\( \delta(G) \geq 2 \) implies \( G \) contains a cycle.

**Theorem (Corradi and Hajnal)**

If \( \delta(G) \geq 2k \) and \( |G| \geq 3k \) then \( G \) contains \( k \) vertex disjoint cycles.
Question

What conditions imply a graph contains a cycle with a chord?

Here a *chord* is an edge between two vertices on the cycle that is not an edge of the cycle.
Question

What conditions imply a graph contains a cycle with a chord?

Here a *chord* is an edge between two vertices on the cycle that is not an edge of the cycle.
First answer by J. Czipzer $\delta(G)$

Theorem

If $G$ has minimum degree at least 3, then $G$ contains a chorded cycle.
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longest path in $G$
Theorem

If $G$ has minimum degree at least 3, then $G$ contains a chorded cycle.
\( \delta(G) \geq 3 \) implies a chorded cycle.

**Theorem**

*If* \( G \) *is a graph on* \( n \geq 4k \) *vertices with minimum degree at least* \( 3k \), *then* \( G \) *contains at least* \( k \) *independent chorded cycles.*

*Note:* A generalization of Corradi-Hajnal.
\( \delta(G) \geq 3 \) implies a chorded cycle.

**Theorem**

*If G is a graph on \( n \geq 4k \) vertices with minimum degree at least \( 3k \), then G contains at least k independent chorded cycles.*

Note: A generalization of Corradi-Hajnal.

**Theorem**

*Let G be a graph of order \( n \geq 3k \) and suppose that \( \delta(G) \geq 2k \), then G contains k disjoint cycles.*
That \( n \geq 4k \) is clearly needed as the cycles need at least 4 vertices each.

For \( m \geq 6k - 2 \), the graph \( K_{3k-1,m-3k+1} \) has \( \delta = 3k - 1 \) and no collection of \( k \) independent chorded cycles, as chorded cycles here require 3 vertices from each partite set.
Theorem

If $G$ is a graph on $n \geq 4k$ vertices such that for any pair of non-adjacent vertices $x, y$,

$$|N(x, y)| \geq 4k + 1,$$

then $H$ contains at least $k$ independent chorded cycles.
Theorem

If \( G \) is a graph on \( n \geq 6k \) vertices with \( \sigma_2(G) \geq 6k - 1 \), then \( G \) contains \( k \) vertex disjoint doubly chorded cycles.
A special case: Cliques

\[ K_5: \text{4-regular but with 5 chords} \]
In general:

Given a $K_{k+1}$: It is $k$-regular with

$$f(k) = \frac{(k - 2)(k + 1)}{2}$$

chords. We think of $f(k)$ chorded cycles as "loose $K_{k+1}$ cliques".
Note: There are no single chorded cliques.

**Theorem**

*Ali, Staton - 1999*

If $\delta(G) = k$, then $G$ contains a

$$\left\lceil \frac{k(k - 2)}{2} \right\rceil$$

— chorded cycle.
Question: Where do chorded cycles fit in???

Note: There are no single chorded cliques.

**Theorem**

Ali, Staton - 1999

*If* $\delta(G) = k$, *then* $G$ *contains a*

$$\left\lceil \frac{k(k - 2)}{2} \right\rceil - \text{chorded cycle.}$$

**Corollary**

*If* $\delta(G) \geq 3$, *then* $G$ *contains a doubly chorded cycle - that is, a* loose $K_4$. 

Ron Gould Emory University  On Cycles with Many Chords
Theorem

If $\delta(G) = k$, then $G$ contains an

$$f(k) = \frac{(k + 1)(k - 2)}{2} - \text{chorded cycle.}$$
Theorem

If \( \delta(G) = k \), then \( G \) contains an

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longest path in $G$
The story so far:

Lower End: \( \delta(G) \geq k \) implies an \( f(k) = \frac{(k-1)(k+2)}{2} \)-chorded cycle.

Upper End:

Theorem

Hajnal and Szemerédi

If \( \delta(G) \geq kt, |G| = (k + 1)t \), then \( G \) can be covered by \( t \) vertex disjoint \( K_{k+1} \)'s.
Conjecture:

If $\delta(G) \geq kt$, and $|G| \geq (k+1)t$ then $G$ contains $t$

$$f(k) = \frac{(k+1)(k-2)}{2}$$ – chorded cycles.
Conjecture:

If $\delta(G) \geq kt$, and $|G| \geq (k + 1)t$ then $G$ contains $t$

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Tight End: Hajnal-Szemerédi Theorem.
Conjecture:

If $\delta(G) \geq kt$, and $|G| \geq (k + 1)t$ then $G$ contains $t$ chorded cycles.

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Tight End: Hajnal-Szemerédi Theorem.
If $t = 1$, this is our first Theorem.
Conjecture:

If $\delta(G) \geq kt$, and $|G| \geq (k + 1)t$ then $G$ contains $t$

$$f(k) = \frac{(k + 1)(k - 2)}{2}$$

— chorded cycles.

Tight End: Hajnal-Szemerédi Theorem.
If $t = 1$, this is our first Theorem.
We show it is true for some classes of graphs, and for graphs with some extra "room".
Theorem

There exist $k_0$, $t_0$ such that if $\delta(G) \geq kt$, where $k \geq k_0$, $t \geq t_0$ and $n \geq n_0(k, t)$, then $G$ contains $t$ chorded cycles.

$$f(k) = \frac{(k + 1)(k - 2)}{2}$$
Theorem

There exist $k_0$, $t_0$ such that if $\delta(G) \geq kt$, where $k \geq k_0$, $t \geq t_0$ and $n \geq n_0(k, t)$, then $G$ contains $t$

$$f(k) = \frac{(k + 1)(k - 2)}{2} - \text{chorded cycles.}$$

- Bounds for $k_0$ and $t_0$ show a tradeoff.
Theorem

There exist $k_0$, $t_0$ such that if $\delta(G) \geq kt$, where $k \geq k_0$, $t \geq t_0$ and $n \geq n_0(k, t)$, then $G$ contains $t$

$$f(k) = \frac{(k + 1)(k - 2)}{2}$$ - chorded cycles.

- Bounds for $k_0$ and $t_0$ show a tradeoff.
- Bounds for $n_0$ quite large.
What can we say about $f(k)$-chorded cycle free graphs?

- minimum degree $\leq k - 1$, 
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What can we say about $f(k)$-chorded cycle free graphs?

- minimum degree $\leq k - 1$,
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Problem: Not very useful!
Theorem

Let \( d = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil \).

- If \( G \) has average degree at least \( 2d \), then \( G \) contains a \( f(k) = \frac{(k+1)(k-2)}{2} \)-chorded cycle.
- There exist graphs with average degree \( 2d - o(1) \) with no \( f(k) \)-chorded cycle.
The harmonic average degree is

$$\hat{d} = \frac{n}{\sum_v 1/\text{deg}(v)}.$$ 

If $\hat{d} > k + 1$, does $G$ contain an $f(k)$-chorded cycle?
Question

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If $\hat{d} > k + 1$, does $G$ contain an $f(k)$-chorded cycle?

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If $G$ is $f(k)$-chorded cycle free, is it true that

$$| \{ v : \text{deg}(v) \leq k - 1 \} | \geq | \{ v : \text{deg}(v) \geq 2k - 1 \} |.$$
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The harmonic average degree is

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Question

If $G$ is $f(k)$-chorded cycle free, is it true that

$$| \{ v : \deg(v) \leq k - 1 \} | \geq | \{ v : \deg(v) \geq 2k - 1 \} |.$$ 

Remark: Can’t replace $2k - 1$ with $2k - 2$. 
Theorem

Let $d = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil$.

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- There exist graphs with average degree $2d - o(1)$ with no $f(k)$-chorded cycle.
More

**Theorem**

Let \( d = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil \).

- If \( G \) has average degree at least \( 2d \), then \( G \) contains a \( f(k) = \frac{(k+1)(k-2)}{2} \)-chorded cycle.
- There exist graphs with average degree \( 2d - o(1) \) with no \( f(k) \)-chorded cycle.

Sharpness: Bipartite graph \( K_{d,n} \).
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- Sharpness: Bipartite graph $K_{d,n}$.
- $k = 2, 3, 4$: Trivial induction removing vertex of lowest degree if $< \delta$. 
Theorem

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- There exist graphs with average degree \( 2d - o(1) \) with no \( f(k) \)-chorded cycle.

Sharpness: Bipartite graph \( K_{d,n} \).

- \( k = 2, 3, 4 \): Trivial induction removing vertex of lowest degree if \( < \delta \).
- \( k \geq 5 \) much tougher induction.