On \( k \)-Ordered Bipartite Graphs

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Abstract

In 1997, Ng and Schultz introduced the idea of cycle orderability. For a positive integer \( k \), a graph \( G \) is \( k \)-ordered if for every ordered sequence of \( k \) vertices, there is a cycle that encounters the vertices of the sequence in the given order. If the cycle is also a hamiltonian cycle, then \( G \) is said to be \( k \)-ordered hamiltonian. We give minimum degree conditions and sum of degree conditions for nonadjacent vertices that imply a balanced bipartite graph to be \( k \)-ordered hamiltonian. For example, let \( G \) be a balanced bipartite graph on \( 2n \) vertices, \( n \) sufficiently large. We show that for any positive integer \( k \), if the minimum degree of \( G \) is at least \( (2n+k-1)/4 \), then \( G \) is \( k \)-ordered hamiltonian.

1 Introduction

Over the years, hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example hamiltonian connectedness. Recently a new strong hamiltonian property was introduced in [3].

We say a graph \( G \) on \( n \) vertices, \( n \geq 3 \), is \( k \)-ordered for an integer \( k \), \( 1 \leq k \leq n \), if for every sequence \( S = (x_1, x_2, \ldots, x_k) \) of \( k \) distinct vertices in \( G \) there exists a cycle that contains all the vertices of \( S \) in the designated order. A graph is \( k \)-ordered hamiltonian if for every sequence \( S \) of \( k \) vertices there exists a hamiltonian cycle which encounters the vertices in \( S \) in the designated order. We will always let \( S = (x_1, x_2, \ldots, x_k) \) denote the ordered \( k \)-set. If we say a cycle \( C \) contains \( S \), we mean \( C \) contains \( S \) in the designated order.
order under some orientation. The neighborhood of a vertex \( v \) will be denoted by \( N(v) \),
the degree of \( v \) by \( d(v) \), the degree of \( v \) to a subgraph \( H \) by \( d_H(v) \), and the minimum
degree of a graph \( G \) by \( \delta(G) \). A graph on \( n \) vertices is said to be \( k \)-linked if \( n \geq 2k \) and
for every set \( \{x_1, \ldots, x_k, y_1, \ldots, y_k\} \) of \( 2k \) distinct vertices there are vertex disjoint paths
\( P_1, \ldots, P_k \) such that \( P_i \) joins \( x_i \) to \( y_i \) for all \( i \in \{1, \ldots, k\} \). Clearly, a \( k \)-linked graph is
also \( k \)-ordered.

In the process of finding bounds implying a graph to be \( k \)-ordered hamiltonian, Ng
and Schultz [3] showed the following:

**Proposition 1.** [3] Let \( G \) be a hamiltonian graph on \( n \) vertices, \( n \geq 3 \). If \( G \) is \( k \)-ordered,
\( 3 \leq k \leq n \), then \( G \) is \((k−1)\)-connected.

**Theorem 2.** [3] Let \( G \) be a graph of order \( n \geq 3 \) and let \( k \) be an integer with \( 3 \leq k \leq n \).
If
\[
d(x) + d(y) \geq n + 2k - 6
\]
for every pair \( x, y \) of nonadjacent vertices of \( G \), then \( G \) is \( k \)-ordered hamiltonian.

Faudree et al. [4] improved the last bound as follows.

**Theorem 3.** [4] Let \( G \) be a graph of sufficiently large order \( n \). Let \( k \geq 3 \). If
\[
\delta(G) \geq \begin{cases} \frac{n+k-3}{2}, & \text{if } k \text{ is odd} \\ \frac{n+k-2}{2}, & \text{if } k \text{ is even} \end{cases}
\]
then \( G \) is \( k \)-ordered hamiltonian.

**Theorem 4.** [4] Let \( G \) be a graph of sufficiently large order \( n \). Let \( k \geq 3 \). If for any two
nonadjacent vertices \( x \) and \( y \),
\[
d(x) + d(y) \geq n + \frac{3k - 9}{2},
\]
then \( G \) is \( k \)-ordered hamiltonian.

**Theorem 5.** [4] Let \( k \) be an integer, \( k \geq 2 \). Let \( G \) be a \((k+1)\)-connected graph of
sufficiently large order \( n \). If
\[
|N(x) \cup N(y)| \geq \frac{n+k}{2}
\]
for all pairs of distinct vertices \( x, y \in V(G) \), then \( G \) is \( k \)-ordered hamiltonian.

Much like results for hamiltonicity, smaller bounds are possible if we restrict \( G \) to be
a balanced bipartite graph. In fact, we get the following results:

**Theorem 6.** Let \( G(A \cup B, E) \) be a balanced bipartite graph of order \( 2n \geq 618 \). Let
\( 3 \leq k \leq \frac{4n}{105} \). If \( \delta(G) \geq 4k - 1 \) and for any two nonadjacent vertices \( x \in A \) and \( y \in B \),
\[
d(x) + d(y) \geq n + \frac{k-1}{2},
\]
then \( G \) is \( k \)-ordered hamiltonian.
The bound on the degree sum is sharp, as will be shown later with an example. The upper bound on $k$ comes out of the proof, the correct bound should be a lot larger and possibly as large as $n/4$.

**Corollary 7.** Let $G$ be a balanced bipartite graph of order $2n \geq 618$. Let $3 \leq k \leq \frac{n}{103}$. If
\[ \delta(G) \geq \frac{2n + k - 1}{4} \]
then $G$ is $k$-ordered hamiltonian.

**Theorem 8.** Let $G(A \cup B, E)$ be a balanced bipartite graph of order $2n \geq 618$. Let $3 \leq k \leq \min\{\frac{n}{103}, \frac{\sqrt{n}}{4}\}$. If for any two nonadjacent vertices $x \in A$ and $y \in B$, $d(x) + d(y) \geq n + k - 2$, then $G$ is $k$-ordered hamiltonian.

The last bound is sharp, as the following graph $G$ shows:

Let the vertex set $V := A_1 \cup A_2 \cup B_1 \cup B_2 \cup B_3$, with $|A_1| = |B_1| = k/2$, $|B_2| = k - 1$, $|A_2| = n - k/2$, $|B_3| = n - 3k/2 + 1$. Let the edge set consist of all edges between $A_1$ and $B_1$ minus a $k$-cycle, all edges between $A_1$ and $B_2$, and all edges between $A_2$ and the $B_i$ for $i \in \{1, 2, 3\}$. Then $G$ has minimum degree $\delta(G) = 3k/2 - 3$, minimal degree sum $n + k - 3$, and $G$ is not $k$-ordered, as there is no cycle containing the vertices of $A_1 \cup B_1$ in the same order as the cycle whose edges were removed between $A_1$ and $B_1$. This example further suggests the following conjecture, strengthening Theorem 6 to a sharp result:

**Conjecture 9.** Let $G(A \cup B, E)$ be a balanced bipartite graph of order $2n$. Let $k \geq 3$. If $\delta(G) \geq \frac{3k - 1}{2} - 2$ and for any two nonadjacent vertices $x \in A$ and $y \in B$, $d(x) + d(y) \geq n + \frac{k - 1}{2}$, then $G$ is $k$-ordered hamiltonian.

In some of the proofs the following theorem of Bollobás and Thomason[1] comes in handy.

**Theorem 10.** [1] Every $22k$-connected graph is $k$-linked.

## 2 Proofs

In this section we will prove Theorem 6 and Theorem 8.

From now on, $A$ and $B$ will always be the partite sets of the balanced bipartite graph $G$, and for a subgraph $H \subseteq G$, $H^A := H \cap A$ and $H^B := H \cap B$ will be its corresponding parts.

The following result and its corollary, which give sufficient conditions for $k$-ordered to imply $k$-ordered hamiltonian, will make the proofs easier.

**Theorem 11.** Let $k \geq 3$ and let $G(A \cup B, E)$ be a balanced bipartite, $k$-ordered graph of order $2n$. If for every pair of nonadjacent vertices $x \in A$ and $y \in B$
\[ d(x) + d(y) \geq n + \frac{k - 1}{2}, \]
then $G$ is $k$-ordered hamiltonian.
Proof: Let \( S = \{x_1, x_2, \ldots, x_k\} \) be an ordered subset of the vertices of \( G \). Let \( C \) be a cycle of maximum order 2 containing all vertices of \( S \) in appropriate order. Let \( L := G - C \). Notice that \( L \) is balanced bipartite, since \( C \) is. Let \( l := |L|/2 = |L^A| = |L^B| \).

Claim 1. Either \( L \) is connected or \( L \) consists of the union of two complete balanced bipartite graphs.

To prove the claim, it suffices to show that \( d_L(u) + d_L(v) \geq l \) for all nonadjacent pairs \( u \in L^A, v \in L^B \). Suppose the contrary, that is, there are two such vertices \( u, v \) with \( d_L(u) + d_L(v) < l \). Since \( d(u) + d(v) \geq n + (k - 1)/2 \), it follows that \( d_C(u) + d_C(v) \geq c + (k + 1)/2 \). There are no common neighbors of \( u \) and \( v \) on \( C \), hence there are at least \( k + 1 \) edges on \( C \) with both endvertices adjacent to \( \{u, v\} \). Fix a direction on \( C \). Say there are \( r \) edges on \( C \) directed from a \( u \)-neighbor to a \( v \)-neighbor, and \( t \) edges from a \( v \)-neighbor to a \( u \)-neighbor. Without loss of generality, let \( r \geq t \). On \( C \), between any two of the \( r \geq (k + 1)/2 \) edges of that type, there have to be at least two vertices of \( S \), else \( C \) could be enlarged (see Figure 1). Thus \( |S| \geq k + 1 \), a contradiction, which proves the claim. \( \diamond \)

![Figure 1:](image)

In particular, the claim shows that there are no isolated vertices in \( L \) and that all of \( L \)'s components are balanced.

Suppose \( l \geq 1 \). Let \( L_1 \) be a component of \( L \), \( L_2 := L - L_1 \), \( l_1 := |L_1|/2 \), and \( l_2 := |L_2|/2 \). The \( k \) vertices of \( S \) split the cycle \( C \) into \( k \) intervals: \([x_1, x_2], [x_2, x_3], \ldots, [x_k, x_1]\). Assume there are vertices \( x, y \in L_1 \) (\( x = y \) is possible) with distinct neighbors in one of the intervals of \( C \) determined by \( S \), say \([x_i, x_{i+1}]\). Let \( z_1 \) and \( z_2 \) be the immediate successor and predecessor on \( C \) to the neighbors of \( x \) and \( y \) respectively according to the orientation of \( C \). Observe that we can choose \( x \) and \( y \) and their neighbors in \( C \) such that none of the vertices on the interval \([z_1, z_2]\) have neighbors in \( L_1 \). We can also assume that \( z_1 \neq z_2 \), otherwise \( x = y \) by the maximality of \( C \), and bypassing \( z_1 \) through \( x \) would lead to a cycle of the same order, but the new outside component \( L_1 - x \) would not be balanced, a contradiction to claim 1. Let \( z \) be either \( z_2 \) or its immediate predecessor such that \( z_1 \) and \( z \) are from different parts. Since \( x \) and \( y \) are in the same component of \( L \), there is an \( x, y \)-path through \( L \). Let \( \tilde{y} \) be either \( y \) or its immediate predecessor on the path such that \( x \) and \( \tilde{y} \) are from different parts. If \( x = y \), let \( \tilde{y} \) be any neighbor of \( x \) in \( L \). Let \( R \) be the path on \( C \) from \( z_1 \) to \( z_2 \) and \( r := |R| \). Since \( C \) is maximal, the \( x, \tilde{y} \)-path
can’t be inserted, and since neither $x$ nor $\bar{y}$ have neighbors on $R$,
\[ d(x) + d(\bar{y}) \leq 2l_1 + \frac{2c - r + 1}{2}. \]
Further, the $z_1, z$-path can’t be inserted anywhere on $C - R$, else $C$ could be enlarged by inserting it and going through $L$ instead (or in the case $x = y$ we would get a same length cycle with unbalanced outside components). Since $z_1$ and $z$ have no neighbors in $L_1$, we get
\[ d(z_1) + d(z) \leq 2l_2 + r + \frac{2c - r + 1}{2}. \]
Hence
\[ d(x) + d(\bar{y}) + d(z_1) + d(z) \leq 2l_2 + 2l_1 + 2c + 1 = 2n + 1, \]
which contradicts (with $k \geq 3$) that
\[ d(x) + d(z) \geq n + \frac{k - 1}{2} \]
and
\[ d(\bar{y}) + d(z_1) \geq n + \frac{k - 1}{2}. \]
Thus, there is no interval $[x_i, x_{i+1}]$ with two independent edges to $L_1$. By Proposition 1, $G$ is $(k - 1)$-connected, thus all but possibly one of the segments $(x_i, x_{i+1})$ have exactly one vertex with a neighbor in $L_1$.

Since $|N_C(L_1)| \leq k$, we assume without loss of generality that $|N_C(L_1^B)| \leq k/2$. Let $x \in L_1^B$ and let $|N_C(x)| = d \leq k/2$. Thus, for every $v \in C$ that is not adjacent to $L_1$ the degree sum condition implies:
\[ d(v) \geq n + \frac{k - 1}{2} - (l_1 + d) = c + l_2 + (\frac{k}{2} - d - \frac{1}{2}). \]
On the other hand, we know $d(v) \leq c + l_2 - 1$. Thus, $d \geq 2$. Now we have shown that $N_{L_1}(C)$ includes vertices from both $L_1^A$ and $L_1^B$. So, without loss of generality, assume $L_1$ has neighbors $y$ and $z$ in $(x_1 \ldots x_2)$ and $(x_2 \ldots x_3)$ respectively and such that $y$ and $z$ are in different partite sets.

Let $y, z$ be the unique vertices in $(x_1, x_2)$ and $(x_2, x_3)$ respectively, which have neighbors in $L_1$. Since the successors of $y$ and $z$ are from different parts and not adjacent to $L_1$, they must be adjacent to each other. But now $C$ can be extended, which is a contradiction.

This proves that $L$ has to be empty. Therefore $C$ is hamiltonian.

An immediate Corollary to Theorem 11 is the following:

**Corollary 12.** Let $k \geq 3$ and let $G$ be a $k$-ordered balanced bipartite graph of order $2n$. If $\delta(G) \geq \frac{n}{2} + \frac{k-1}{4}$, then $G$ is $k$-ordered hamiltonian.
To see that these bounds are sharp, consider the following graph $G(A \cup B, E)$:

$$A := A_1 \cup A_2, B := B_1 \cup B_2,$$

with

$$|A_1| = |B_1| = \left\lceil \frac{n}{2} + \frac{k - 1}{4} \right\rceil - 1,$$

$$|A_2| = |B_2| = n - |A_1|,$$

and

$$E := \{ab | a \in A_1, b \in B \} \cup \{ab | a \in A, b \in B_1 \}.$$ 

For $n$ sufficiently large, $G$ is obviously a $k$-connected, $k$-ordered, and balanced bipartite graph. The minimum degree is $\delta(G) = d(v) = |A_1|$ for any vertex $v \in B_2 \cup A_2$, thus the minimum degree condition is just missed. But $G$ is not $k$-ordered hamiltonian, for if we consider $S = \{x_1, x_2, \ldots, x_k\}, \{x_1, x_3, \ldots\} \subseteq A_2, \{x_2, x_4, \ldots\} \subseteq B_2$. Let $C$ be a cycle that picks up $S$ in the designated order. Then $C \cap (A_1 \cup B_2)$ consists of at least $\lfloor k/2 \rfloor$ paths, all of which start and end in $A_1$. Therefore $|C \cap A_1| \geq |C \cap B_2| + (k - 1)/2$. If $C$ was hamiltonian, it would follow that $|A_1| \geq |B_2| + (k - 1)/2$, which is not true.

The following easy lemmas will be useful.

**Lemma 13.** Let $G$ be a graph, let $k \geq 1$ be an integer and let $v \in V(G)$ with $d(v) \geq 2k - 1$ for some $k$. If $G - v$ is $k$-linked, then $G$ is $k$-linked.

**Proof:** This is an easy exercise. \(\square\)

**Lemma 14.** Let $G$ be a $2k$-connected graph with a $k$-linked subgraph $H \subset G$. Then $G$ is $k$-linked.

**Proof:** Let $S := \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ be a set of $2k$ vertices in $G$, not necessarily disjoint from $H$. Since $G$ is $2k$-connected, there are $2k$ disjoint paths from $S$ to $H$, including the possibility of one-vertex paths. Since $H$ is $k$-linked, those paths can be joined in a way that $k$ paths arise which connect $x_i$ with $y_i$ for $1 \leq i \leq k$. \(\square\)

**Lemma 15.** Let $k \geq 1$. Let $G(A \cup B, E)$ be a bipartite graph with $d(v) \geq \frac{|B|}{2} + \frac{3k}{2}$ for all $v \in A$, and $d(w) \geq 2k$ for all $w \in B$. Then $G$ is $k$-linked.

**Proof:** Let $S := \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ be a set of $2k$ vertices in $G$. Pick a set $S' := \{x_1', \ldots, x_k', y_1', \ldots, y_k'\} \subset A$ as follows: If $x_i \in A$ set $x_i' = x_i$. Otherwise let $x_i'$ be a neighbor of $x_i$ not in $S$. Similarly pick the $y_i'$. It is possible to pick $2k$ different vertices for $S'$ since $d(w) \geq 2k$ for all $w \in B$.

Now find disjoint paths of length 2 between $x_i'$ and $y_i'$ avoiding all the other vertices of $S$ for $1 \leq i \leq k$. This is possible since $|N(x_i') \cap N(y_i')| \geq d(x_i') + d(y_i') - |B| \geq 3k$. \(\square\)

**Proof of Theorem 6:** By Theorem 11, it suffices to show that $G$ is $k$-ordered.

Let $K$ be a minimal cutset. If $|K| \geq 22k$, then $G$ is $k$-linked by Theorem 10. Therefore it is $k$-ordered. Assume now that $|K| < 22k$. We have to deal with two cases.
Case 1. There is an isolated vertex \( v \in G - K \).

Since \( |K| = |N(v)| \geq \delta(G) \geq 4k - 1 \), \( G \) is \( 2k \)-connected, thus by Lemma 14 it suffices to find a \( k \)-linked subgraph. Without loss of generality, let \( v \in B \). Let \( R = G - K - v \). Then \( d(w) > n - 22k \) for all \( w \in R^A \). So there are at least \( (n - 22k)^2 \) edges in \( R \), resulting in less than \( 23k \) vertices \( u \in R^B \) with \( d_R(u) < 2k \). Let \( H \) be the subgraph of \( R \) induced by \( R^A \) and the vertices of \( R^B \) with \( d_R(u) \geq 2k \). For \( w \in R^A \), we have \( d_H(w) \geq n - 45k \geq \frac{|H^B|}{2} + \frac{3k}{2} \), since \( n > 100k \). By Lemma 15, \( H \) is \( k \)-linked.

Case 2. There are no isolated vertices in \( G - K \).

First, observe that \( G - K \) has exactly two components. Otherwise, for the three components \( C_1, C_2, C_3 \) choose vertices \( v_i \in C_i^A, w_i \in C_i^B, 1 \leq i \leq 3 \). Then we can bound their degree sum as follows:

\[
2n + 2|K| \geq (|C_1| + |K|) + (|C_2| + |K|) + (|C_3| + |K|)
\geq (d(v_1) + d(w_1)) + (d(v_2) + d(w_2)) + (d(v_3) + d(w_3))
= (d(v_1) + d(w_2)) + (d(v_2) + d(w_3)) + (d(v_3) + d(w_1))
\geq 3(n + \frac{k - 1}{2}),
\]

a contradiction.

Call the two components \( L \) and \( R \). Without loss of generality, let \( |R| \geq |L| \) and \( |L^A| \geq |L^B| \). Let \( v \in L^A, w \in L^B, x \in R^A, y \in R^B \). Then

\[
|L^A| + |R^A| + |K^A| = |L^B| + |R^B| + |K^B| = n,
\]

\[
|L^B| + |R^A| + |K| \geq d(w) + d(x) \geq n + \frac{k - 1}{2},
\]

\[
|L^A| + |R^B| + |K| \geq d(v) + d(y) \geq n + \frac{k - 1}{2}.
\]

Thus, the inequalities above imply the parts of the components are of similar size:

\[
|L^A| - |L^B| \leq |K^B| - \frac{k - 1}{2},
\]

\[
|R^A| - |R^B| \leq |K^B| - \frac{k - 1}{2},
\]

\[
|R^B| - |R^A| \leq |K^A| - \frac{k - 1}{2}.
\]

Further, we get the following bounds for the degrees inside the components:

\[
d_R(y) \geq n + \frac{k - 1}{2} - d(v) - |K^A| \geq n + \frac{k - 1}{2} - |L^B| - |K^B| - |K^A| = |R^B| - (|K^A| - \frac{k - 1}{2}),
\]

\[
d_R(x) \geq |R^A| - (|K^B| - \frac{k - 1}{2}),
\]

\[
d_L(w) \geq |L^B| - (|K^A| - \frac{k - 1}{2}),
\]

\[
d_L(v) \geq |L^A| - (|K^B| - \frac{k - 1}{2}).
\]
Claim 1. \(R\) is \(k\)-linked.

By symmetry of the argument, we may assume that \(|R^B| \geq |R^A|\), thus

\[ |R^B| \geq \frac{|R|}{2} \geq \frac{2n - |K| - |L|}{2} \geq \frac{n}{2} - \frac{|K|}{4}. \]

Now,

\[
d_R(y) \geq \frac{|R^B| - (|K^A| - \frac{k-1}{2})}{2} \geq \frac{|R^A|}{2} + \frac{|R^B|}{2} - |K| + \frac{k-1}{2} \geq \frac{103k}{4} - \frac{9(2k-1)}{8} + \frac{k-1}{2}.
\]

Further,

\[
d_R(x) \geq \frac{|R^A| - (|K^B| - \frac{k-1}{2})}{2} \geq |R^B| - |K| + \frac{k-1}{2} > 2k.
\]

Hence, the conditions of Lemma 15 are satisfied for \(R\), and \(R\) is \(k\)-linked. \(\diamondsuit\)

If \(|K| \geq 2k\), then \(G\) is \(k\)-linked by Lemma 14 and we are done. So assume from now on \(|K| < 2k\).

Claim 2. \(L\) is \(k\)-linked.

If \(|L| > n - 2k\), the proof is similar to the last case:

\[
d_L(v) \geq |L^A| - |K^B| + \frac{k-1}{2} > \frac{|L^B|}{2} + \frac{n - 2k}{4} - 2k + \frac{k-1}{2} > \frac{|L^B|}{2} + \frac{3k}{2},
\]

and

\[
d_L(w) \geq |L^A| - (|K^B| - \frac{k-1}{2}) > |L^B| - |K| > 2k.
\]

Applying Lemma 15 to \(L\) gives the result.

If \(|L| \leq n - 2k\), \(L\) is complete bipartite from the degree sum condition. Further, \(|L^A| \geq |L^B| \geq d(v) - |K^B| \geq 2k\) from the minimum degree condition, hence \(L\) is \(k\)-linked. \(\diamondsuit\)

Let \(S := \{x_1, x_2, \ldots, x_t\}\) be a set in \(V(G)\). We want to find a cycle passing through \(S\) in the prescribed order. Note that the minimum degree condition forces \(|R| \geq |L| \geq |K|\).

Assume \(|K| = \kappa(G) = k + t\) where \(t \geq -1\). Using the fact that \(K\) is a minimal cut set, by Hall’s Theorem (see for instance [2]) there is a matching of \(K\) into \(L\) and respectively \(K\) into \(R\), which together produce \(k + t\) pairwise disjoint \(P_3\)'s. Of all such matchings, pick one on either side with the fewest intersections with the set \(S\).

Observe that a vertex \(s \in K^B\) is either adjacent to every vertex of \(L^A\) or \(d(s) > n/4\). Otherwise there would be a vertex \(v \in L^A\) not connected to \(s\), and \(d(v) + d(s) \leq |L^B| + |K^B| + n/4 \leq n/2 - k + 2k + n/4\), a contradiction. A similar argument shows that the analog statement is true for \(s \in K^A\), since \(|L^A|\) and \(|L^B|\) differ by less than \(|K| < 2k\). Hence, each vertex \(s \in K\) has large degree to at least one of \(L\) or \(R\), in fact large enough that either \((L \cup \{s\})\) or \((R \cup \{s\})\) is \(k\)-linked.
Assign every vertex of $K$ one by one to either $L$ or $R$ such that the new subgraphs $\bar{L}$ and $\bar{R}$ are still $k$-linked, applying Lemma 13 repeatedly. Left over from the $P_3$’s is now one matching with $k + t$ edges between $\bar{L}$ and $\bar{R}$. We call an edge of this matching a double if both its endvertices are in $S$ and a single if exactly one endvertex is in $S$. If an edge is disjoint from $S$, we call it free.

We claim that the number of doubles is at most $t$ if $k$ is even and at most $t + 1$ if $k$ is odd. Let $l^A$ (and respectively $r^A$) be the number of doubles which are edges between $L^A$ and $K^B$ (respectively between $R^A$ and $K^B$). Define $l^B$ and $r^B$ similarly. Note that this means $d := l^A + l^B + r^A + r^B$ is the number of doubles. Let $v \in L^A - S$, $w \in L^B - S$, $x \in R^A - S$ and $y \in R^B - S$ such that none of those vertices are on an edge of the matching (this is possible since $|L^A| - |K^B| \geq 2k$, $|L^B| - |K^A| \geq 2k$ from the minimum degree condition). Then

$$2n + 2 \left\lfloor \frac{k - 1}{2} \right\rfloor \leq d(v) + d(w) + d(x) + d(y) \leq 2n + k + t - l^A - l^B - r^A - r^B.$$  

If $d \geq t + 1$ for $k$ even or $t + 2$ for $k$ odd, we obtain a contradiction to the above inequality.

Let $c$ be the number of elements of $S$ that are not vertices on any of the $k + t$ edges of the matching. Then $t + d + c$ of the edges are free. We are now prepared to construct the cycle containing the set $\{x_1, x_2, \cdots, x_k\}$ by constructing a set of disjoint $x_i, x_{i+1}$-paths, using that $\bar{L}$ and $\bar{R}$ are $k$-linked. Note that in constructing each $x_i, x_{i+1}$-path, using a free edge is only necessary if (1) $x_i$ is not on a single and (2) $x_i$ and $x_{i+1}$ are on different sides.

If $k$ is even, these two conditions can occur at most $2d + c$ times. If $k$ is odd, these two conditions can occur at most $2d - 1 + c$ times (because of the parity, condition 2 cannot occur for every vertex). But neither ever exceeds $t + d + c$, the number of free edges. Hence, we may form a cycle containing the elements of $S$ in the appropriate order. \qed

**Proof of Theorem 8:** By Theorem 11 it suffices to show that $G$ is $k$-ordered.

If the minimum degree $\delta(G) \geq 4k - 1$, then we are done by Theorem 6. Thus, suppose that $s \in A$ is a vertex with $d(s) < 4k - 1$. Let $R$ be the induced subgraph of $G$ on the following vertex set:

$$R^B := \{ v \in B : sv \notin E \},$$  
$$R^A := \{ w \in A : d_{Rw} \geq 2k \}.$$  

The degree sum condition guarantees $d(v) \geq n - 3k$ for all $v \in R^B$. Further, $|R^B| = n - d(s) \geq n - 4k + 2$. It is easy to see that $|R^A| > n - 4k$ and that all the conditions for Lemma 15 are satisfied. Hence, $R$ is $k$-linked.

Let $H$ be the biggest $k$-linked subgraph of $G$. If $G = H$, we are done. Otherwise, let $L := G - H$. The size of $L$ is $|L| = 2n - |H| \leq 2n - |R| \leq 8k$. Observe that no vertex $v \in L$ has $d_H(v) > 2k - 2$, otherwise $V(H) \cup \{v\}$ would induce a bigger $k$-linked subgraph by Lemma 13. Hence, no vertex in $L$ has degree greater than $10k$, and therefore, $L$ is complete bipartite.

Define

$$\alpha := \min\{d_H(v) | v \in L^A \} \cup \{2k\},$$  

THE ELECTRONIC JOURNAL OF COMBINATORICS 10 (2003), #R11

9
\[
\beta := \min\{d_H(v) | v \in L^B \} \cup \{2k\}.
\]

Since \(L\) is small, there are vertices \(x \in H^A, y \in H^B\), with \(N(x) \cup N(y) \subset H\). If \(L^A = \emptyset\), then \(\alpha = 2k\), and if \(L^B = \emptyset\), then \(\beta = 2k\). Either way, we get \(\alpha + \beta \geq 2k\).

Now assume that \(L^A \neq \emptyset\) and \(L^B \neq \emptyset\). Let \(v \in L^A\) such that \(d_H(v) = \alpha\). Then
\[
n + k - 2 \leq d(v) + d(y) \leq |H^A| = d(v) + n - |L^A|.
\]
Thus, \(d(v) \geq |L^A| + k - 2\), and
\[
|L^B| + \alpha = d(v) \geq |L^A| + k - 2.
\]
Analogously, let \(w \in L^B\) with \(d_H(w) = \beta\), then
\[
n + k - 2 \leq d(w) + d(x) \leq |H^B| = d(w) + n - |L^B|,
\]
and thus \(d(w) \geq |L^B| + k - 2\) and
\[
|L^A| + \beta = d(w) \geq |L^B| + k - 2.
\]

Therefore,
\[
\alpha + \beta \geq 2k - 4.
\]

Let \(S := \{x_1, x_2, \ldots, x_k\}\) be a set in \(V(G)\). From now on, all the indices are modulo \(k\). To build the cycle, we need to find paths from \(x_i\) to \(x_{i+1}\) for all \(1 \leq i \leq k\).

If \(x_i\) and \(x_{i+1}\) are neighbors, just use the connecting edge as path. Now, for all other \(x_i \in L\) we find two neighbors \(y_i\) and \(z_i\) not in \(S\). If \(x_i\) and \(x_{i+i}\) have a common neighbor \(v\) which is not already used, set \(z_i = y_{i+1} = v\). Afterwards, we can find distinct \(y_i\) and \(z_i\) by the following count: Suppose \(x_i \in L^A\), so we need to find \(y_i, z_i \in N(x_i) - U_i\), where
\[
U_i := N(x_i) \cap \{x_j, y_j, z_j : |i - j| > 1\} \cup \{z_{i+1}, y_{i-1}\}.
\]
For every \(x_j \in L^A, |i - j| > 1\), there can be at most two vertices in \(U_i\). For \(x_j \in L^A, |i - j| = 1\), there can be at most one vertex in \(U_i\). For \(x_j \in B, |i - j| > 1\), there can be at most one vertex in \(U_i\). Hence,
\[
|U_i| \leq 2|L^A \cap S - \{x_{i-1}, x_i, x_{i+1}\}| + 2 + |B \cap S - \{x_{i-1}, x_i, x_{i+1}\}| \leq |L^A| + k - 4,
\]
and since \(d(x_i) \geq |L^A| + k - 2\), we can pick \(y_i\) and \(z_i\).

Try to choose as few \(y_i, z_i\) out of \(L\) as possible (i.e. pick as many as possible in \(H\)). Now for all \(y_i, z_j\), where \(y_i \neq z_{i-1}, z_j \neq y_{j+1}\), choose vertices \(y'_i, z'_i \in H\) as follows: If \(y_i \in H\), let \(y'_i = y_i\), if \(z_i \in H\), let \(z'_i = z_i\). Otherwise, let \(y'_i\) be a neighbor of \(y_i\) in \(H\), and let \(z'_i\) be a neighbor of \(z_i\) in \(H\), which is not already used. We need to check if there is a vertex in \(N(y_i) \cap H\) available.

Let \(O_i = (N(x_i) \cup N(y_i)) \cap H\). We know that
\[
|O_i| = d_H(x_i) + d_H(y_i) \geq \alpha + \beta \geq 2k - 4.
\]
For every $j \notin \{i-1, i, i+1\}$, $|O_i \cap \{x_j, y_j, z_j, y'_j, z'_j\}| \leq 2$, and for $j = i+1$, $|O_i \cap \{x_j, y_j, y'_j\}| \leq 1$. This is a total count of at most $2k-5$, at least one is left over for $y'_i$. Observe that $y'_i \notin N(x_i)$, otherwise we would have chosen it to be $y_i$, so in fact $y'_i \in N(y_i)$. A similar count shows the availability of a vertex for $z'_i$, with one possible exception: The one vertex left over could be $y'_i$. This is only a problem if the count for $y'_i$ gave us exactly one available vertex, otherwise we can just pick a different $y'_i$. But now we can switch the vertices $y_i$ and $z_i$, and choose $y'_i$ from $\{x_{i+1}, y_{i+1}, y'_{i+1}\}$ (one of those is in $N(x_i) \cup N(y_i)$, since the count of used vertices gave exactly $2k-5$), and choose $z'_i$ from $\{x_{i-1}, y_{i-1}, y'_{i-1}\}$.

For all $x_i \in H$, set $y'_i = z'_i = x_i$. Since $H$ is $k$-linked, we can now find $z'_i, y'_{i+1}$-paths inside $H$ for all needed indices to complete the cycle. \qed

### 3 Further Results

We also looked at the following closely related property:

**Definition 1.** We say a graph $G$ is $k$-ordered connected if for every sequence $S = (x_1, x_2, ..., x_k)$ of $k$ distinct vertices in $G$, there exists a path from $x_1$ to $x_k$ that contains all the vertices of $S$ in the given order. A graph is $k$-ordered hamiltonian connected if there is always a hamiltonian path from $x_1$ to $x_k$ which encounters $S$ in the designated order.

Along the lines of the proofs in [4], you can show the following theorems for this property:

**Theorem 16.** Let $G$ be a graph of sufficiently large order $n$. Let $k \geq 3$. If

$$\delta(G) \geq \frac{n + k - 3}{2},$$

then $G$ is $k$-ordered hamiltonian connected.

**Theorem 17.** Let $G$ be a graph of sufficiently large order $n$. Let $k \geq 3$. If for any two nonadjacent vertices $x$ and $y$, $d(x) + d(y) \geq n + \frac{3k-6}{2}$, then $G$ is $k$-ordered hamiltonian connected.

The proofs do not give any new insights, so we will not present them here.

### References

