant 4(n-3)-16=4n-28>3(n-8),$$

a contradiction.

If $m=9$, a similar count shows the existence of chords. But if there is a chord, claw-freeness forces the appearance of a K_4 of the form $v_iv_{i+1}v_{i+4}v_{i+5}$ inside $\langle C\rangle$, say at $v_1v_2v_5v_6$.

Now v_8 has no neighbors outside C: Suppose $x\in V-C$, $xv_8\in E$. To prevent a claw at v_8, x has to be adjacent to v_7 or v_9. But then the 7-cycle $C'=v_2v_5Cv_2$ can be extended to a C_8 through x.

If $v_8v_3\in E$, then v_3 is adjacent to either v_7 or v_9 to avoid a claw at v_8. But then again, C' can be extended through v_3. The symmetric argument shows that $v_8v_4\notin E$. Further, if $v_8v_2\in E$, then $v_8v_3\in E$ to prevent a claw at v_2, which is not possible. The symmetric argument shows that $v_8v_5\notin E$. So $d(v_8)=2$. But this implies that $d(v_3)\geqslant n-5$. We know that v_3 is not adjacent to v_8, v_1 and v_5. Further, v_3 cannot be adjacent to v_9 without creating a claw at v_9. Thus, v_3 is adjacent to all other vertices, in particular v_3v_6, $v_3v_7\in E$. But now, C' can be extended through v_3, a contradiction.

If $m\geqslant 10$, a chord is guaranteed, again. Consider a chord v_iv_j, such that $|v_iCv_j|$ is minimal. Now find a chord v_rv_s on v_jCv_i, such that there is no other chord within v_rCv_s. Either all vertices in v_rCv_s or all vertices in V_iCv_j have chords, since there is at most one vertex outside C, and all vertices with degree at most 3 have to be pairwise adjacent. Say all vertices in v_rCv_s have chords. Now, similar to the first case in the proof of Lemma 1, insert all but one of $v_{r+1}Cv_{s-1}$ into $v_sCv_rv_s$ to construct a C_{m-1}.

Case 2: *Suppose* $n\geqslant 12$, $m\geqslant \delta_2/2+3$.

By Lemma 1, C has no chords ($n\geqslant 12$ guarantees $\delta_2\geqslant 9$). Thus, there are

$$\sum_{v\in V(C)}(d(v)-2)\geqslant m\left(\frac{\delta_2}{2}-2\right)$$

edges from C to $G-C$. On the other hand, every vertex in $G-C$ can have at most three neighbors on C, otherwise C has a shortening path, which is impossible by Lemma 1. So

$$m\left(\frac{\delta_2}{2}-2\right)\leqslant 3(n-m),$$

thus

$$3n\geqslant m\left(\frac{\delta_2}{2}+1\right)\geqslant\left(\frac{\delta_2}{2}+3\right)\left(\frac{\delta_2}{2}+1\right)$$
$$>(\sqrt{3n+1}+1)(\sqrt{3n+1}-1)=3n,$$

a contradiction.

Case 3: *Suppose* $n \geqslant 12$, $m < \delta_2/2 + 3$.

Let $d := \lceil \delta_2/2 \rceil$, so $m \leqslant d+2$. By Corollary 9, we know that $m \geqslant \Delta/2 + 3 \geqslant d/2 + 3$, particularly $m \geqslant 6$. By Lemma 1, C has no chords. Let $C = v_1 v_2 \cdots v_m v_1$. Since all vertices of degree $<d$ have to be pairwise adjacent, we may assume that $d(v_i) \geqslant d$ for $3 \leqslant i \leqslant m$. For $i = 1, 2, \ldots, m-1$, let $N_i := N(v_i) \cap N(v_{i+1})$, let $N_m := N(v_m) \cap N(v_1)$. Since G is claw-free, every vertex adjacent to C lies in some N_i. Note, that if $d(v_i) \geqslant d$, then $N_{i-1} \cap N_i = \emptyset$, and N_{i-1} and N_i induce complete subgraphs, otherwise, $\langle N(v_i) \rangle$ is traceable by Theorem 8, so we can find cycles of any length up to $d(v_i) + 1$ in $\langle N(v_i) \cup v_i \rangle$, in particular one of length $m-1$.

Now we claim that there cannot be any edges or 2-paths between N_i and N_j, for $3 \leqslant i < j \leqslant m-1$. If $j - i \geqslant 4$, an edge or 2-path leads to a shortening path of C, a contradiction to Lemma 1. If $j - i \leqslant 3$ and $m \geqslant 7$, one can easily find a cycle of length at most 6 through that edge or 2-path, v_{i+1} and v_j, which we can then extend to a C_{m-1}, using any number of vertices out of $N(v_j)$. If $m = 6$, then $j - i \leqslant 2$, and one can easily find a cycle of length at most 5 through that edge or 2-path, v_{i+1} and v_j, which we can then extend to a C_5, using any number of vertices out of $N(v_j)$.

Since all vertices of degree $<d$ have to be pairwise adjacent, we can now guarantee, after possibly renumbering the vertices of C, that all such vertices in $H := \bigcup N_i \cup C$ must lie in $N_m \cup N_1 \cup N_2 \cup \{v_1, v_2\}$.

Our next claim is, that for two vertices $x, y \in N_i$, $3 \leqslant i \leqslant m-1$, their neighborhoods intersect as follows: $N(x) \cap N(y) = N_i \cup \{v_i, v_{i+1}\} - \{x, y\}$. We already established that it is at least of that size, since $\langle N_i \rangle$ is complete. But it cannot be bigger; for suppose, there is a $z \in (N(x) \cap N(y)) - H$. Then, z is not adjacent to v_i. Therefore, the neighborhood of x is traceable by Theorem 8, and since $d(x) \geqslant d$, we can find a C_{m-1} in $\langle N(x) \cup x \rangle$.

Let $M_i := \{z \in V - H\colon\ zx \in E \text{ for some } x \in N_i\}$. Since $|N_i| \leqslant m-4$ for all $3 \leqslant i \leqslant m-1$ (else you can find a C_{m-1} in $\langle N_i \cup \{x_i, x_{i+1}\} \rangle$, since $\langle N_i \rangle$ is complete), and the degree of vertices in N_i is at least d, every $x \in N_i$ has at least $d - m + 3$ neighbors outside $N_i \cup \{x_i, x_{i+1}\}$. Thus, $|M_i| \geqslant (d - m + 3)|N_i|$ for $3 \leqslant i \leqslant m-1$. Further, the M_i are disjoint, otherwise there would be 2-paths between the N_i.

But now we see that

$$
\begin{aligned}
n &\geqslant |C| + |N_m \cup N_1 \cup N_2| + \sum_{i=3}^{m-1} |N_i \cup M_i| \\
&\geqslant |C| + |N_m \cup N_1 \cup N_2| + \sum_{i=3}^{m-1} (d - m + 4)|N_i| \\
&\geqslant^* m + \frac{(d(v_m) - 2) + (d(v_1) - 2) + (d(v_3) - 2) + (d - m + 4) \sum_{i=4}^{m-1} (d(v_i) - 2)}{2} \\
&\geqslant \frac{d}{2} + 3 + \frac{d - 2 + \delta_2 - 4 + (d - m + 4)(m - 4)(d - 2)}{2} \\
&\geqslant^{**} \frac{4d - 1 + (2d - 4)(d - 2)}{2} \\
&> d^2 - 2d + 3 \\
&> n,
\end{aligned}
$$

where $\geqslant^*$ results from a count that counts every vertex in the N_i at most twice, and $\geqslant^{**}$ comes from the fact, that for $d \geqslant 2$,

$$\min_{d/2+3 \leqslant m \leqslant d+2} ((d-m+4)(m-4)) = 2d-4$$

and $\delta_2 \geqslant 2d-1$. This contradiction concludes the proof. □

4. Sharpness

In this section, we demonstrate the sharpness of some of the results.

The following family of graphs (see also Fig. 1) demonstrates the sharpness of the bound on δ_2 in Lemma 1. Let $k \geqslant 4$, and let $H_1, \ldots, H_{2k}$ be $2k$ disjoint copies of K_5, and $u_i v_i$ an edge of H_i $(i=1,\ldots,2k)$. Now the graph F_k is obtained from $\bigcup_{i=1}^{2k} H_i - u_i v_i$ by adding the edges $v_1u_2, v_2u_3, \ldots, v_{2k-1}u_{2k}, v_{2k}u_1$ and the edges u_1v_k, u_1u_{k+1}, $u_{2k}v_k$, $u_{2k}v_{k+1}$. We have $\delta_2(F_k) = 8$, and there is a C_{6k} with chords, but no C_p for $5k+2 < p < 6k$.

The graph G in Fig. 2 shows that in Theorem 5, P_7-free cannot be replaced by P_8-free. This graph is $\{K_{1,3}, P_8\}$-free with $\delta_2 = 10$, and G contains a C_8 but no C_7.

The degree bounds in Theorems 6 and 7 are sharp. Consider the following family of graphs from [2]:

For any integer $p \geqslant 2$, we define the graph G_p as follows. Let $H_1, \ldots, H_p$ be p-disjoint copies of K_{3p-2}, and $u_i v_i$ an edge of H_i $(i=1,\ldots,p)$. Now G_p is obtained from $\bigcup_{i=1}^{p} H_i - u_i v_i$ by adding the edges $v_1u_2, v_2u_3, \ldots, v_{p-1}u_p, v_pu_1$.

The graph G_p is both hamiltonian and claw-free. Furthermore, we have $\delta(G_p) = 3p-3$ and $|V(G_p)| = p(3p-2)$, implying that $\delta(G_p) = \sqrt{3n+1} - 2$. It is obvious that G_p does not contain C_{3p-1} and hence G_p is not (sub)pancyclic.

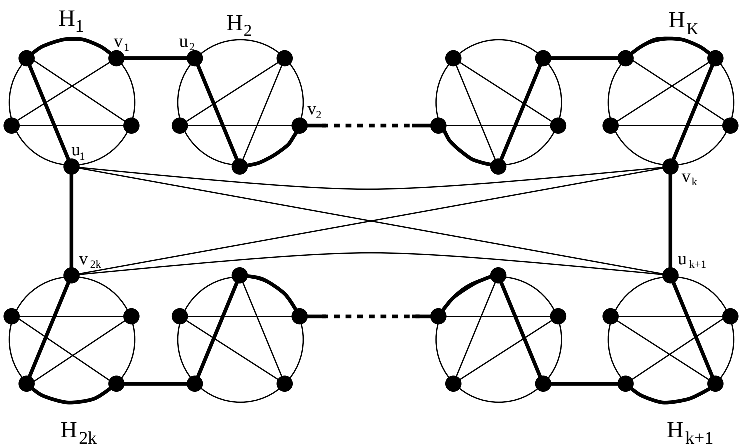

Fig. 1. Graph F_k.

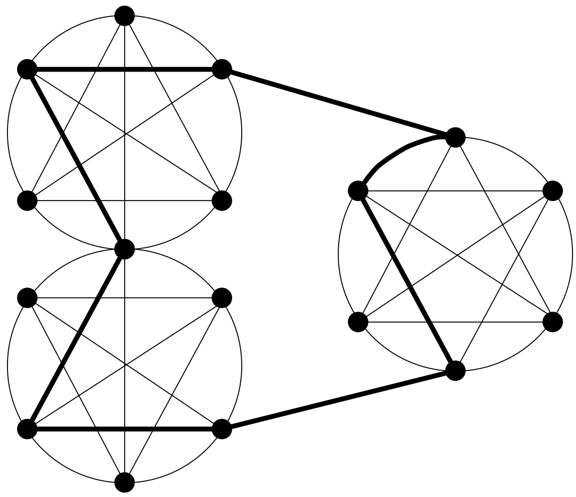

Fig. 2. Graph G.

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