

# Chapter 8

## Independence

### Section 8.1 Vertex Independence and Coverings

Next, we consider a problem that strikes close to home for us all, final exams. At the end of each term, students are required to take final exams in each of their classes. Each exam is to be given once during some specified period, and the time allowed for each exam (no matter what the class) is the same. The question of interest is: What is the minimum number of examination periods needed to ensure there are no conflicts, that is, that no student has two exams during the same period. Of course, as you well know, this is a completely fictional problem since no school has ever tried to determine this number.

As usual, we desire a graph model for this problem. Thus, we seek a graph  $G = (V, E)$  where each vertex of  $V$  represents an examination and  $xy \in E$  if, and only if, there is some student that must take both examination  $x$  and examination  $y$ . Two examinations can be scheduled in the same period only if there is no edge between the corresponding vertices in our model. Thus, we seek sets of mutually nonadjacent vertices in  $G$ ; that is, we seek independent sets of vertices. A solution to our problem is a partitioning of  $V$  into sets of mutually independent vertices where the number of such sets is a minimum. Vertices in the same set of this partition represent exams that can be scheduled during the same period without conflict. Thus, it is clear that we need to study independence if we are to solve this scheduling problem.

There are several ideas that are related to vertex independence that will be helpful. We have already studied one such idea, namely matchings. In our study, we sought independent sets of edges; now we seek independent sets of vertices. Another useful and related notion was also introduced in Chapter 7, the idea of a covering. In Chapter 7 we also saw that there is a relation between independent edges and coverings. In this section we wish to show a similar relationship between independent vertices and coverings. It is easy to see that given any independent set  $I$  in  $V$ , the vertices of  $V - I$  form a covering of  $G$ . Conversely, if  $V - I$  forms a covering, then  $\langle I \rangle$  must be empty; hence,  $I$  must be independent. Thus, we have shown the following useful result.

**Proposition 8.1.1** In a graph  $G = (V, E)$ , a subset  $I$  of  $V$  is independent if, and only if,  $V - I$  is a covering of  $G$ .

An independent set in a graph  $G$  is called a *maximum* independent set provided no other independent set in  $G$  has larger cardinality; it is called *maximal* if it is contained in no larger independent set. Recall that the number of vertices in a maximum independent set in  $G$  is called the *independence number* of  $G$  and is denoted  $\beta(G)$ . Analogously, the number of vertices in a minimum covering of a graph  $G$  is called the *covering number* of

$G$  and is denoted by  $\alpha(G)$ . Natural analogs of the independence number and covering number also exist for edges. The *edge independence number*, denoted  $\beta_1(G)$ , is the size of a maximum matching in  $G$ , and the *edge covering number*, denoted  $\alpha_1(G)$ , is the minimum size of a set  $L$  of edges with the property that every vertex is an end vertex of some edge in  $L$ .

In Chapter 7, we proved a result from König and Egerváry (Theorem 7.1.3) that showed that in a bipartite graph,  $\beta_1(G) = \alpha(G)$ . Our next result, from Gallai [11], shows several other relations among these parameters.

**Theorem 8.1.1** If  $G$  is a graph of order  $p$  with  $\delta(G) > 0$ , then

$$\begin{aligned}\alpha(G) + \beta(G) &= p \quad \text{and} \\ \alpha_1(G) + \beta_1(G) &= p.\end{aligned}$$

**Proof.** In order to establish the first equality, let  $I$  be an independent set of vertices in  $G$  with  $|I| = \beta(G)$ . Since  $I$  is independent,  $V - I$  is a cover of  $G$ . Therefore,

$$\alpha(G) \leq |V - I| \leq p - \beta(G).$$

If  $C$  is a set of vertices that covers  $E(G)$  and  $|C| = \alpha(G)$ , then  $V - C$  is an independent set of vertices, and so

$$\beta(G) \geq |V - C| \geq p - \alpha(G).$$

But these two inequalities together show that  $\alpha(G) + \beta(G) = p$ .

The proof of the second equality is left as an exercise.  $\square$

With the aid of the last result, we can now prove a theorem that looks very much like the König-Egerváry theorem.

**Theorem 8.1.2** If  $G$  is a bipartite graph with  $\delta(G) > 0$ , then  $\beta(G) = \alpha_1(G)$ .

**Proof.** Let  $G$  be a bipartite graph with  $\delta(G) > 0$ . By Gallai's theorem (Theorem 8.1.1),

$$\alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G).$$

However, by the König-Egerváry theorem,  $\beta_1(G) = \alpha(G)$ ; thus,  $\beta(G) = \alpha_1(G)$ .  $\square$

## Section 8.2 Vertex Colorings

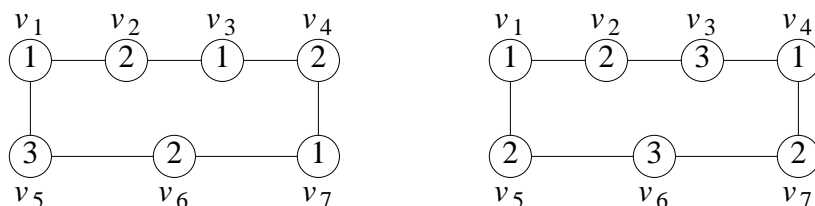
Recall that we wish to partition the vertices of a graph into independent sets in such a way that we minimize the number of these sets. This problem is usually described in a more visual manner. We say that an assignment of colors to the vertices of a graph  $G$  (one color per vertex) so that adjacent vertices are assigned different colors is a (legal) *coloring* of  $G$ . Colorings always exist since we can assign each vertex a different color if necessary. In a given coloring of a graph  $G$ , the set of all those vertices assigned the same color is called a *color class*. Clearly, a coloring of  $G$  produces a partition of  $V(G)$  into color classes, and each of the color classes is an independent set of vertices. A coloring that uses  $n$  colors is called an  $n$ -*coloring*, and a graph whose vertices can be colored with  $n$  or fewer colors is called  $n$ -*colorable*. The minimum number of colors in a coloring of  $G$ , where the minimum is taken over all colorings of  $G$ , is called the *chromatic number* of  $G$  and is denoted by  $\chi(G)$ . If  $G$  is a graph for which  $\chi(G) = n$ , then we say  $G$  is  $n$ -*chromatic*.

Note that in most cases, there is nothing unique about colorings. By this we mean that color classes may easily vary according to the particular coloring at hand. For example, suppose we consider the two colorings of  $C_7$  shown in Figure 8.2.1. We obtain very different color classes for each of these colorings. The first coloring has color classes

$$\{ v_1, v_3, v_7 \} \quad , \quad \{ v_2, v_4, v_6 \} \quad \text{and} \quad \{ v_5 \}.$$

The second coloring yields color classes of

$$\{ v_1, v_4 \}, \quad \{ v_2, v_5, v_7 \} \quad \text{and} \quad \{ v_3, v_6 \}.$$



**Figure 8.2.1.** Different color classes formed from different colorings.

For several well-known classes of graphs, chromatic numbers are readily determined. For example,

$$\chi(C_{2p}) = 2, \quad \chi(C_{2p+1}) = 3, \quad \text{and} \quad \chi(K_p) = p.$$

Further, it is also easy to show that  $\chi(K_{p_1, p_2, \dots, p_n}) = n$ . In general, if  $G$  is a  $k$ -partite graph, then  $\chi(G) \leq k$ . Thus, if  $G$  is a 2-chromatic graph, then  $G$  must be bipartite.

Despite the fact that a great deal of effort has gone into the study of colorings, there is no known general formula or method for finding the chromatic number of a graph. Thus, we can only hope for general bounds or formulas in certain special cases. In the study of graph colorings, certain types of graphs are often helpful. A graph  $G$  is *critically  $n$ -chromatic* or simply,  *$n$ -critical*, (if the context of coloring is clear), if  $\chi(G) = n$  and  $\chi(G - x) = n - 1$  for every vertex  $x$  in  $G$ . Similarly, we say  $G$  is *minimally  $n$ -chromatic*, or simply  *$n$ -minimal*, if  $\chi(G) = n$  and  $\chi(G - e) = n - 1$  for every edge  $e$  in  $G$ . Dirac [7] began the investigation of critically  $n$ -chromatic graphs. Several of his results are structural in nature and will be useful in our study of chromatic numbers of graphs. Our goal is to use this structural information to establish the sharpest possible general upper bound on the chromatic number of a graph. We begin with a look at degrees in critical graphs.

**Theorem 8.2.1** If  $G$  is a critically  $n$ -chromatic graph, then  $\delta(G) \geq n - 1$ .

**Proof.** Suppose that this is not the case; that is, let  $G$  be a critically  $n$ -chromatic graph with  $\delta(G) < n - 1$ . We will produce a coloring of  $G$  using fewer than  $n$  colors. To do this, let  $v$  be a vertex of degree  $\delta(G)$ . Since  $G$  is critically  $n$ -chromatic,  $G - v$  is  $(n - 1)$ -colorable. Color the vertices of  $G - v$  with  $n - 1$  colors, and let  $V_1, V_2, \dots, V_{n-1}$  be the corresponding color classes. Since  $\deg v = \delta(G) < n - 1$ , there must exist a color class  $V_i$  with the property that  $v$  is nonadjacent with every vertex in  $V_i$ . Thus,  $v$  can be assigned color  $i$ , producing an  $n - 1$  coloring of  $G$  and the desired contradiction.  $\square$

There are several useful facts that follow directly from Theorem 8.2.1. These are stated next, with the proofs left for the exercises.

**Corollary 8.2.1**

1. Every  $n$ -chromatic graph has at least  $n$  vertices of degree at least  $n - 1$ .
2. For any graph  $G$ ,  $\chi(G) \leq \Delta(G) + 1$ .

We have established an upper bound on  $\chi(G)$ , but with a little more effort, we can improve upon this bound. Thus, we continue our investigation of the structure of critically  $n$ -chromatic graphs by examining the nature of vertex cut sets in critical graphs. In order to do this, we adopt the following notation. Let  $S$  be a vertex cut set in a connected graph  $G$ . Let the components of  $G - S$  have vertex sets  $V_1, V_2, \dots, V_t$ . Then the subgraphs  $G_i = \langle V_i \cup S \rangle$  are called the  *$S$ -components* of  $G$ . Moreover, we say that colorings of  $G_1, G_2, \dots, G_t$  *agree on  $S$*  if each vertex of  $S$  is assigned the same color in each of the colorings of the  $G_i$  ( $i = 1, \dots, t$ ).

**Theorem 8.2.2** If  $G$  is a critically  $n$ -chromatic graph ( $n \geq 4$ ), then no vertex cut set

induces a complete graph and, hence,  $G$  must be 2-connected.

**Proof.** We proceed by contradiction. Let  $G$  be a critically  $n$ -chromatic graph and suppose that  $G$  has a vertex cut set  $S$  that induces a complete graph. Denote the  $S$ -components of  $G$  by  $G_1, G_2, \dots, G_k$ . Since  $G$  is critically  $n$ -chromatic, each  $G_i$  is  $(n - 1)$ -colorable. Furthermore, since  $\langle S \rangle$  is complete, the vertices of  $S$  must receive different colors in any  $(n - 1)$ -coloring of some  $G_i$ . By permuting colors in  $G_2, \dots, G_k$  we see that there are  $(n - 1)$ -colorings of  $G_1, G_2, \dots, G_k$  that agree on  $S$ . These colorings together produce an  $(n - 1)$ -coloring of  $G$ , which contradicts the fact that  $G$  is  $n$ -critical. Therefore, no vertex cut set of  $G$  induces a complete graph.

Now, suppose that  $\{v\}$  is a vertex cut set of  $G$ . Then clearly  $\langle v \rangle$  is a complete graph. But we just showed that no critical graph could have a cut set that induced a complete graph and, hence,  $\{v\}$  cannot be a cut set. That is,  $G$  must be 2-connected.  $\square$

We continue the investigation of critical graphs with a look at their edge connectivity. The next result is again from the work of Dirac [7].

**Theorem 8.2.3** Every critically  $n$ -chromatic graph ( $n \geq 2$ ) is  $(n - 1)$ -edge connected.

**Proof.** Suppose that  $G$  is a critically  $n$ -chromatic graph ( $n \geq 2$ ). If  $n = 2$ , then  $G$  is  $K_2$ , while if  $n = 3$ , then  $G$  is an odd cycle. Thus,  $G$  is 1-edge or 2-edge connected, respectively.

Now, we assume that  $n \geq 4$  and that  $G$  is not  $(n - 1)$ -edge connected. Thus, there must exist a partition of  $V(G)$  into subsets  $W_1$  and  $W_2$  such that there are fewer than  $n - 1$  edges joining  $W_1$  and  $W_2$ . Call the set of these edges  $E_W$ . Since  $G$  is critically  $n$ -chromatic, we know that  $\langle W_1 \rangle$  and  $\langle W_2 \rangle$  are both  $(n - 1)$ -colorable. Let them both be colored with  $(n - 1)$  colors. If the edges in  $E_W$  are all incident to vertices assigned different colors, then we have an  $(n - 1)$ -coloring of  $G$ , a contradiction. Suppose this does not happen. Our strategy now is to permute colors so that the edges of  $E_W$  do have end vertices assigned different colors, producing a contradiction.

Let  $V_1, \dots, V_k$  be the color classes of  $\langle W_1 \rangle$  with at least one edge to  $\langle W_2 \rangle$ . Further suppose that there are  $q_i$  edges from  $V_i$  to  $\langle W_2 \rangle$ . Thus, we see from our assumptions that  $\sum_{i=1}^k q_i \leq n - 2$ .

We now try to permute colors to obtain the desired coloring. If each vertex  $v_1 \in V_1$  is adjacent only with vertices of  $W_2$  with different colors, then we do nothing. If however, there is some  $v_1$  that is adjacent to some vertex of  $W_2$  of the same color, then in  $\langle W_1 \rangle$  we permute the colors so that no vertex of  $V_1$  is adjacent to a vertex of  $W_2$  having the same color. This is possible since the vertices of  $V_1$  may be assigned any one of at least  $n - 1 - q_1 (> 0)$  colors.

Now, with this new coloring, if each vertex  $v_2 \in V_2$  is adjacent only to vertices in  $W_2$  assigned different colors, then again we do nothing. But, if some vertex  $v_2 \in V_2$  is adjacent to a vertex of  $W_2$  assigned the same color, then in  $W_1$  we again permute the  $n - 1$  colors, leaving the color assigned to  $V_1$  fixed, until no vertex in  $V_1 \cup V_2$  is adjacent to a vertex in  $W_2$  having the same color. This is possible since the vertices of  $V_2$  can be assigned any of  $(n - 1) - (q_2 + q_1)$  colors and this value is greater than zero. Continuing this process, we arrive at an  $(n - 1)$ -coloring of  $G$  and the desired contradiction.  $\square$

Analogous to the idea of a critical graph is the edge concept of a minimally  $n$ -chromatic graph. Since every connected, minimally  $n$ -chromatic graph is critically  $n$ -chromatic, there is a strong relationship between these two ideas. This relationship will aid us in our study of the structure of critical graphs. Theorems 8.2.3 and 8.2.1 have the following immediate Corollary.

### Corollary 8.2.2

1. If  $G$  is a connected,  $n$ -minimal graph ( $n \geq 2$ ), then  $G$  is  $(n - 1)$ -edge connected.
2. If  $G$  is  $n$ -critical or connected and  $n$ -minimal, then  $\delta(G) \geq n - 1$ .

We are now ready to complete our study of the structure of critical and minimal graphs. If an  $n$ -critical graph  $G$  has a two vertex cut set  $\{u, v\}$ , then we know  $u$  and  $v$  cannot be adjacent. We say that an  $S = \{u, v\}$ -component  $H$  of  $G$  is *color-unique* if every  $(n - 1)$ -coloring of  $H$  assigns the same color to both  $u$  and  $v$  and that it is *color-distinct* if every  $(n - 1)$ -coloring of  $H$  assigns different colors to  $u$  and  $v$ . The following result is again from Dirac [8].

**Theorem 8.2.4** Let  $G$  be an  $n$ -critical graph with a two vertex cut set  $S = \{u, v\}$ . Then:

1.  $G = H_1 \cup H_2$ , where  $H_1$  is a color-unique  $S$ -component and  $H_2$  is a color-distinct  $S$ -component.
2. Both  $H_1 + uv$  and the graph obtained from  $H_2$  by identifying  $u$  and  $v$  are  $n$ -critical.

**Proof.** Let  $G$  be an  $n$ -critical graph with a two vertex cut set  $S = \{u, v\}$ . Then, since  $G$  is critical, each  $S$ -component of  $G$  is  $(n - 1)$ -colorable. There exists no  $(n - 1)$ -colorings of these  $S$ -components which all agree on  $S$  or else there would be an  $(n - 1)$ -coloring of  $G$ . Thus, there are two  $S$ -components, say  $H_1$  and  $H_2$ , such that no  $(n - 1)$ -coloring of  $H_1$  agrees with any  $(n - 1)$ -coloring of  $H_2$ . Clearly, then, one component must be color-unique and the other color-distinct. Further, if there were more  $S$ -components, then some two would agree, and then deleting one of these would

contradict the fact  $G$  is  $n$ -critical. Without loss of generality, say  $H_1$  is color-unique. Since  $H_1$  and  $H_2$  are of different types, the subgraph  $H_1 \cup H_2$  is not  $(n - 1)$ -colorable. Thus, since  $G$  is  $n$ -critical, we must have that  $G = H_1 \cup H_2$ .

A proof of (2) is left to the exercises.  $\square$

**Corollary 8.2.3** Let  $G$  be an  $n$ -critical graph with a two vertex cut set  $\{ u, v \}$ . Then

$$\deg u + \deg v \geq 3n - 5.$$

The proof of the corollary is also left to the exercises.

We conclude this section with the most fundamental result dealing with vertex colorings. This theorem is from Brooks [5], and it provides the general upper bound on the chromatic number we have been seeking.

**Theorem 8.2.5** If  $G$  is a connected graph that is neither an odd cycle nor a complete graph, then  $\chi(G) \leq \Delta(G)$ .

**Proof.** Let  $G$  be a connected  $n$ -chromatic graph which is neither an odd cycle nor a complete graph. Without loss of generality, we may assume that  $G$  is  $n$ -critical. By Theorem 8.2.2,  $G$  is 2-connected. Further, since 1-critical and 2-critical graphs are complete and 3-critical graphs are odd cycles (see the exercises), we must have that  $n \geq 4$ .

If  $G$  has a 2-vertex cut set, say  $\{ u, v \}$ , then Corollary 8.2.3 implies that

$$2\Delta(G) \geq \deg u + \deg v \geq 3n - 5 \geq 2n - 1.$$

But this implies that  $\chi(G) = n \leq \Delta(G)$  since since  $2\Delta(G)$  is even.

Now assume that  $G$  is a 3-connected graph. Since  $G$  is not complete, there are three vertices,  $u, v$ , and  $w$ , in  $G$  such that both  $uv$  and  $vw$  are edges of  $G$  but  $uw$  is not an edge of  $G$ . Let  $u = v_1$  and  $w = v_2$  and let  $v_3, \dots, v_p = v$  be an ordering of the vertices of  $G - \{ u, w \}$  with the property that each  $v_i$  is adjacent to some  $v_j$  where  $j > i$ . This ordering can be accomplished by arranging the vertices in nonincreasing order of their distances from  $v$  in  $G - \{ u, w \}$ .

We wish to color the vertices of  $G$  using at most  $\Delta(G)$  colors. To do this, assign both  $v_1$  and  $v_2$  color 1. Then, successively color  $v_3, v_4, \dots, v_p$  with the smallest available color in the numerically ordered colors  $1, 2, \dots, \Delta(G)$ . From our construction of the ordering on the vertices, each vertex  $v_i$  ( $i \geq 3$ ), except  $v_p$ , is adjacent to some vertex  $v_j$  where  $j > i$ . Therefore,  $v_i$  is adjacent to at most  $\Delta(G) - 1$  vertices preceding it in the vertex ordering. Thus, when  $v_i$  is to be colored (in its turn according to our listing), there will be a color available to assign to it. Also, since  $v_p$  is adjacent to two vertices colored 1, then it will also have a color available when it is time for it to be colored. Thus, we

have colored the vertices of  $G$  with at most  $\Delta(G)$  colors, completing the proof.  $\square$

Can you find examples of graphs for which the bound in Brooks's theorem is sharp? Can you find examples for which the bound from Brooks's theorem is arbitrarily bad, that is, for which the difference between the actual chromatic number and the bound grows larger as the order grows larger?

### Section 8.3 Approximate Coloring Algorithms

The general question of determining the chromatic number of a graph is another NP-complete problem (see [12]). Some exhaustive search algorithms have been developed (for example, see [6] or [4]). These algorithms will find the chromatic number of very small graphs effectively. However, in most practical cases, we are interested in finding the chromatic number of large graphs, certainly of order 100 or more and often of order 1000 or more. Thus, we again turn to a variety of heuristics to help us develop approximation algorithms providing good bounds.

The first (and usually worst) heuristic one thinks of involves the greedy approach. The idea is simply to color the vertices, one by one, as they are encountered using any available color, that is, any color not already assigned to a neighboring vertex. Unfortunately, as you might imagine, this algorithm can be very bad. In fact, this approach can provide a very poor bound on the chromatic number.

Manvel [17] described several heuristics typical of those motivating most approximation algorithms. We state these ideas for later reference.

1. A vertex of high degree is harder to color than a vertex of low degree.
2. Vertices with the same neighborhood should be colored alike.
3. Coloring many vertices with the same color is a good idea.

The algorithms we are about to examine are motivated by some combination of these ideas. We can also classify them as falling into two fundamental categories. The first group of algorithms can be thought of as being in the category of sequentially based methods, where the order in which the vertices will be colored is decided before we begin to color them. This technique is essentially the technique we used in the proof of Brooks's theorem. Given any ordering of the vertices of a graph, sequential coloring algorithms usually try to assign the minimum color possible to the next vertex. That is, if we are to color vertex  $v$ , then having ordered the colors numerically, we assign  $v$  the smallest color according to this ordering that does not appear in  $N(v)$ . We formally state the generic sequential coloring algorithm next.

#### Algorithm 8.3.1 Generic Sequential Coloring Algorithm.

**Input:** Any ordering of the vertices of a graph  $G$ .  
**Output:** A coloring of the vertices.  
**Method:** Use the minimum available color.

1. Assign color 1 to vertex  $v_1$ .
2. If  $H_{i-1} = \langle v_1, \dots, v_{i-1} \rangle$  has been colored with  $j$  colors, then assign  $v_i$  color  $k$ , where  $k \leq j + 1$  is the minimum available color (according to some numerical ordering of the colors, say  $1, 2, \dots, n$ ).

Heuristic 1 is the prime motivation for the *largest first* heuristic, which orders the vertices in descending order based on their degrees (hence, it is a sequential coloring algorithm). The vertex of highest degree is colored first, the vertex of next highest degree second, and so forth in a greedy manner. In each case, the color selected for the vertex is the smallest possible legal color. Thus, a fast and simple method for coloring vertices is available. In fact, the largest first heuristic generally provides a reasonable bound on the chromatic number of small-order graphs.

Using the largest first heuristic and the sequential coloring algorithm, Welsh and Powell [22] obtained the following max-min result.

**Theorem 8.3.1** Let  $G$  be a graph with  $V(G) = \{ v_1, \dots, v_n \}$  and where  $\deg v_i \geq \deg v_{i+1}$  for  $i = 1, \dots, n - 1$ . Then

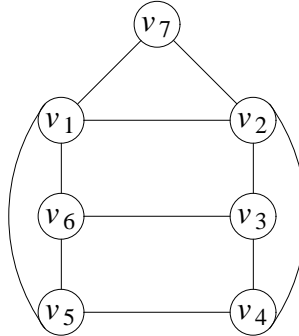
$$\chi(G) \leq \max_i \min \{ i, \deg v_i + 1 \}.$$

Many variations on heuristic 1 are also possible. A reversal in strategy provides us with a somewhat more effective use of the same basic idea. This is known as the *smallest last* algorithm of Matula, Marble and Isaacson [18]. In this algorithm we again want to determine an ordering of the vertices, this time based on recursively examining the vertices of smallest degree and removing them from the graph. The order of removal of the vertices is the reverse of the order in which they are to be colored. Thus, we select the vertex of lowest degree in the graph and remove it from the graph, effectively placing it as the last vertex on the list to be colored. In the subgraph that remains, we repeat the process, again selecting the vertex of smallest degree and removing it (hence, it will be the next to last vertex colored). We continue in this manner until all vertices have been ordered. Next, we again sequentially color this listing in a greedy manner. You should note that this is not necessarily the same as selecting the vertices of minimum degree in the graph itself; hence, this is potentially a different ordering from the largest first ordering.

The fundamental difference between the smallest last and largest first heuristics is their views of the graph. The ordering in the smallest last method is based on the degrees in the subgraphs obtained by removing the vertices of smallest degree, rather than simply on the degrees in the graph itself. Thus, the ordering of vertices obtained via each of

these methods may well be different. Tests by Brelaz [4] show that, typically, the smallest last method is somewhat better than the largest first method.

**Example 8.3.1.** Suppose we apply each of our heuristics to the graph of Figure 8.3.1.



**Figure 8.3.1.** A graph to test coloring heuristics.

Given the arbitrary ordering of  $v_1, v_2, v_4, v_3, v_5, v_6, v_7$ , the generic sequential coloring algorithm provides the following color assignments:

$$\begin{aligned} v_1 &\leftarrow 1, & v_2 &\leftarrow 2, & v_4 &\leftarrow 1, \\ v_3 &\leftarrow 3, & v_5 &\leftarrow 2, & v_6 &\leftarrow 4, & v_7 &\leftarrow 3. \end{aligned}$$

Hence, this arbitrary ordering provides a bound of 4 on the chromatic number of the graph.

If we use the largest first ordering of  $v_1, v_2, v_3, v_6, v_5, v_4, v_7$ , we obtain the following coloring:

$$\begin{aligned} v_1 &\leftarrow 1, & v_2 &\leftarrow 2, & v_3 &\leftarrow 1, & v_6 &\leftarrow 2, \\ v_5 &\leftarrow 3, & v_4 &\leftarrow 4, & v_7 &\leftarrow 3. \end{aligned}$$

Thus, we again obtain a bound of 4 on the chromatic number of the graph.

Finally, a smallest last ordering of  $v_1, v_5, v_6, v_4, v_3, v_2, v_7$  provides the following coloring:

$$\begin{aligned} v_1 &\leftarrow 1, & v_5 &\leftarrow 2, & v_6 &\leftarrow 3, & v_4 &\leftarrow 1, \\ v_3 &\leftarrow 2, & v_2 &\leftarrow 3, & v_7 &\leftarrow 2. \end{aligned}$$

Hence, this ordering provides a bound of 3, and since the graph contains a  $K_3$ , we see this is the value of the chromatic number.  $\square$

Note that the sequential ordering used in each of the above algorithms need not be unique for the particular graph and that a different ordering used in the same algorithm can provide a different value. Can you find a legal largest first ordering that also provides a bound of 3 for the chromatic number of this graph?

In each of the previous examples, we have implicitly broken ties by a random selection of vertices, since the only information we were using was the degree of the vertex itself. Certainly, more involved heuristic tests can be applied. For example, suppose we try a two-step approach to constructing our vertex ordering. That is, let's include a bit more information than merely the degrees of the vertices. Suppose that we include the sum of the degrees of all neighbors of a vertex as well. The idea is that a vertex of high degree whose neighbors together also have high degree sum would possibly present us with problems later. Certainly, you can devise other such tests for deciding on the vertex ordering and any of these tests can also be used to break ties.

One of the best known two-step heuristics for fairly small graphs (up to about 100 vertices) is from Breaz [4]. It is motivated by a combination of heuristics 1 and 2. Define the *color-degree* of a vertex  $v$  to be the number of colors used to color the vertices adjacent to  $v$ . A sequential order of the vertices is then decided primarily by the color-degree, with ties being broken by selecting the vertex with largest degree in the uncolored subgraph.

**Algorithm 8.3.2 Breaz Color-Degree Algorithm.**

**Input:** A graph  $G$ .

**Output:** An approximate coloring of the vertices of  $G$ .

**Method:** Break ties based on the smallest color-degree.

1. Order the vertices in decreasing order of degrees.
2. Color a vertex of largest degree with color 1.
3. Select a vertex with maximum color-degree. If there is a tie, choose any of these vertices of largest degree in the uncolored subgraph.
4. Color the vertex selected in step 3 with the least possible color.
5. If all vertices are colored,  
    then stop;  
    else go to step 3.

We can prove that there is an instance when this algorithm provides the chromatic number exactly.

**Theorem 8.3.2** If  $G$  is a 2-connected bipartite graph of order at least 3, then the coloring obtained from Algorithm 8.3.2 determines the chromatic number for  $G$ .

**Proof.** Let  $G$  be a 2-connected bipartite graph of order at least 3 and suppose that  $G$  has been colored by Algorithm 8.3.2. Assume that vertex  $x$  has color-degree 2. In this case, assume that it has two neighbors with different colors. Now, using these two colors, construct color alternating paths from these vertices. Since  $G$  is finite, a cycle must be

formed. Since  $G$  is bipartite, this cycle must be even, and the neighbors of  $x$  must have the same color, contradicting our assumption.  $\square$

**Example 8.3.2.** Suppose we perform Algorithm 8.3.2 on the graph of our previous example. Initially, all vertices have color-degree zero; hence, we first select the vertex of highest degree, say  $v_1$ , and we assign  $v_1$  the color 1. Since  $v_2$ ,  $v_6$  and  $v_7$  now have color-degree 1, select  $v_2$  since it has the largest uncolored degree (in this case 3). Then  $v_2$  is assigned the color 2. Now,  $v_7$  is the only vertex with color-degree 2, so it is selected and assigned color 3. All the remaining vertices have color-degree 1 and uncolored degree 2, so we randomly select  $v_3$  and assign it color 1. Next,  $v_4$  has color-degree 2, and it is then assigned color 3. This is followed by assigning  $v_5$  color 2 and  $v_6$  color 3. Thus, the bound obtained from this algorithm is 3, which we already know is the chromatic number of the graph.  $\square$

Brelaz [4] conducted some tests on his algorithm and compared the results with the smallest last heuristic, among others. Without increasing the level of sophistication of these algorithms, he found that Algorithm 8.3.2 was generally the best. However, since we are never, or at least rarely, content, let's increase the level of sophistication somewhat.

Suppose that a graph  $G$  is colored with  $k$  colors. Let  $J_1, \dots, J_k$  be the color classes determined by this coloring. Then, if we consider the graph induced by the union of any two of these color classes, say  $\langle J_i \cup J_j \rangle$ , we see that it may not be connected. We term any component of  $\langle J_i \cup J_j \rangle$  an  *$i$ - $j$  component*. If the sets  $J_i$  and  $J_j$  are interchanged, that is, the vertices in  $J_i$  are recolored  $j$  and the vertices of  $J_j$  are recolored  $i$ , then we say we have performed an  *$i$ - $j$  interchange*. Clearly, the graph  $G$  is still  $k$ -colored. At times though, we can gain some flexibility after performing an interchange. The following algorithm introduces interchanges into our sequentially based methods.

**Algorithm 8.3.3 Interchange Coloring Algorithm.**

**Input:** A sequential ordering on the vertices of  $G$ .

**Output:** A coloring of  $G$ .

**Method:** We try to perform interchanges before using additional colors.

1. Assign  $v_1$  the color 1.
2. If  $H_{i-1} = \langle v_1, \dots, v_{i-1} \rangle$  has been colored with  $j$  colors, and if  $m$  is the least color not occurring on a neighbor of  $v_i$  in  $H_{i-1}$ , then
  - a. if  $m \leq j$ , then assign  $v_i$  color  $m$ ;
  - b. if  $m = j + 1$ , then let  $C^1$  be the set of colors that occur on exactly one vertex in  $N_{H_{i-1}}(v_i)$ .

If some distinct pair  $b, c \in C^1$  has a  $b, c$ -component of  $H_{i-1}$  with only one

neighbor of  $v_i$  in  $H_i$ , then perform a  $b - c$  -interchange on one such component of  $H_{i-1}$ . Now, color  $v_i$  with the available color, producing a  $j$  coloring of  $H_i$ . If no such interchange is possible, color  $v_i$  with color  $j + 1$ , and, hence,  $j + 1$  coloring  $H_i$ .

This color interchange approach can be applied in combination with any of the sequential ordering algorithms we have seen. Brelaz's [4] testing showed that improvements could be made when such an enhancement was incorporated into any of these algorithms.

In order to motivate the second category of coloring algorithms, we need to speculate about the reasons for our failure to be able to color graphs exactly. One reason might be the difference in the way in which the chromatic number seems to be determined in large graphs as opposed to small graphs. There appear to be two different lower bounds that drive up the chromatic number. It is certainly the case that  $\chi(G) \geq \omega(G)$  (the *clique number*, that is, the order of the largest complete subgraph of  $G$ ). It seems that for small graphs,  $\chi(G)$  and  $\omega(G)$  tend to be very close. On the other hand, no color class can contain more than  $\beta(G)$  vertices, so it is also clear that  $\chi(G) \geq \frac{|V(G)|}{\beta(G)}$ . For small graphs, this lower bound tends to be much less than  $\chi(G)$ , while for large graphs, this lower bound seems to be much better than  $\omega(G)$ .

Matula has calculated the expected value of  $\beta(G)$  for random graphs with edge probability 0.5. His estimates seem to predict the value of  $\beta(G)$  very well and suggest the following approach. Based on the order and edge density of  $G$ , locate an independent set with the expected number of vertices. Now, delete this set and in the graph that remains, repeat this process. Continue until all vertices are colored.

Johri and Matula [14] have produced a variety of algorithms based on this approach. We shall briefly describe their attack. In small graphs we can carry out an exhaustive search for the desired independent sets and perform the algorithm basically as described. However, in large graphs this is not practical. Thus, they turned to a two-step approach. The first step is to find an independent set of vertices, say  $I_1$ , that is fairly large with respect to the the desired size. That is,  $I_1$  is within some tolerance  $t_1$  of the expected value of  $\beta(G)$ . The second step is to search in the remaining collection of vertices with no adjacencies to  $I_1$  for the largest expected independent set, say  $I_2$ . The only catch here is that we do not want to drastically change the edge density by removing independent sets carelessly; hence,  $I_2$  is selected to cover as many edges as possible. Then,  $I_1 \cup I_2$  is deleted from the graph and is used as the next color class.

It is clear that this general method is very flexible and offers a great many variations for experimentation. Thus, with more complex testing and conditions, progressively slower versions can be created that color with progressively fewer colors.

Johri and Matula tried several variations of this algorithm. The first, called GE1, used random selection of the vertices in step 1 to find  $I_1$ . Their second algorithm, GE2, tried to increase the size of  $I_1 \cup I_2$  at each stage by sampling various sets. If  $I_1 \cup I_2$  was not as large as the desired independent set, then the step was repeated in an effort to find better candidates. After a certain number of failures, the expectation would be readjusted downward and the search would be continued. Their third algorithm, GE3, selects vertices to add to  $I_1$  by taking those whose neighbors have the largest average degree. This algorithm also reverts to a simple exhaustive search when 80 or fewer vertices remain to be colored.

The following table is extracted from their work. It is based on tests they performed on ten random graphs of order 1000 with edge probability 0.5. The time is in CPU seconds on a CDC 6600.

Algorithm Tested	Average No. of Colors	Average Time
Largest First	122.7	101.3
Smallest Last	124.3	126.7
Brelaz Algorithm	115.8	111.7
GE1	105.2	432.8
GE2	100.1	1128.2
GE3	95.9	3212.2

## Section 8.4 Edge Colorings

A natural analog to coloring the vertices of a graph is coloring the edges. We define the *edge chromatic number*, sometimes called the *chromatic index*, to be the least number of colors needed to color the edges of a graph  $G$  so that no two adjacent edges are assigned the same color. Denote the edge chromatic number of the graph  $G$  as  $\chi_1(G)$ . An immediate observation is that  $\chi_1(G) = \chi(L(G))$ . Another easy observation is that if  $G$  contains a vertex of degree  $k$ , then  $\chi_1(G) \geq k$ . Some examples of edge chromatic numbers of special classes of graphs are also easy to see. For instance,

$$\chi_1(C_p) = \begin{cases} 2 & \text{if } p \text{ is even,} \\ 3 & \text{if } p \text{ is odd.} \end{cases}$$

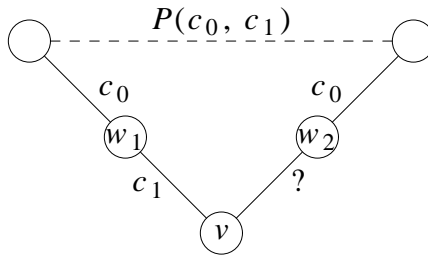
$$\chi_1(K_p) = \begin{cases} p - 1 & \text{if } p \text{ is even,} \\ p & \text{if } p \text{ is odd.} \end{cases}$$

It turns out that we can bound the edge chromatic number fairly tightly for graphs and somewhat less effectively, but still reasonably, for multigraphs. We now turn our attention to developing these bounds. The bound for graphs is from Vizing [20].

**Theorem 8.4.1** If  $G$  is a graph, then  $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$ .

**Proof.** It is clear that for any graph  $G$ ,  $\chi_1(G) \geq \Delta(G)$ . Thus, we need only show that the upper bound also holds. To do this, we use induction on the number of edges in  $G$ . The result is clear if  $G$  has only one edge; thus, assume that all but one edge of  $G$  has been colored using at most  $\Delta(G) + 1$  colors. Say the remaining uncolored edge is  $e_1 = vw_1$ . There must be at least one color unused at  $v$  and at least one color unused at  $w_1$ . If the same color is unused at both vertices, then we color  $e_1$  with this color and we have the desired edge coloring of  $G$ . If this is not the case, then let  $c_0$  be an unused color at  $v$  and let  $c_1$  ( $\neq c_0$ ) be an unused color at  $w_1$ . We now consider a three-step algorithm that will complete the proof.

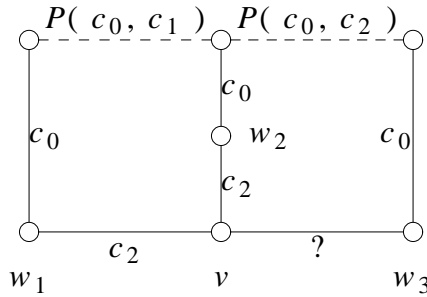
**Step 1.** Let  $e_2 = vw_2$  be the edge incident to  $v$  which has been assigned the color  $c_1$ . Since  $c_1$  was not used at  $w_1$ , we know that such an edge must exist. Remove this color from  $e_2$  and assign it instead to  $e_1$ . We may also assume that  $v$ ,  $w_1$  and  $w_2$  all belong to the same component induced by the edges colored  $c_0$  and  $c_1$  or else we could interchange the colors of the edges in the component containing  $w_2$  without changing the color of  $e_1$ . But if that were the case, we could color  $e_2$  with  $c_0$  and obtain a proper coloring of  $G$ . Let  $P(c_0, c_1)$  be the bicolored path joining  $w_1$  and  $w_2$  in this component (see Figure 8.4.1).



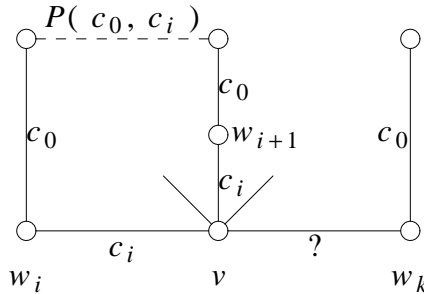
**Figure 8.4.1.** The configuration of step 1.

**Step 2.** Let  $c_2$  ( $\neq c_1$ ) be any unused color at  $w_2$ . We may assume that  $c_2$  is used at  $v$  or else we could complete the proof by coloring  $e_2$  with  $c_2$ . Thus, let  $e_3 = vw_3$  be the edge incident to  $v$  with color  $c_2$ . Then, we can remove color  $c_2$  from  $e_3$  and assign it to  $e_2$ . By the argument used in step 1, we may assume that  $v$ ,  $w_2$  and  $w_3$  all belong to the same two-color component of  $G$  induced by  $c_0$  and  $c_2$ . Let  $P(c_0, c_2)$  be the bicolored path joining  $w_2$  and  $w_3$  in this component (see Figure 8.4.2).

**Step 3.** If we repeat the procedure of step 2, we eventually reach a vertex  $w_k$  that is adjacent to  $v$ , but the edge  $vw_k$  is uncolored and some color  $c_i$  ( $i < k - 1$ ) is unused at  $w_k$ . Again, we may assume that  $v$ ,  $w_i$  and  $w_{i+1}$  all belong to the same two-color component  $H$  of  $G$  obtained by using  $c_0$  and  $c_i$ . Since  $c_0$  is missing at  $v$  and  $c_i$  is missing at  $w_{i+1}$ , then  $H$  must be a path from  $v$  to  $w_{i+1}$  that passes through  $w_i$  and consists entirely of edges alternately colored  $c_0$  and  $c_i$ . This path does not contain  $w_k$



**Figure 8.4.2.** Colorings in step 2.



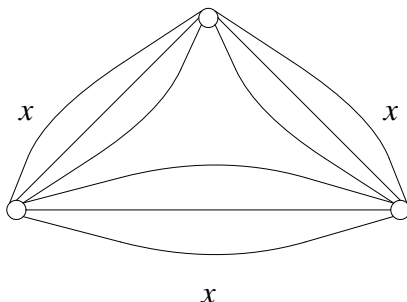
**Figure 8.4.3.** Step 3.

since  $c_i$  does not appear at  $w_k$  (see Figure 8.4.3). Thus, if  $H_k$  is the two-color component of  $G$  obtained by using  $c_0$  and  $c_i$  and containing the vertex  $w_k$ , then  $H_k$  and  $H$  must be disjoint. We can then interchange the colors of the edges in  $H_k$  and then color  $vw_k$  with  $c_0$ . This completes the proof.  $\square$

Our next two results provide upper bounds on the edge chromatic number of a multigraph. The first is from Vizing [21] and the second from Shannon [19]. Vizing's bound is sometimes better than the bound of Shannon. To state Vizing's result, let the *maximum multiplicity*  $m(G)$  be defined as the maximum number of edges joining any pair of vertices in a multigraph  $G$ . Vizing's bound for multigraphs can now be stated. It can also be viewed as a generalization of his bound for graphs since for graphs  $m(G) = 1$ .

**Theorem 8.4.2** If  $G$  is a multigraph, then  $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + m(G)$ .

**Example 8.4.1.** The upper bound in Theorem 8.4.2 is sharp. This can be seen from the multigraph of Figure 8.4.4. It has maximum degree  $2x$  and multiplicity  $x$ . Since any two edges are adjacent, we see that  $\chi_1$  must be  $3x = q$ .



**Figure 8.4.4.** Sharpness example for Theorem 8.4.2.

We now present Shannon's [19] bound for multigraphs. We shall use Vizing's generalization to aid in our proof.

**Theorem 8.4.3** If  $G$  is a multigraph, then  $\chi_1(G) \leq \frac{3}{2} \Delta(G)$ .

**Proof.** Let  $G$  be a multigraph with  $\chi_1(G) = k$ , where  $k > \frac{3}{2} \Delta(G)$ . By removing sufficient edges from  $G$ , we can obtain a minimal multigraph  $M$  from  $G$ . By Theorem 8.4.2, we know that  $\chi_1(M) \leq \Delta(M) + m(M)$ . Thus, there must be vertices  $v$  and  $w$  that are joined by at least  $k - \Delta(M) > \Delta(M)/2$  edges.

Now, color all the edges of  $M$  except one of the edges between  $v$  and  $w$ . Since  $M$  is minimal, this can be done using only  $k - 1$  colors. The number of colors unused at  $v$  or  $w$  (or both) cannot exceed

$$(k - 1) - (\Delta(M) - 1) \leq m(M)$$

since  $k \leq \Delta(M) + m(M)$ . But, the number of colors unused at  $v$  (or  $w$ ) is also at least

$$(k - 1) - (\Delta(M) - 1) = k - \Delta(M) > \Delta/2.$$

Then, we see that the number of colors unused at both  $v$  and  $w$  is at least

$$2(k - \Delta(M)) > 0.$$

By assigning one of these unused colors to the uncolored edge from  $v$  to  $w$  we obtain a coloring of  $M$  with only  $k - 1$  colors. But this contradicts the fact that  $\chi_1(M) = k$ , and the result is proved.  $\square$

Vizing's theorem (8.4.1) has set off a rather extensive study attempting to classify graphs according to their edge chromatic number. A graph  $G$  is said to be of *class 1* if  $\chi_1(G) = \Delta(G)$  and of *class 2* otherwise. From the examples we have seen, we know that  $K_{2n}$  is of class 1 and that  $K_{2n+1}$  is of class 2. However, the general problem of deciding which graphs are class 1 and which are class 2 (sometimes called the

classification problem) remains unsolved. Evidence exists to show that class 2 graphs are fairly rare. In fact, Erdős and Wilson [9] have shown that almost all graphs are class 1; that is, if  $Pr(n)$  is the probability that a random graph of order  $n$  is class 1, then  $Pr(n) \rightarrow 1$  as  $n \rightarrow \infty$ . However, it also seems natural to expect that the more edges a graph contains, the more likely it is to be in class 2. The following result from Beineke and Wilson [2] confirms this idea.

**Theorem 8.4.4** Let  $G$  be a  $(p, q)$  graph. If  $q > \Delta(G) \beta_1(G)$ , then  $G$  is of class 2.

**Proof.** If  $G$  is class 1, then any  $\Delta(G)$ -coloring of the edges of  $G$  partitions the edges into  $\Delta(G)$  independent sets. Since the number of edges in each such set cannot exceed  $\beta_1(G)$ , then  $q \leq \Delta(G) \beta_1(G)$ , a contradiction.  $\square$

Further results about the classification problem will be explored in the exercises.

## Section 8.5 The Four Color Theorem

The idea of coloring can be traced to Francis Guthrie. In 1852, while he was a student of Augustus De Morgan, Guthrie asked De Morgan to verify the "fact" that any map (consider it a map of countries if you wish) drawn in the plane could be colored with at most four colors, so that adjacent (that is, sharing a boundary) countries received different colors. De Morgan responded by saying he did not know that this was a "fact," and he proceeded to ask other mathematicians (like Hamilton) about this problem. Both Guthrie and De Morgan believed this statement was indeed a fact, yet neither could verify it.

If we consider a map drawn in the plane and insert a vertex in each country and join two vertices by an edge if the corresponding countries share a common boundary, then we have created a graph model of the map, and this model is easily seen to be planar. We created this model in much the same way as the geometric dual of a graph and, in fact, the graph model can be thought of as the dual of the map. The problem of coloring the countries of this map can then be stated as a graph-coloring problem. The "fact" that De Morgan and Guthrie could not prove can now be stated.

**The Four Color Conjecture** Every planar graph can be colored with four or fewer colors.

The four color conjecture was to become one of the most famous of all mathematical problems. It is sometimes called the four color disease. By this we mean that many good mathematicians spent a great deal of time (in fact, for some a lifetime) working on this problem without complete success. This problem has generated a strange history filled

with attempts at proofs, publication of incorrect proofs and, in general, a great deal of unrewarded efforts.

The first and most famous attempt at a proof was provided by Alfred Bay Kempe [15]. His proof appeared in 1880 (it was announced in 1879), and for ten years the problem was believed to be settled. Then in 1890, Heawood [13] discovered an error in Kempe's proof. Heawood was able to modify Kempe's argument to produce the following result.

**Theorem 8.5.1** Every planar graph is 5-colorable.

**Proof.** We proceed by induction on the order of the graph. Let  $G$  be a graph of order  $p$ . If  $p \leq 5$ , the result is clear. Assume that  $p \geq 6$  and inductively assume that all planar graphs of order  $p - 1$  are 5-colorable. By Corollary 6.1.5, we know that  $G$  contains a vertex  $v$  of degree at most 5. Then, by our assumptions,  $G - v$  is planar, has order  $p - 1$  and is 5-colorable. Consider a 5-coloring of  $G - v$ . If this coloring does not use all 5 colors on the vertices in  $N(v)$ , then we can assign  $v$  one of the missing colors and obtain a 5-coloring of  $G$ . Thus, suppose all five colors are used in  $N(v)$  (hence,  $\deg v = 5$ ).

Without loss of generality, we can assume the vertices adjacent to  $v$  in  $G$  are  $v_i$  and that each has color  $i$ ,  $i = 1, 2, 3, 4, 5$ . Further assume that these vertices are arranged cyclically about  $v$ . Consider any two colors assigned to nonconsecutive vertices, say 1 and 3, and let  $H$  be the subgraph of  $G - v$  induced by the vertices colored 1 and 3. If the vertices  $v_1$  and  $v_3$  belong to different components of  $H$ , then by interchanging colors in one of these components, we can free a color to use for  $v$ .

Thus, we suppose that  $v_1$  and  $v_3$  belong to the same component of  $H$ . Thus, there exists a path  $P$  from  $v_1$  to  $v_3$  that has its vertices alternately colored 1 and 3. The path  $P$ , along with the path  $v_1, v, v_3$ , produces a cycle that completely encloses  $v_2$  or both  $v_4$  and  $v_5$ . Thus, there exists no path alternately colored 2 and 4 joining  $v_2$  and  $v_4$  in  $G$ . Let  $H_1$  be the subgraph of  $G$  induced by those vertices colored 2 and 4. Interchanging the colors in the component of  $H_1$  containing  $v_2$  frees the color 2 for use on  $v$ , producing the desired 5-coloring of  $G$ .  $\square$

In the years that followed Heawood's work, a great deal of time and effort went into the study of graph colorings. Finally, in 1976, Appel and Haken [1], with the computer aid of Koch, verified the four color conjecture. Their general strategy was very similar to Kempe's original idea. However, their proof is very long and has many cases, and it required nearly 1200 hours of computer time to check that these cases all worked. The somewhat amazing fact here is that they were able to build a theory that reduced the infinitely many possible structures to a finite collection of cases, regardless of the number of these cases.

**Theorem 8.5.2** Every planar graph is 4-colorable.

### Section 8.6 Chromatic Polynomials

In an attempt to study the four color conjecture, Birkhoff [3] found that studying the number of colorings could be helpful. Two colorings of  $G$  are regarded as distinct provided some vertex is assigned different colors in the two colorings. Suppose we denote by  $c_k(G)$  the number of distinct  $k$  colorings of  $G$ . By our definition,  $c_k(G) > 0$  if, and only if,  $G$  is  $k$  colorable.

**Example 8.6.1.** It is easy to see that  $K_3$  has six colorings. First, color one vertex and note that there are two colorings possible on the remaining two vertices. These colorings are obtained by interchanging the remaining two colors on these two vertices. Two of the six colorings are shown in Figure 8.6.1; the rest come about by permuting the roles of the colors.



**Figure 8.6.1.** Two distinct colorings of  $K_3$ .

It is straightforward to see that if  $G = K_n$  ( $n \leq k$ ), then there are  $k$  choices for coloring the first vertex,  $k - 1$  choices for coloring the second vertex, and so forth. Hence,

$$c_k(K_n) = k(k - 1) \cdots (k - n + 1).$$

Also, if  $G$  is empty, then any vertex can be assigned any one of the  $k$  colors and so

$$c_k(G) = k^n.$$

In general, we can determine a recurrence relation for  $c_k(G)$  that is similar to our formula for the number of spanning trees of  $G$ . We let  $G/e$  denote the simple graph (with loops or multiple edges removed) obtained by identifying the end vertices of the edge  $e$ .

**Theorem 8.6.1** If  $G$  is a graph, then  $c_k(G) = c_k(G - e) - c_k(G/e)$  for any edge  $e$  of  $G$ .

**Proof.** Let  $e = uv$ . For each  $k$ -coloring of  $G - e$  that assigns the same color to  $u$  and  $v$ , there corresponds a  $k$ -coloring of  $G/e$  in which the vertex of  $G/e$  formed by identifying  $u$  and  $v$  is assigned the common color of  $u$  and  $v$ . Therefore,  $c_k(G/e)$  is just the number of

$k$ -colorings of  $G - e$  in which  $u$  and  $v$  have the same color.

Since each  $k$ -coloring of  $G - e$  that assigns different colors to  $u$  and  $v$  is also a legal  $k$ -coloring of  $G$ , and conversely, then  $c_k(G)$  is the number of  $k$ -colorings in  $G - e$  in which  $u$  and  $v$  are assigned different colors. Thus,

$$c_k(G - e) = c_k(G) + c_k(G/e),$$

and the result follows.  $\square$

This recurrence relation will allow us to describe chromatic polynomials well enough to actually justify calling them polynomials. The following corollary is again from Birkhoff [3].

**Corollary 8.6.1** For any graph  $G$  of order  $n$ ,  $c_k(G)$  is a polynomial in  $k$  of degree  $n$ . Further, this polynomial has integer coefficients, leading term  $k^n$ , constant term 0 and the coefficients alternate in sign.

**Proof.** We proceed by induction on the size of  $G$ . If  $G$  is empty, then we already know that  $c_k(G) = k^n$ . Thus, suppose the result holds for all graphs with fewer than  $q$  edges and let  $G$  be a graph with  $q$  edges, ( $q \geq 1$ ). Let  $e$  be an arbitrary edge of  $G$ . Then both  $G - e$  and  $G/e$  have size  $q - 1$ , and from the induction hypothesis, we know that there are nonnegative integers  $r_1, \dots, r_{n-1}$  and  $s_1, \dots, s_{n-2}$  such that

$$c_k(G - e) = \sum_{i=1}^{n-1} (-1)^{n-i} r_i k^i + k^n \quad \text{and}$$

$$c_k(G/e) = \sum_{i=1}^{n-2} (-1)^{n-i-1} s_i k^i + k^{n-1}.$$

Now, by Theorem 8.6.1, we see that

$$\begin{aligned} c_k(G) &= c_k(G - e) - c_k(G/e) \\ &= \sum_{i=1}^{n-2} (-1)^{n-i} (r_i + s_i) k^i - (r_{n-1} + 1) k^{n-1} + k^n. \end{aligned}$$

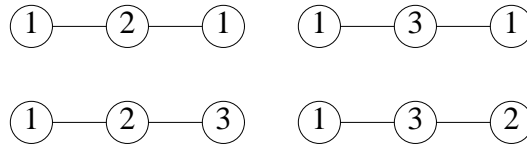
But then  $G$  also satisfies the conditions of the corollary, and so the result follows.  $\square$

We now have a means of calculating the chromatic polynomial of a graph using our recursive formula. In fact, there are two possible approaches. We can either use the recursion formula as a difference of terms, which allows us to reduce from our graph  $G$  to empty graphs or we can view our graph as the graph  $G - e$  and use the sum formula to reduce to complete graphs. In the following example, we take both approaches for the path  $P_3$ .

**Example 8.6.2.** Given the graph  $P_3$ , we first use the difference formula:

$$\begin{aligned} c_k(P_3) &= c_k(P_3 - e) - c_k(G/e) = c_k(P_3 - e) - c_k(K_2) \\ &= c_k(3K_1) - c_k(2K_1) - (c_k(2K_1) - c_k(K_1)) \\ &= c_k(3K_1) - 2c_k(2K_1) + c_k(K_1) = k^3 - 2k^2 + k. \end{aligned}$$

If  $k = 2$ , then  $c_2(P_3) = 2$ . This is easily verified while if  $k = 3$ , then  $c_3(K_3) = 12$ . In Figure 8.6.2, four of the twelve colorings are shown. The rest can be obtained by permuting colors.



**Figure 8.6.2.** Four colorings of  $P_3$ .

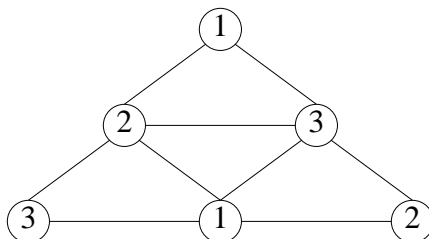
Now, to apply the recursion as a sum formula, we view  $P_3$  as  $G - e$  and, thus, we compute

$$\begin{aligned} c_k(G - e) &= c_k(G) + c_k(G/e) = c_k(K_3) + c_k(K_2) \\ &= k(k - 1)(k - 2) + k(k - 1) = k^3 - 2k^2 + k. \end{aligned}$$

Thus, either method effectively produces the chromatic polynomial for  $P_3$ .  $\square$

## Section 8.7 Perfect Graphs

Recall that a complete subgraph is called a *clique*, and that the maximum order of a clique of  $G$  is called the *clique number* of  $G$  and is denoted  $\omega(G)$ . Clearly,  $\omega(G) \leq \chi(G)$ . One might consider the class of graphs in which  $\omega(G) = \chi(G)$ , but this class has too little structure to be of much use. Berge introduced a related class of graphs, in which there is enough structure to gain valuable information. In fact, for this class the independence number and chromatic number are computable in polynomial time. A graph  $G$  is *perfect* if  $G$  and each of its induced subgraphs have the property that their chromatic number equals their clique number. The graph of Figure 8.7.1 is perfect. Its clique number is easily seen to be 3, and a 3-coloring of its vertices is shown. It is straightforward to convince oneself that this graph is perfect.



**Figure 8.7.1.** A perfect graph.

It is also easy to see that every bipartite graph is perfect. It requires a little more effort to see that the complement of a bipartite graph is also perfect.

**Theorem 8.7.1** The complement  $\bar{G}$  of any bipartite graph  $G$  is perfect.

**Proof.** Let  $G$  be a bipartite graph. Then each induced subgraph of  $\bar{G}$  has the form  $\bar{H}$ , where  $H$  is an induced subgraph of  $G$ . If  $H$  has no isolated vertices, then we know that  $\alpha_1(H) = \beta(H)$  (see Theorem 8.1.2). Since it is clear that  $\beta(H) = \omega(\bar{H})$ , we need only show that  $\chi(\bar{H}) = \alpha_1(H)$  in order to establish the fact that  $\bar{G}$  is perfect.

Clearly, the chromatic number of  $\bar{H}$  equals the minimum number of elements in a partition of  $V(H)$  such that each element of the partition induces a complete subgraph in  $H$ . Since  $H$  contains no triangles, each such complete subgraph has order 1 or 2. It follows, then, that such a partition contains  $\alpha_1(H)$  elements and, thus,  $\chi(\bar{H}) = \alpha_1(H)$ . Since a similar argument can be applied if  $H$  has isolated vertices, the proof is complete.  $\square$

There are many other classes of graphs that are perfect. An interesting class is obtained in the next result.

**Theorem 8.7.2** If a graph  $G$  is  $P_4$ -free, then  $G$  is perfect.

**Proof.** We proceed by induction on the order  $p$  of  $G$ . The result is clear if  $p = 1$ . Assume the result holds for all graphs of order less than  $p$ , ( $p \geq 2$ ) and let  $G$  be a graph of order  $p$  that is  $P_4$ -free. From exercise 28 in this chapter, for every nontrivial subset  $S$  of  $V(G)$ , either  $\langle S \rangle_G$  or  $\langle S \rangle_{\bar{G}}$  is disconnected. In particular, this implies that either  $G$  or  $\bar{G}$  is disconnected.

Suppose that  $G$  is disconnected and say  $C_1, C_2, \dots, C_k$  are the components of  $G$  ( $k \geq 2$ ). Since  $G$  is  $P_4$ -free, so are each of its components. Further, since each component has order less than  $p$ , by the inductive hypothesis, each component is perfect. But then

$$\chi(G) = \max_i \chi(C_i)$$

and

$$\omega(G) = \max_i \omega(C_i).$$

Thus,  $\chi(G) = \omega(G)$ . Since the same argument applies to any induced subgraph of  $G$ , we see in this case that  $G$  is perfect.

Now, assume that  $\bar{G}$  is disconnected. Let  $B_1, \dots, B_t$  be the components of  $\bar{G}$ . Each  $\bar{B}_i$  is a subgraph of  $G$  of order less than  $p$ ; hence, by the inductive hypothesis

$$\chi(\bar{B}_i) = \omega(\bar{B}_i) \quad \text{for } i = 1, 2, \dots, t.$$

Furthermore,  $G$  is the join of the  $\bar{B}_i$ s. Thus,

$$\chi(G) = \sum_{i=1}^t \chi(\bar{B}_i)$$

and

$$\omega(G) = \sum_{i=1}^t \omega(\bar{B}_i) \quad (\text{see exercise 25 in this chapter}).$$

Thus,  $\chi(G) = \omega(G)$ . Since the same argument applies to any induced subgraph of  $G$ , we see  $G$  is perfect.  $\square$

Berge conjectured that the complement of any perfect graph was also perfect. This result was proved independently by Fulkerson [10] and Lovász [16].

**Theorem 8.7.3** (The Perfect Graph Theorem) The complement of a perfect graph is perfect.

The odd cycle  $C_{2k+1}$  (for  $k \geq 2$ ) is not a perfect graph, since  $\omega(C_{2k+1}) = 2$  and  $\chi(C_{2k+1}) = 3$ . However, every proper subgraph of  $C_{2k+1}$  is perfect. Thus, in a sense,  $C_{2k+1}$  is minimally imperfect! Such graphs have come to be called *p-critical*. The same fact holds for the complement of  $C_{2k+1}$ . To date, these are the only two known graphs with this property. This led Berge to the following conjecture, which can be stated in several equivalent ways.

### The Strong Perfect Graph Conjecture

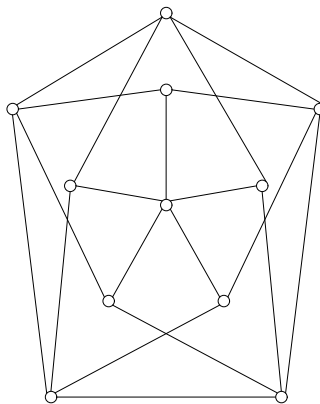
1. A graph  $G$  is perfect if, and only if, neither  $G$  nor  $\bar{G}$  contains as an induced subgraph an odd cycle of length at least 5.
2. A graph  $G$  is perfect if and only if in  $G$  and  $\bar{G}$  every odd cycle of length at least 5 has a chord.

3. The only  $p$ -critical graphs are  $C_{2k+1}$  and  $\bar{C}_{2k+1}$ .

### Exercises

1. Show that  $\chi_1(K_{m,n}) = \max\{m, n\}$ .
2. Show that  $\chi(K_{2n}) = \Delta(K_{2n})$  and that  $\chi(K_{2n+1}) = \Delta(K_{2n+1}) + 1$ .
3. Show that if  $G$  is a bipartite graph, then  $\chi_1(G) = \Delta(G)$ .
4. Prove that if  $G$  is a graph of order  $p$  with  $\delta(G) > 0$ , then  $\alpha_1(G) + \beta_1(G) = p$ .
5. Prove that  $\chi(K_{p_1, \dots, p_n}) = n$ .
6. Prove that if  $G$  is  $k$ -partite, then  $\chi(G) \leq k$ .
7. Prove Corollary 8.2.1.
8. Prove Corollary 8.2.2.
9. Prove Corollary 8.2.3.
10. Prove Theorem 8.3.1.
11. Show that every  $k$ -chromatic graph is a subgraph of some complete  $k$ -partite graph.
12. Determine the  $n$ -critical graphs for  $n = 1, 2, 3$ .
13. Show that a critically  $n$ -chromatic graph need not be  $(n - 1)$ -connected.
14. Characterize graphs whose line graphs are 2-colorable.
15. Show that for every graph  $G$ ,  $\chi(G) \leq 1 + \max \delta(H)$  where the maximum is taken over all induced subgraphs  $H$  of  $G$ .
16. If  $m(G)$  denotes the length of a longest path in  $G$ , prove that  $\chi(G) \leq 1 + m(G)$ .
17. Find a largest first ordering of the vertices of the graph in Example 8.3.1 that produces a sequential coloring using three colors.
18. Show that every regular graph of odd order is class 2.
19. Show that if  $H$  is a regular graph of odd order and if  $G$  is any graph obtained from  $H$  by deleting at most  $\frac{1}{2}\delta(G) - 1$  edges, then  $G$  is of class 2.
20. Show that if  $H$  is a regular graph of even order and if  $G$  is any graph obtained from  $H$  by subdividing any edge of  $H$ , then  $G$  is class 2.
21. Show that if  $G$  is any graph obtained from an odd cycle  $C_{2k+1}$  by adding no more than  $2k - 2$  independent edges, then  $G$  is class 2.

22. Show that if  $G$  is a regular graph containing a cut vertex, then  $G$  is of class 2.
23. Show that there are no regular  $\delta(G)$ -minimal graphs with  $\delta(G) \geq 3$ .
24. Show that every bipartite graph is perfect.
25. Let  $G_1, G_2, \dots, G_k$  be pairwise disjoint graphs. Also let  $G = G_1 + G_2 + \dots + G_k$ . Prove that  $\chi(G) = \sum \chi(G_i)$  and that  $\omega(G) = \sum_{i=1}^k \omega(G_i)$ .
26. Use the largest first, smallest last and color-degree algorithms to bound the chromatic number of each of the following graphs.
- $K_{1,3}$
  - $K_4 - e$
  - The Petersen graph
  - The Grötsch graph shown below



27. Find the chromatic polynomial for  $K_{1,3}$  and for  $K_4 - e$ . How many 5-colorings are there for each of these graphs?
28. Let  $G$  be a graph. For every nontrivial subset  $S$  of  $V(G)$ , either  $\langle S \rangle_G$  or  $\langle S \rangle_{\bar{G}}$  is disconnected if, and only if,  $G$  is  $P_4$ -free.

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