Section 3.7 More About Counting Trees – Using Generating Functions

In this section, our fundamental goal is to introduce one of the principal tools used to count combinatorial objects, namely generating functions.

To begin with, what is a generating function? Suppose we have a sequence of numbers (c_i); then the power series $\sum_{i=0}^{\infty} c_i x^i$ is called the *generating function* for the sequence. Examples from calculus and algebra demonstrate that a power series can be used to approximate a function of x. Our goal is to use the tools already developed for series to help us count objects. Our purpose is not to present a complete development of generating functions, but rather to show the reader already familiar with series how they can be used for counting.

Suppose we consider the question of the number of binary trees T possible on a set of n vertices. Let's call this number b_n. Clearly, if $n = 1$, $b_1 = 1$. For $n > 1$, we can select one vertex to be the root, and the remaining $n - 1$ vertices can be partitioned into those in the left and right subtrees of T. If there are j vertices in the left subtree and $n - 1 - j$ vertices in the right subtree, then the number of binary trees we can form on n vertices depends on the number of ways we can build the left and right subtrees; that is b_n depends on b_j and b_{n-1-j}. To determine exactly how many binary trees b_n there are, we must sum the products $b_j b_{n-1-j}$ for $j = 0, 1, \ldots, n - 1$. This gives us the following *recurrence relation:*

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0.$$

We now illustrate how to solve this recurrence relation using generating functions.

Suppose our generating function $g(x) = \sum_{i=0}^{\infty} b_i x^i$. When we square $g(x)$, we obtain the following:

$$(g(x))^2 = g(x) \times g(x) = \sum_{n \geq 0} \Big(\sum_{0 \leq j \leq n} b_j \; b_{n-j} \Big) x^n$$

$$= \sum_{n \geq 0} (b_0 b_n + b_1 b_{n-1} + b_2 b_{n-2} + \cdots + b_n b_0) \, x^n.$$

But on careful examination we see that the coefficient of x^n in $g^2(x)$ is nothing but b_{n+1}. Hence, we obtain another relationship, namely

$$1 + x g^2(x) = g(x).$$

But this equation is quadratic in $g(x)$, and so it yields the solution

$$g(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Now, to obtain a series expansion for $g(x)$, we use the binomial generating function

$$(1 + z)^r = 1 + rz + r\left(\frac{r-1}{2}\right)z^2 + r\left(\frac{(r-1)(r-2)}{6}\right)z^3 + \cdots$$

We write the coefficients of this power series using the definition of generalized binomial coefficients, where for any real number r and integer k,

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}, \text{for } k > 0,$$

while it has value zero if $k < 0$ and 1 when $k = 0$. Thus, we can rewrite the binomial generating function as

$$(1 + z)^r = \sum_{k\geq 0} \binom{r}{k} z^k.$$

Substituting this function in our expression for $g(x)$, we obtain

$$g(x) = \frac{1}{2x}\left(1 - \sum_{k\geq 0} \binom{½}{k} (-4x)^k\right).$$

Changing the dummy variable k to $n + 1$ and simplifying yields

$$\begin{aligned} g(x) &= \frac{1}{2x}\left(1 - \sum_{n+1\geq 0} \binom{½}{n+1} (-4x)^{n+1}\right) \\ &= \frac{1}{2x} + \sum_{n+1\geq 0} \binom{½}{n+1} (-1)^n 2^{2n+1} x^n \\ &= \sum_{n\geq 0} \binom{½}{n+1} (-1)^n 2^{2n+1} x^n. \end{aligned}$$

But this process yields coefficients of x^n which match b_n in our original definition of $g(x)$. Thus, we have that,

$$b_n = \binom{½}{n+1} (-1)^n 2^{2n+1}.$$

Using exercise 26, we can simplify this expression for b_n to obtain:

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

Exercises

1. Show that if $T = (V, E)$ is a tree, then for any $e \in E$, $T - e$ has exactly two components.
2. Show that any connected graph on p vertices contains at least $p - 1$ edges.
3. Show that if T is a tree with $\Delta(T) \geq k$, then T has at least k leaves.
4. In a connected graph G, a vertex v is called *central* if $\max_{u \in V(G)} d(u, v) = rad(G)$. Show that for a tree T, the set of central vertices consists of either one vertex or two adjacent vertices.
5. Show that the sequence $d_1, d_2, \ldots, d_p$ of positive integers is the degree sequence of a tree if, and only if, the graph is connected and $\sum_{i=1}^{p} d_i = 2(p - 1)$.
6. Show that the number of end vertices in a nontrivial tree of order n equals $2 + \sum_{deg\ v_i \geq 3} (deg\ v_i - 2)$.
7. Determine the time complexity of Kruskal's algorithm.
8. What happens to the time complexity of Kruskal's Algorithm if we do not presort the edges in nondecreasing order of weight?
9. Apply Kruskal's algorithm to the graph: