On Vertex-Disjoint Chorded Cycles and Degree Sum Conditions

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Abstract

In this paper, we consider a degree sum condition sufficient to imply the existence of $k$ vertex-disjoint chorded cycles in a graph $G$. Let $\sigma_4(G)$ be the minimum degree sum of four independent vertices of $G$. We prove that if $G$ is a graph of order at least $11k + 7$ and $\sigma_4(G) \geq 12k - 3$ with $k \geq 1$, then $G$ contains $k$ vertex-disjoint chorded cycles. We also show that the degree sum condition on $\sigma_4(G)$ is sharp.

Keywords: Vertex-disjoint chorded cycles, Minimum degree sum, Degree sequence.
1 Introduction

The study of cycles in graphs is a rich and an important area. One question of particular interest is to find conditions that guarantee the existence of \( k \) vertex-disjoint cycles. Corrádi and Hajnal [4] first considered a minimum degree condition to imply a graph must contain \( k \) vertex-disjoint cycles, proving that if \( |G| \geq 3k \) and the minimum degree \( \delta(G) \geq 2k \), then \( G \) contains \( k \) vertex-disjoint cycles. For an integer \( t \geq 1 \) and an independent vertex set \( X \) with \( |X| = t \), let

\[
\sigma_t(G) = \min \left\{ \sum_{v \in X} d_G(v) \right\},
\]

and \( \sigma_t(G) = \infty \) when the independence number \( \alpha(G) < t \). Enomoto [5] and Wang [13] independently extended the Corrádi and Hajnal result, requiring a weaker condition on the minimum degree sum of any two non-adjacent vertices. They proved that if \( |G| \geq 3k \) and \( \sigma_2(G) \geq 4k - 1 \), then \( G \) contains \( k \) vertex-disjoint cycles. In 2006, Fujita et al. [7] proved that if \( |G| \geq 3k + 2 \) and \( \sigma_3(G) \geq 6k - 2 \), then \( G \) contains \( k \) vertex-disjoint cycles, and in [10], this result was extended to \( \sigma_4(G) \geq 8k - 3 \).

An extension of the study of vertex-disjoint cycles is that of vertex-disjoint chorded cycles. A chord of a cycle is an edge between two non-adjacent vertices of the cycle. We say a cycle is chorded if it contains at least one chord. In 2008, Finkel proved the following result on the existence of \( k \) vertex-disjoint chorded cycles.

**Theorem 1.** (Finkel [6]) Let \( k \geq 1 \) be an integer. If \( G \) is a graph of order at least \( 4k \) and \( \delta(G) \geq 3k \), then \( G \) contains \( k \) vertex-disjoint chorded cycles.

In 2010, Chiba et al. proved Theorem 2. Since \( \sigma_2(G) \geq 2\delta(G) \), Theorem 2 is stronger than Theorem 1.

**Theorem 2** (Chiba, Fujita, Gao, Li [1]). Let \( k \geq 1 \) be an integer. If \( G \) is a graph of order at least \( 4k \) and \( \sigma_2(G) \geq 6k - 1 \), then \( G \) contains \( k \) vertex-disjoint chorded cycles.
Recently, Theorem 2 was extended as follows. Since \(\sigma_3(G) \geq 3\sigma_2(G)/2\), when the order of \(G\) is sufficiently large, Theorem 3 is stronger than Theorem 2.

**Theorem 3** (Gould, Hirohata, Keller [11]). *Let \(k \geq 1\) be an integer. If \(G\) is a graph of order at least \(8k + 5\) and \(\sigma_3(G) \geq 9k - 2\), then \(G\) contains \(k\) vertex-disjoint chorded cycles.*

**Remark 1.** We note if \(k = 1\) in Theorem 3, then Theorem 3 holds under the condition that \(|G| \geq 7\).

In this paper, we consider a similar extension for chorded cycles, as, in [10], the existence of \(k\) vertex-disjoint cycles was proved under the condition \(\sigma_4(G)\). In particular, we first show the following.

**Theorem 4.** *If \(G\) is a graph of order at least 15 and \(\sigma_4(G) \geq 9\), then \(G\) contains a chorded cycle.*

**Remark 2.** We consider the following graph \(G\) of order 14. (See Fig. 1.) The white vertex (◦) shows degree 2, and the black vertex (●) shows degree 3. Then \(G\) satisfies the \(\sigma_4(G)\) condition in Theorem 4. However, \(G\) does not contain a chorded cycle. Thus \(|G| \geq 15\) is necessary.

![Fig. 1. The graph \(G\) of order 14.](image)

**Theorem 5.** *Let \(k \geq 1\) be an integer. If \(G\) is a graph of order \(n \geq 11k + 7\) and \(\sigma_4(G) \geq 12k - 3\), then \(G\) contains \(k\) vertex-disjoint chorded cycles.*

**Remark 3.** Theorem 5 is sharp with respect to the degree sum condition. Consider the complete bipartite graph \(G = K_{3k-1,n-3k+1},\)
where large \( n = |G| \). Then \( \sigma_4(G) = 4(3k - 1) = 12k - 4 \). However, \( G \) does not contain \( k \) vertex-disjoint chorded cycles, since any chorded cycle must contain at least three vertices from each partite set, in particular, from the \( 3k - 1 \) partite set. Thus \( \sigma_4(G) \geq 12k - 3 \) is necessary.

For related results on vertex-disjoint chorded cycles in graphs and bipartite graphs, we refer the reader to see [2, 3, 8, 12].

Let \( G \) be a graph, \( H \) a subgraph of \( G \) and \( X \subseteq V(G) \). For \( u \in V(G) \), the set of neighbors of \( u \) in \( G \) is denoted by \( N_G(u) \), and we denote \( d_G(u) = |N_G(u)| \). For \( u \in V(G) \), we denote \( N_H(u) = N_G(u) \cap V(H) \) and \( d_H(u) = |N_H(u)| \). Also we denote \( d_H(X) = \sum_{u \in X} d_H(u) \). If \( H = G \), then \( d_G(X) = d_H(X) \). Furthermore, \( N_G(X) = \bigcup_{u \in X} N_G(u) \) and \( N_H(X) = N_G(X) \cap V(H) \). Let \( A, B \) be two vertex-disjoint subgraphs of \( G \). Then \( N_G(A) = N_G(V(A)) \) and \( N_B(A) = N_G(A) \cap V(B) \). The subgraph of \( G \) induced by \( X \) is denoted by \( \langle X \rangle \). Let \( G - X = \langle V(G) - X \rangle \) and \( G - H = \langle V(G) - V(H) \rangle \). If \( X = \{x\} \), then we write \( G - x \) for \( G - X \). If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For two disjoint graphs \( G_1 \) and \( G_2 \), \( G_1 \cup G_2 \) denotes the union of \( G_1 \) and \( G_2 \). Let \( Q \) be a path or a cycle with a given orientation and \( x \in V(Q) \). Then \( x^+ \) denotes the first successor of \( x \) on \( Q \) and \( x^- \) denotes the first predecessor of \( x \) on \( Q \). If \( x, y \in V(Q) \), then \( Q[x, y] \) denotes the path of \( Q \) from \( x \) to \( y \) (including \( x \) and \( y \)) in the given direction. The reverse sequence of \( Q[x, y] \) is denoted by \( Q^{-}[y, x] \). We also write \( Q(x, y) = Q[x^+, y] \), \( Q[x, y] = Q[x, y^-] \) and \( Q(x, y) = Q[x^+, y^-] \). If \( Q \) is a path (or a cycle), say \( Q = \langle x_1, x_2, \ldots, x_t(x_1) \rangle \), then we assume an orientation of \( Q \) is given from \( x_1 \) to \( x_t \) (if \( Q \) is a cycle, then the orientation is clockwise). If \( P \) is a path connecting \( x \) and \( y \) of \( V(G) \), then we denote the path \( P \) as \( P[x, y] \). If \( G \) is one vertex, that is, \( V(G) = \{x\} \), then we simply write \( x \) instead of \( G \). For an integer \( r \geq 1 \) and two vertex-disjoint subgraphs \( A, B \) of \( G \), we denote by \( (d_1, d_2, \ldots, d_r) \) a degree sequence from \( A \) to \( B \) such that \( d_B(v_i) \geq d_i \) and \( v_i \in V(A) \) for each \( 1 \leq i \leq r \). In this paper, since it is sufficient to consider the case of equality in the above inequality, when we write \( (d_1, d_2, \ldots, d_r) \), we assume \( d_B(v_i) = d_i \) for each \( 1 \leq i \leq r \). For two disjoint \( X, Y \subseteq V(G) \), \( E(X, Y) \) denotes the set of edges of \( G \).
connecting a vertex in \( X \) and a vertex in \( Y \). For a graph \( G \), \( \text{comp}(G) \) is the number of components of \( G \). A cycle of length \( \ell \) is called a \( \ell \)-cycle. For terminology and notation not defined here, see [9].

2 Preliminaries

**Definition 1.** Suppose \( C_1, \ldots, C_r \) are \( r \) vertex-disjoint chorded cycles in a graph \( G \). We say \( \{C_1, \ldots, C_r\} \) is minimal if \( G \) does not contain \( r \) vertex-disjoint chorded cycles \( C'_1, \ldots, C'_r \) such that

\[
|\bigcup_{i=1}^{r} V(C'_i)| < |\bigcup_{i=1}^{r} V(C_i)|.
\]

**Definition 2.** Let \( C = v_1, \ldots, v_t, v'_1 \) be a cycle with chord \( v_i v_j, i < j \). We say a chord \( vv' \neq v_i v_j \) is parallel to \( v_i v_j \) if either \( v, v' \in C[v_i, v_j] \) or \( v, v' \in C[v_j, v_i] \). Note if two distinct chords share an endpoint, then they are parallel. We say two distinct chords are crossing if they are not parallel.

**Definition 3.** Let \( u_i v_j \) and \( u_\ell v_m \) be two distinct edges between two vertex-disjoint paths \( P_1 = u_1, \ldots, u_s \) and \( P_2 = v_1, \ldots, v_t \). We say \( u_i v_j \) and \( u_\ell v_m \) are parallel if either \( i \leq \ell \) and \( j \leq m \), or \( \ell \leq i \) and \( m \leq j \). Note if two distinct edges between \( P_1 \) and \( P_2 \) share an endpoint, then they are parallel. We say two distinct edges between two vertex-disjoint paths are crossing if they are not parallel.

**Definition 4.** Let \( v_i v_j \) and \( v_\ell v_m \) be two distinct edges between vertices of a path \( P = v_1, \ldots, v_t \), with \( j \geq i + 2 \) and \( m \geq \ell + 2 \). We say \( v_i v_j \) and \( v_\ell v_m \) are nested if either \( i \leq \ell < m \leq j \) or \( \ell \leq i < j \leq m \).

**Definition 5.** Let \( P = v_1, \ldots, v_t \) be a path. We say a vertex \( v_i \) on \( P \) has a left edge if there exists an edge \( v_i v_j \) for some \( j < i - 1 \), that is not an edge of the path. We also say \( v_i \) has a right edge if there exists an edge \( v_i v_j \) for some \( j > i + 1 \), that is not an edge of the path.

3 Lemmas

The following lemmas will be needed.
Lemma 1 ([11]). Let \( r \geq 1 \) be an integer, and let \( \mathcal{C} = \{C_1, \ldots, C_r\} \) be a minimal set of \( r \) vertex-disjoint chorded cycles in a graph \( G \). If \( |C_i| \geq 7 \) for some \( 1 \leq i \leq r \), then \( C_i \) has at most two chords. Furthermore, if the \( C_i \) has two chords, then these chords must be crossing.

Lemma 2 ([11]). Let \( r \geq 1 \) be an integer, and let \( \mathcal{C} = \{C_1, \ldots, C_r\} \) be a minimal set of \( r \) vertex-disjoint chorded cycles in a graph \( G \). Then \( d_{C_i}(x) \leq 4 \) for any \( 1 \leq i \leq r \) and any \( x \in V(G) - \bigcup_{i=1}^{r} V(C_i) \). Furthermore, for some \( C \in \mathcal{C} \) and some \( x \in V(G) - \bigcup_{i=1}^{r} V(C_i) \), if \( d_{C}(x) = 4 \), then \( |C| = 4 \), and if \( d_{C}(x) = 3 \), then \( |C| \leq 6 \).

Lemma 3 ([11]). Suppose there exist at least three mutually parallel edges or at least three mutually crossing edges connecting two vertex-disjoint paths \( P_1 \) and \( P_2 \). Then there exists a chorded cycle in \( \langle P_1 \cup P_2 \rangle \).

Lemma 4 ([11]). Suppose there exist at least five edges connecting two vertex-disjoint paths \( P_1 \) and \( P_2 \) with \( |P_1 \cup P_2| \geq 7 \). Then there exists a chorded cycle in \( \langle P_1 \cup P_2 \rangle \) not containing at least one vertex of \( \langle P_1 \cup P_2 \rangle \).

Lemma 5 ([11]). Let \( P_1, P_2 \) be two vertex-disjoint paths, and let \( u_1, u_2 \ (u_1 \neq u_2) \) be in that order on \( P_1 \). Suppose \( d_{P_2}(u_i) \geq 2 \) for each \( i \in \{1, 2\} \). Then there exists a chorded cycle in \( \langle P_1[u_1,u_2] \cup P_2 \rangle \).

Lemma 6 ([11]). Let \( H \) be a graph containing a path \( P = v_1, \ldots, v_t \) \( (t \geq 3) \), and not containing a chorded cycle. If \( v_1v_i \in E(H) \) for some \( i \geq 3 \), then \( d_P(v_j) \leq 3 \) for any \( j \leq i - 1 \) and in particular, \( d_P(v_{i-1}) = 2 \). And if \( v_1v_i \in E(H) \) for some \( i \leq t-2 \), then \( d_P(v_j) \leq 3 \) for any \( j \geq i + 1 \) and in particular, \( d_P(v_{i+1}) = 2 \).

Lemma 7 ([11]). Let \( H \) be a graph containing a path \( P = v_1, \ldots, v_t \) \( (t \geq 6) \), and not containing a chorded cycle. If \( d_P(v_1) = 1 \), then \( d_P(v_i) = 2 \) for some \( 3 \leq i \leq 5 \), and if \( v_1v_3 \in E(H) \), then \( d_P(v_i) = 2 \) for some \( 4 \leq i \leq 6 \).

Lemma 8 ([11]). Let \( H \) be a graph containing a path \( P = v_1, \ldots, v_t \) \( (t \geq 6) \), and not containing a chorded cycle. If \( d_P(v_1) = 1 \), then \( d_P(v_i) = 2 \) for some \( t-4 \leq i \leq t-2 \), and if \( v_tv_{t-2} \in E(H) \), then \( d_P(v_i) = 2 \) for some \( t-5 \leq i \leq t-3 \).
Lemma 9. Let \( H \) be a connected graph of order at least 6. Suppose \( H \) contains neither a chorded cycle nor a Hamiltonian path. Let \( H = (P_1 \cup P_2) \), where \( P_1 = u_1, \ldots, u_s \) (\( s \geq 5 \)) is a longest path in \( H \) and \( P_2 = v_1, \ldots, v_t \) (\( t \geq 1 \)) is a longest path in \( H - P_1 \). If \( u_i \in V(P_1) \) for some \( 2 \leq i \leq s - 3 \) is adjacent to an endpoint \( v \) of \( P_2 \) and \( u_j \in V(P_1) \) for some \( i + 2 \leq j \leq s - 1 \) is adjacent to an endpoint \( v' \) of \( P_2 \) (possibly, \( v = v' \)), then \( d_H(u_i) = 2 \) for some \( \ell \in \{i + 1, j - 1\} \).

Proof. Let \( v, v' \) be as in the lemma, and we may assume \( v = v_1 \) and \( v' = v_t \) (possibly, \( v = v' \)). Suppose \( d_H(u_i) \geq 3 \) for each \( \ell \in \{i + 1, j - 1\} \). If \( u_{i+1} \) has a left edge, say \( u_{i+1}u_h \) with \( h < i \), then \( P_1[u_h, u_i, v_1, P_2[v_1, v_t], u_j, P_1^{-1}[u_j, u_{i+1}], u_h] \) is a cycle with chord \( u_iu_{i+1} \), a contradiction. By symmetry, \( u_{j-1} \) does not have a right edge. Since \( u_iv_1, u_jv_t \in E(H) \), \( N_{P_1}(u_{i\ell}) = \emptyset \) for each \( \ell \in \{i + 1, j - 1\} \), otherwise, since consecutive vertices on \( P_1 \) each have adjacencies on \( P_2 \), there exists a longer path than \( P_1 \) in \( H \), a contradiction. Note that even if \( v = v' \), \( N_{P_1}(u_\ell) = \emptyset \) for each \( \ell \in \{i + 1, j - 1\} \). Since \( d_H(u_i) \geq 3 \) for each \( \ell \in \{i + 1, j - 1\} \), \( u_{i+1} \) has a right edge and \( u_{j-1} \) has a left edge. No vertex in \( P_1[u_i, u_j] \) can have an edge that does not lie on \( P_1 \) to some other vertex in \( P_1[u_i, u_j] \), otherwise, this edge is a chord of the cycle \( P_1[u_i, u_j, v_1, P_2^{-1}[v_1, v_t], u_i] \). Thus we have edges \( u_{i+1}u_h \) with \( h > j \) and \( u_{j-1}u_{h'} \) with \( h' < i \). Then \( P_1[u_h, u_i, v_1, P_2[v_1, v_t], u_j, P_1[u_j, u_h], u_{i+1}, P_1[u_{i+1}, u_{j-1}], u_{h'}] \) is a cycle with chord \( u_iu_{i+1} \) (and \( u_{j-1}u_j \)), a contradiction. Thus the lemma holds. \( \Box \)

Lemma 10 ([11]). Let \( H \) be a graph of order at least 13. Suppose \( H \) does not contain a chorded cycle. If \( H \) contains a Hamiltonian path, then there exists an independent set \( X \) of four vertices in \( H \) such that \( d_H(X) \leq 8 \).

Lemma 11 ([11]). Let \( H \) be a connected graph of order at least 4. Suppose \( H \) contains neither a chorded cycle nor a Hamiltonian path. Let \( P_1 = u_1, \ldots, u_s \) (\( s \geq 3 \)) be a longest path in \( H \), and let \( P_2 = v_1, \ldots, v_t \) (\( t \geq 1 \)) be a longest path in \( H - P_1 \). Then the following statements hold.

(i) \( N_{H - P_1}(u_i) = \emptyset \) for each \( i \in \{1, s\} \).
(ii) \( d_H(u_i) = d_{P_1}(u_i) \leq 2 \) for each \( i \in \{1, s\} \).
(iii) \( N_{H - (P_1 \cup P_2)}(v_j) = \emptyset \) for each \( j \in \{1, t\} \).
(iv) \( d_{P_2}(v_j) \leq 2 \) for each \( j \in \{1, t\} \).
(v) \( d_{P_1}(z) \leq 2 \) for each \( z \in V(H) - V(P_1) \) and each \( i \in \{1, 2\} \).
(vi) \( d_{P_1}(\{v_1, v_i\}) \leq 3 \) for each \( t \geq 2 \).

Proofs of (v) and (vi). Note parts (i) to (iv) are from [11], hence we only prove parts (v) and (vi). Since \( H \) does not contain a chorded cycle, (v) holds. Suppose \( d_{P_1}(\{v_1, v_i\}) \geq 4 \). By (v), \( d_{P_1}(v_j) = 2 \) for each \( j \in \{1, t\} \). Then, by Lemma 5, \( H \) has a chorded cycle, a contradiction. Thus (vi) holds.

Lemma 12. Let \( H \) be a connected graph of order at least 15. Suppose \( H \) contains neither a chorded cycle nor a Hamiltonian path. Let \( P_1 = u_1, \ldots, u_s \) (\( s \geq 3 \)) be a longest path in \( H \), and let \( P_2 = v_1, \ldots, v_t \) (\( t \geq 1 \)) be a longest path in \( H - P_1 \) such that \( d_{P_2}(v_1) \leq d_{P_2}(v_t) \). Then there exists an independent set \( X \) of four vertices in \( H \) such that \( \{u_1, u_s, v_1\} \subseteq X \) and \( d_H(X) \leq 8 \).

Remark 4. Let \( H \) be a graph of order 14 shown in Fig. 1 (Remark 2, Theorem 4), \( P_1 = u_1, \ldots, u_{11} \), and \( P_2 = v_1, v_2, v_3 \). Then \( H \) satisfies all the conditions except for the order in Lemma 12. However, the conclusion does not hold. Thus \( |H| \geq 15 \) is necessary.

Proof. Suppose \( u_1u_s \in E(H) \). Since \( H \) is connected and \( V(H - P_1) \neq \emptyset \), there exists a longer path than \( P_1 \), a contradiction. Thus \( u_1u_s \notin E(H) \). Let \( R = H - (P_1 \cup P_2) \). If \( t = 1 \), that is, \( v_1 = v_t \), then \( d_{P_2}(v_1) \leq 2 \) by Lemma 11 (v). If \( t \geq 2 \), then \( d_{P_1}(\{v_1, v_t\}) \leq 3 \) by Lemma 11 (vi). Then \( d_{P_1}(v_1) \leq 1 \) by the assumption \( (d_{P_1}(v_1) \leq d_{P_1}(v_t)) \), and \( d_{P_1}(v_t) \leq 2 \) by Lemma 11 (v).

Claim 1. If \( |P_2| \leq 3 \), then \( H = \langle P_1 \cup P_2 \rangle \).

Proof. Suppose \( H \neq \langle P_1 \cup P_2 \rangle \). Now we prove the following two subclaims.

Subclaim 1.1. For any \( v \in V(P_2) \), \( N_R(v) = \emptyset \).

Proof. By Lemma 11 (iii), \( N_R(v_j) = \emptyset \) for each \( j \in \{1, t\} \). If \( |P_2| \leq 2 \), then the subclaim holds. Thus we may assume \( |P_2| = 3 \). Suppose
\[ N_R(v') \neq \emptyset \text{ for some } v' \in V(P_2). \text{ Then } v' = v_2. \text{ Let } w_1 \in N_R(v_2). \]

If \( v_1v_3 \in E(H) \), then the subclaim holds, otherwise, there exists a longer path than \( P_2 \) in \( H - P_1 \), a contradiction. Thus \( v_1v_3 \notin E(H) \). Since \( d_{P_1}(v_1) \leq 1 \) and \( d_{P_1}(v_3) \leq 2 \), we have \( d_H(v_1) \leq 2 \) and \( d_H(v_3) \leq 3 \). Suppose a vertex on \( P_2 \) has a neighbor \( w_1 \) in \( R \). Then \( v_2w_1 \in E(H) \). Recall \( u_1u_s \notin E(H) \), and note \( u_iv_j \notin E(H) \) for any \( i \in \{1, s\} \) and any \( j \in \{1, 3\} \) by Lemma 11 (i). We also note \( d_H(u_i) \leq 2 \) for any \( i \in \{1, s\} \) by Lemma 11 (ii). If \( d_H(\{v_1, v_3\}) \leq 4 \), then \( X = \{u_1, u_s, v_1, v_3\} \) is an independent set in \( H \) and \( d_H(X) \leq 8 \), and \( X \) is the desired set. Thus we may assume \( d_H(\{v_1, v_3\}) = 5 \), that is, \( d_H(v_1) = 2 \) and \( d_H(v_3) = 3 \). Then \( d_{P_1}(v_1) = 1 \) and \( d_{P_1}(v_3) = 2 \). Recall \( w_1 \in N_R(v_2) \). Clearly, \( N_R(w_1) = \emptyset \), otherwise, there exists a longer path than \( P_2 \) in \( H - P_1 \), a contradiction. If \( d_H(w_1) \leq 2 \), then \( X = \{u_1, u_s, v_1, w_1\} \) is the desired set. Thus \( d_H(w_1) \geq 3 \), that is, \( d_{P_1}(w_1) \geq 2 \). Note \( w_1 \) and \( v_3 \) lie on a path \( P = w_1, v_2, v_3, w_1 \), and \( w_1, v_3 \) send at least two edges each to \( P_1 \). By Lemma 5, there exists a chorded cycle in \( \langle P_1 \cup P \rangle \), a contradiction. \( \square \)

**Subclaim 1.2.** For any \( u \in V(P_1) \), \( N_R(u) = \emptyset \).

**Proof.** We first prove \( d_H(v_1) \leq 2 \). Suppose not, that is, \( d_H(v_1) \geq 3 \). Recall \( d_{P_1}(v_1) \leq 1 \). By Subclaim 1.1 and Lemma 11 (iv), \( d_{P_1}(v_1) = 1 \) and \( d_{P_1}(v_3) = 2 \). Thus \( |P_2| = 3 \) and \( v_1v_3 \in E(H) \). Since \( d_{P_1}(v_1) \leq d_{P_1}(v_3) \) by the assumption, \( d_{P_1}(v_3) \geq 1 \). Then \( \langle P_1 \cup P_2 \rangle \) contains a cycle with chord \( v_1v_3 \), a contradiction. Thus \( d_H(v_1) \leq 2 \). Suppose there exists a vertex in \( P_1 \) with a neighbor \( w_1 \) in \( R \). If \( d_H(w_1) \leq 2 \), then \( X = \{u_1, u_s, v_1, w_1\} \) is the desired set. Thus \( d_H(w_1) \geq 3 \).

First suppose \( d_{P_1}(w_1) \geq 2 \). Then \( d_{P_1}(w_1) = 2 \) by Lemma 11 (v), and \( d_{R}(w_1) \geq 1 \) by Subclaim 1.1. Let \( w_2 \in N_R(w_1) \). If \( d_H(w_2) \leq 2 \), then \( X = \{u_1, u_s, v_1, w_2\} \) is the desired set. Thus \( d_H(w_2) \geq 3 \). If \( d_{P_1}(w_2) \geq 2 \), then we have two vertices on a path \( P = w_1, w_2, \) each sending at least two edges to another path \( P_1 \), and by Lemma 5, a chorded cycle exists in \( \langle P_1 \cup P \rangle \), a contradiction. Thus \( d_{P_1}(w_2) \leq 1 \), and by Subclaim 1.1, \( d_{R}(w_2) \geq 2 \). Let \( w_3 \in N_{R-\{v_2\}}(w_2) \). If \( d_H(w_3) \leq 2 \), then \( X = \{u_1, u_s, v_1, w_3\} \) is the desired set. Thus \( d_H(w_3) \geq 3 \). Suppose \( d_{P_1}(w_3) \geq 2 \). Then consider the path \( P = w_1, w_2, w_3 \). Since \( w_1 \) and \( w_3 \) send at least two edges to another path \( P_1 \), a chorded cycle exists in \( \langle P_1 \cup P \rangle \) by Lemma 5, a contradiction. Thus \( d_{P_1}(w_3) \leq 1 \).
Also, \( N_{R-\{w_1, w_2\}}(w_3) = \emptyset \), otherwise, there exists a longer path than \( P_2 \) in \( H - P_1 \), a contradiction. By Subclaim 1.1, \( N_{P_2}(w_3) = \emptyset \). Thus \( d_{P_1}(w_3) = 1 \) and \( w_1, w_2 \in N_H(w_3) \). Then \( \langle P_1 \cup P \rangle \) contains a cycle with chord \( w_1w_3 \), a contradiction.

Next suppose \( d_{P_1}(w_1) = 1 \). Then \( d_R(w_1) \geq 2 \) by Subclaim 1.1. Let \( w_2, w_3 \in N_R(w_1) \). If \( d_H(w_i) \leq 2 \) for some \( i \in \{2, 3\} \), then \( X = \{u_1, u_s, v_1, w_1\} \) is the desired set. Thus \( d_H(w_i) \geq 3 \) for each \( i \in \{2, 3\} \). Suppose \( d_R(w_i) \geq 3 \) for some \( i \in \{2, 3\} \). Without loss of generality, we may assume \( i = 2 \). Then \( w_2 \) has a neighbor \( w_4 \) in \( R \) distinct from \( w_1 \) and \( w_3 \), and hence \( w_3, w_1, w_2, w_4 \) is a longer path than \( P_2 \) in \( H - P_1 \), a contradiction. Thus for each \( i \in \{2, 3\} \), \( d_R(w_i) \leq 2 \), and then \( d_{P_1}(w_i) \geq 1 \) by Subclaim 1.1. Note \( w_i \) for each \( i \in \{2, 3\} \) does not have a neighbor in \( R \) distinct from \( w_1, w_2, w_3 \), otherwise, there exists a longer path than \( P_2 \) in \( H - P_1 \), a contradiction. Now suppose \( d_R(w_i) = 2 \) for some \( i \in \{2, 3\} \). Then \( w_2w_3 \in E(H) \). Let \( P = w_2, w_1, w_3 \). Since \( d_{P_1}(w_i) \geq 1 \) for each \( i \in \{2, 3\} \), there exists a cycle with chord \( w_2w_3 \) in \( \langle P_1 \cup P \rangle \), a contradiction. Thus \( d_R(w_i) \leq 1 \) for each \( i \in \{2, 3\} \), and then \( d_{P_1}(w_i) \geq 2 \) by Subclaim 1.1. By Lemma 5, a chorded cycle exists in \( \langle P_1 \cup P \rangle \), a contradiction. \(\square\)

Since \( H \) is connected, we get a contradiction by Subclaims 1.1 and 1.2. Thus Claim 1 holds. \(\square\)

**Claim 2.** We have \( d_{P_1}(v_t) \geq 1 \).

*Proof.* Suppose \( d_{P_1}(v_t) = 0 \). By the assumption \( (d_{P_1}(v_t) \leq d_{P_1}(v_t)) \), we have \( d_{P_1}(v_t) = 0 \). Then we may assume \( |P_2| = t \geq 3 \), otherwise, we get a contradiction by Claim 1 and the connectedness of \( H \). Recall \( u_1u_s \notin E(H) \). By Lemmas 11 (iii) and (iv), \( d_H(v_j) \leq 2 \) for each \( j \in \{1, t\} \). If \( v_1v_t \notin E(H) \), then \( X = \{u_1, u_s, v_1, v_t\} \) is the desired set. Thus \( v_1v_t \in E(H) \).

First suppose \( |P_2| = t = 3 \). By Claim 1, \( H = \langle P_1 \cup P_2 \rangle \). Since \( v_1v_3 \in E(H) \), consider \( P'_2 = v_2, v_1, v_3 \). Then \( v_2 \) can be regarded as an endpoint of \( P'_2 \). Since \( d_{P_1}(v_1) = 0 \), we may assume \( d_{P_1}(v_2) = 0 \) by considering \( v_2 \) instead of \( v_1 \). Since \( N_{P_1}(P_2) = \emptyset \), this contradicts the connectedness of \( H \).
Next suppose $|P_2| = t \geq 4$. Recall $u_1u_s \not\in E(H)$ and $v_1v_t \in E(H)$. Consider $P' := P_2 - [v_{t-1}, v_t]$. Then $v_{t-1}$ can be regarded as an endpoint of $P'$. Thus $N_R(v_{t-1}) = \emptyset$ by Lemma 11 (iii), and $d_{P_2}(v_{t-1}) \leq 2$ by Lemma 11 (iv). Since $d_{P_1}(v_1) = 0$, we may assume $d_{P_1}(v_{t-1}) = 0$ by considering $v_{t-1}$ instead of $v_1$. Thus $d_{H}(v_{t-1}) = 2$. Hence $X = \{u_1, u_s, v_1, v_{t-1}\}$ is the desired set, and Claim 2 holds. \qed

Now we consider the following three cases based on $|P_2|$.

**Case 1.** Suppose $|P_2| = t = 1$.

Then $P_2 = v_1$. By Claim 1, $H = \langle P_1 \cup P_2 \rangle$. Since $|H| \geq 15$, $|P_1| \geq 14$. Recall $d_{P_1}(v_1) \leq 2$ when $t = 1$. By Claim 2, $d_{P_1}(v_1) \in \{1, 2\}$. Note $d_{H}(v_1) = d_{P_1}(v_1)$.

First suppose $d_{P_1}(v_1) = 2$. Let $u_i, u_j \in N_{P_1}(v_1)$ with $i < j$. Note $i \geq 2$ and $j \leq s - 1$ by Lemma 11 (i). If $j = i + 1$, then $H$ contains a Hamiltonian path, a contradiction. Thus $j \geq i + 2$. By Lemma 9, $d_{H}(u_\ell) = 2$ for some $\ell \in \{i + 1, j - 1\}$. Note $u_\ell u_1, u_\ell u_s \not\in E(H)$. Then $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set.

Next suppose $d_{P_1}(v_1) = 1$. Note $d_{P_1}(u_1) \leq 2$. Assume $u_1u_i \in E(H)$ for some $4 \leq i \leq s - 1$. By Lemma 6, $d_{P_1}(u_{i-1}) = 2$. If $v_1u_{i-1} \in E(H)$, then $v_1, u_{i-1}, P_1 - [u_{i-1}, u_1], u_i, P_1 - [u_1, u_s]$ is a Hamiltonian path, a contradiction. Thus $v_1u_{i-1} \not\in E(H)$ and $d_{H}(u_{i-1}) = 2$. Then $X = \{u_1, u_{i-1}, u_s, v_1\}$ is the desired set. Thus if $d_{P_1}(u_1) = 2$, then $u_1u_3 \in E(H)$. Then $d_{P_1}(u_i) = 2$ for some $3 \leq i \leq 6$ by Lemma 7. Similarly, either $d_{P_1}(u_s) = 1$ or $u_au_{s-2} \in E(H)$ by symmetry. Then $d_{P_1}(u_j) = 2$ for some $5 \leq j \leq s - 2$ by Lemma 8. Note $|P_1| = s \geq 14$. Since $d_{P_1}(v_1) = 1$ by our assumption, $v_1u_\ell \not\in E(H)$ for some $\ell \in \{i, j\}$, and $d_{H}(u_\ell) = 2$. Thus $X = \{u_1, u_\ell, u_s, v_1\}$ is the desired set.

**Case 2.** Suppose $|P_2| = t \in \{2, 3\}$.

By Claim 1, $H = \langle P_1 \cup P_2 \rangle$. Recall $d_{P_1}(\{v_1, v_t\}) \leq 3$, $d_{P_1}(v_1) \leq 1$, and $d_{P_1}(v_t) \leq 2$. We also note $d_{P_1}(\{v_1, v_t\}) \geq 1$ by Claim 2. Since $|H| \geq 15$, $|P_1| = s \geq 12$.

First suppose $|N_{P_1}(\{v_1, v_t\})| \in \{2, 3\}$. Let $u_i, u_j \in N_{P_1}(\{v_1, v_t\})$.
with \(i < j\). Assume \(j = i + 1\). Then \(H\) contains a longer path than \(P_1\), a contradiction. Thus \(j \geq i + 2\). Note \(i \geq 2\) and \(j \leq s - 1\) by Lemma 11 (i). By Lemma 9, \(d_H(u_\ell) = 2\) for some \(\ell \in \{i + 1, j - 1\}\). Note \(u_\ell u_1 \notin E(H)\) and \(w_\ell w_s \notin E(H)\). If \(d_H(v_1) \leq 2\), then \(X = \{u_1, u_\ell, u_s, v_1\}\) is the desired set. Thus we may assume that \(d_H(v_1) \geq 3\). Since \(d_{P_1}(v_1) \leq 1\) and \(d_{P_2}(v_1) \leq 2\), we have \(d_{P_1}(v_1) = 1\) and \(d_{P_2}(v_1) = 2\). Then \(t = 3\) and \(v_1v_3 \in E(H)\). Since \(d_{P_1}(v_1) \leq d_{P_2}(v_1) = d_{P_1}(v_3)\) by the assumption, we have \(d_{P_1}(v_3) \geq 1\). Thus \(\langle P_1 \cup P_2 \rangle\) contains a cycle with chord \(v_1v_3\), a contradiction.

Next suppose \(|N_{P_1}(\{v_1, v_3\})| = 1\). Assume \(u_1u_i \in E(H)\) for some \(4 \leq i \leq s - 1\). By Lemma 6, \(d_{P_1}(u_{i-1}) = 2\). Let \(P'_1 = P_1\{u_{i-1}, u_i, P_1[u_i, u_s]\}\). Then \(|P'_1| = |P_1|\) and \(u_{i-1}\) can be regarded as an endpoint of \(P'_1\). By Lemma 11 (i), \(d_{P_1}(u_{i-1}) = 0\). Then \(d_H(u_{i-1}) = d_{P_1}(u_{i-1}) = 2\). If \(d_H(v_1) \leq 2\), then \(X = \{u_1, u_{i-1}, u_s, v_1\}\) is the desired set. Thus we may assume that \(d_H(v_1) \geq 3\). Then \(d_{P_1}(v_1) = 1\), and \(d_{P_2}(v_1) = 2\), that is, \(t = 3\) and \(v_1v_3 \in E(H)\). Also, \(d_{P_1}(v_3) \geq 1\). Thus \(\langle P_1 \cup P_2 \rangle\) contains a cycle with chord \(v_1v_3\), a contradiction. Hence, either \(d_{P_1}(u_1) = 1\) or \(u_1u_3 \in E(H)\). Then \(d_{P_1}(u_i) = 2\) for some \(3 \leq i \leq 6\) by Lemma 7. Similarly, either \(d_{P_1}(u_s) = 1\) or \(u_s u_{s-2} \in E(H)\) by symmetry. Then \(d_{P_1}(u_j) = 2\) for some \(5 \leq j \leq s - 2\) by Lemma 8. Since \(|N_{P_1}(\{v_1, v_t\})| = 1\) by our assumption, \(u_\ell \notin N_{P_1}(\{v_1, v_3\})\) for some \(\ell \in \{i, j\}\). Suppose \(t = 2\). Then \(d_H(v_1) \leq 2\) and \(d_H(u_\ell) = d_{P_1}(u_\ell) = 2\). Thus \(X = \{u_1, u_\ell, u_s, v_1\}\) is the desired set. Hence \(t = 3\). If \(v_1v_3 \notin E(H)\), then \(d_H(v_1) \leq 2\) and \(d_H(v_3) \leq 2\). Thus \(X = \{u_1, u_s, v_1, v_3\}\) is the desired set. Hence we may assume that \(v_1v_3 \in E(H)\). Note \(d_{P_1}(v_1) \leq 1\). Suppose \(d_{P_1}(v_1) = 1\). Since \(d_{P_1}(v_3) \geq 1\), \(\langle P_1 \cup P_2 \rangle\) contains a cycle with chord \(v_1v_3\), a contradiction. Suppose \(d_{P_1}(v_1) = 0\). Then \(d_H(v_1) = 2\). If \(d_H(u_\ell) = 2\), then \(X = \{u_1, u_\ell, u_s, v_1\}\) is the desired set. Thus we may assume that \(d_H(u_\ell) \geq 3\). Then \(u_\ell v_2 \in E(H)\). Since \(d_{P_1}(v_3) \geq 1\), \(\langle P_1 \cup P_2 \rangle\) contains a cycle with chord \(v_2v_3\), a contradiction.

**Case 3.** Suppose \(|P_2| = t \geq 4\).

Recall \(d_{P_1}(v_1) \leq 1\) and \(d_{P_1}(v_t) \leq 2\). We consider two subcases as follows.
Subcase 1. Suppose \( d_{P_1}(v_1) = 1 \).

By Claim 2, \( d_{P_1}(v_1) \geq 1 \). Then \( d_{P_2}(v_1) = d_{P_2}(v_t) = 1 \), otherwise, there exists a cycle in \( \langle P_1 \cup P_2 \rangle \) with chord adjacent to \( v_1 \) or \( v_t \), a contradiction. Thus \( d_H(v_1) = 2 \) by Lemma 11 (iii). If \( d_{P_1}(v_1) = 1 \), then \( d_{H}(v_1) = 2 \) by Lemma 11 (iii). Then \( X = \{u_1, u_s, v_1, v_t\} \) is the desired set. Thus \( d_{P_1}(v_1) = 2 \). Let \( u_i, u_j \in N_{P_1}(v_i) \) with \( i < j \). Consider the vertex \( v_{t-1} \). If \( d_H(v_{t-1}) = 2 \), then \( X = \{u_1, u_s, v_1, v_{t-1}\} \) is the desired set. Thus \( d_H(v_{t-1}) \geq 3 \). If \( d_{P_2}(v_{t-1}) \geq 3 \), then there exists a cycle in \( \langle P_1 \cup P_2 \rangle \) with chord adjacent to \( v_{t-1} \), a contradiction. Thus \( d_{P_2}(v_{t-1}) = 2 \), and then \( N_{P_2}(v_{t-1}) \neq \emptyset \) or \( N_{R}(v_{t-1}) \neq \emptyset \).

First suppose \( N_{P_1}(v_{t-1}) \neq \emptyset \). If \( v_1 \) or \( v_{t-1} \) has a neighbor in \( P_1[u_1,u_j] \cup P_1[u_j,u_s] \), then there exist three parallel edges between \( P_1 \) and \( P_2 \), and by Lemma 3, a chorded cycle exists in \( \langle P_1 \cup P_2 \rangle \), a contradiction. Thus \( N_{P_1}(v_{t-1}) \neq \emptyset \) for each \( \ell \in \{1, t - 1\} \). Then we again have three parallel edges or three crossing edges, and by Lemma 3, a chorded cycle exists in \( \langle P_1 \cup P_2 \rangle \), a contradiction.

Next suppose \( N_{R}(v_{t-1}) \neq \emptyset \). Let \( w \in N_{R}(v_{t-1}) \). If \( d_H(w) \leq 2 \), then \( X = \{u_1, u_s, v_1, w\} \) is the desired set. Thus \( d_H(w) \geq 3 \). Then \( d_{P_1}(w) \leq 1 \), otherwise, since \( d_{P_1}(v_t) = 2 \), there exists a chorded cycle in \( \langle P_1 \cup P_2 \rangle \) by Lemma 5, a contradiction. Since \( P_2 \) is a longest path in \( H - P_1 \), \( N_{R}(w) = \emptyset \). Thus \( d_{P_1}(w) = 1 \) and \( d_{P_2}(w) = 2 \). Let \( u_p \in N_{P_1}(v_1) \) and \( u_q \in N_{P_1}(w) \). Without loss of generality, we may assume \( p \leq q \). By Lemma 11 (iii), \( wv_1, wv_t \notin E(H) \). Thus \( wv_1 \in E(H) \) for some \( 2 \leq \ell \leq t - 2 \). Then \( w, v_{t-1}, P_2^{-}[v_{t-1}, v_1], u_p, P_1[u_p, u_q], w \) is a cycle with chord \( wv_\ell \), a contradiction.

Subcase 2. Suppose \( d_{P_1}(v_1) = 0 \).

Suppose \( v_1v_t \in E(H) \). Then note \( d_H(v_1) = 2 \). Now we consider the path \( P_2' = P_2^{-}[v_{t-1}, v_1], v_t \). Then \( v_{t-1} \) can be regarded as an endpoint of \( P_2' \). Since \( d_{P_1}(v_1) = 0 \) by the assumption, we may assume \( d_{P_1}(v_{t-1}) = 0 \) by considering \( v_{t-1} \) instead of \( v_1 \). Thus \( d_H(v_{t-1}) = 2 \). Recall \( u_1u_s \notin E(H) \). Then \( X = \{u_1, u_s, v_1, v_{t-1}\} \) is the desired set. Thus \( v_1v_t \notin E(H) \). If \( d_H(v_1) \leq 2 \), then \( X = \{u_1, u_s, v_1, v_t\} \) is the desired set. Thus \( d_H(v_1) \geq 3 \). By Lemma 11 (iii), (iv), and (v), we have \( d_H(v_1) \leq 4 \) and \( d_{P_1}(v_t) \in \{1, 2\} \).
First suppose \( d_{P_1}(v_i) = 2 \). Let \( u_i, u_j \in N_{P_1}(v_i) \) with \( i < j \). Note
\( i \geq 2 \) and \( j \leq s - 1 \) by Lemma 11 (i), and \(|P_1| \geq |P_2| \geq 4\). If \( j = i + 1 \), then there exists a longer path than \( P_1 \), a contradiction. Thus \( j \geq i + 2 \). Therefore, \(|P_1| \geq 5\). If \( d_H(u_\ell) = 2 \) for some \( \ell \in \{i + 1, j - 1\} \), then \( X = \{u_1, u_\ell, u_s, v_1\} \) is the desired set. Thus \( d_H(u_\ell) \geq 3 \) for each \( \ell \in \{i + 1, j - 1\} \). By Lemma 9, we may assume \( H \neq (P_1 \cup P_2) \). Now we claim \( N_R(u_\ell) \neq \emptyset \) for some \( \ell \in \{i + 1, j - 1\} \). Assume not. Note \( N_{P_2}(u_\ell) = \emptyset \) since \( P_1 \) is a longest path in \( H \). Since \( H \) does not contain a chorded cycle, there exist edges \( u_{i+1}u_h \) with \( h > j \) and \( u_{j-1}u_{h'} \) with \( h' < i \). Then \( P_1[u_h, u_i], v_1, u_j, P_1[u_j, u_h], u_{i+1}, P_1[u_{i+1}, u_{j-1}], u_{h'} \) is a cycle with chord \( u_iu_{i+1} \) and \( u_{j-1}u_j \), a contradiction. Thus the claim holds. If \( j \geq i + 3 \), then we may assume \( \ell = j - 1 \), that is, \( N_R(u_{j-1}) \neq \emptyset \), otherwise, consider \( P^-[u_s, u_1] \). Let \( w_1 \in N_R(u_{j-1}) \), and let \( P_3 = w_1, \ldots, w_p \) (\( p \geq 1 \)) be a longest path starting from \( w_1 \) in \( R \). If \( d_H(w_p) \leq 2 \), then \( X = \{u_1, u_s, v_1, w_p\} \) is the desired set. Thus \( d_H(w_p) \geq 3 \). If \( N_{P_2}(w) \neq \emptyset \) for some \( w \in V(P_3) \), that is, \( v_\ell \in N_{P_2}(w) \) for some \( 1 \leq \ell \leq t \), then
\[
P_1[u_1, u_{j-1}], w_1, P_3[w_1, w], v_\ell, P_2[v_\ell, v_t], u_j, P_1[u_j, u_s]
\]
is a longer path than \( P_1 \), a contradiction. Thus \( N_{P_2}(w) = \emptyset \) for any \( w \in V(P_3) \). Since \( P_3 \) is a longest path starting from \( w_1 \) in \( R \), \( N_{R-P_3}(w_p) = \emptyset \). Suppose \( |P_3| = p = 1 \). Since \( N_R(w_1) = \emptyset \) and \( d_H(w_p) \geq 3 \), \( d_{P_1}(w_1) \geq 3 \). This contradicts Lemma 11 (v). Suppose \( |P_3| = p = 2 \). Then \( d_H(w_2) \geq 3 \), and by Lemma 11 (v), \( d_{P_1}(w_2) = 2 \). If \( u_\ell \in N_{P_1}(w) \) for some \( j \leq \ell \leq s \), then
\[
P_1[u_i, u_{j-1}], w_1, P_3[w_1, w_2], u_\ell, P_1^-[u_\ell, u_j], v_\ell, u_i
\]
is a cycle with chord \( u_{j-1}u_j \), a contradiction. Thus \( u_\ell, u_{\ell'} \in N_{P_1}(w_2) \) for some \( 1 \leq \ell < \ell' \leq j - 1 \). Then \( P_1[u_\ell, u_{j-1}], w_1, P_3[w_1, w_2], u_\ell \) is a cycle with chord \( w_2u_{\ell'} \), a contradiction. Suppose \( |P_3| = p \geq 3 \). Then \( d_{P_3}(w_p) \leq 2 \). Assume \( d_{P_3}(w_p) = 2 \). Since \( d_{P_1}(w_p) \geq 1 \), there exists a cycle in \( (P_1 \cup P_3) \) with chord adjacent to \( w_p \), a contradiction. Thus \( d_{P_3}(w_p) = 1 \), and \( d_{P_1}(w_p) = 2 \). Then we have a chorded cycle in \( (P_1 \cup P_3) \) as in the case where \( |P_3| = 2 \) by considering \( w_p \) instead of \( w_2 \), a contradiction.

Next suppose \( d_{P_1}(v_i) = 1 \). Let \( u_i \in N_{P_1}(v_i) \) with \( 1 \leq i \leq s \). Note \( i \not\in \{1, s\} \) by Lemma 11 (i). Since \( d_H(v_i) \geq 3 \), \( d_{P_2}(v_i) = 2 \) by Lemmas
Lemma 13 \([\text{[11]}]\). Let \(k \geq 2\) be an integer, and let \(G\) be a graph. Suppose \(G\) does not contain \(k\) vertex-disjoint chorded cycles. Let

11 (iii) and (iv). Let \(v_t \in N_{P_2}(v_i)\) with \(\ell \leq t - 2\). Now we consider the path \(P'_2 = P_2[v_1, v_t], v_t, P'_2[v_t, v_{t+1}]\). Then \(v_{t+1}\) can be regarded as an endpoint of \(P'_2\). Since \(d_{P_2}(v_i) = 1\), we may assume \(d_{P_2}(v_{t+1}) = 1\). Let \(u_j \in N_{P_1}(v_{t+1})\) with \(1 \leq j \leq s\). Note \(j \notin \{1, s\}\) by Lemma 11 (i). Then we may assume \(j \leq i\), otherwise, consider \(P^-[u_s, u_1]\). Suppose \(\ell = t - 2\), that is, \(v_tv_{t-1} \in E(H)\). Then \(P_1[u_j, u_i], v_t, v_{t-2}, v_{t-1}, u_j\) is a cycle with chord \(v_{t-1}v_t\), a contradiction. Thus \(\ell \leq t - 3\). If \(j = i - 1\), then there exists a longer path than \(P_1\), a contradiction.

Suppose \(j = i\). Recall \(v_tv_t \in E(H)\) with \(\ell \leq t - 3\). If \(d_H(v_{t-1}) = 2\), then \(X = \{u_1, u_s, v_1, v_{t-1}\}\) is the desired set. Thus \(d_H(v_{t-1}) \geq 3\). Assume \(u_j \in N_{P_1}(v_{t-1})\) for some \(1 \leq j \leq s\). We may assume \(j \leq i\), otherwise, consider \(P^-[u_s, u_1]\). Then \(P_1[u_j, u_i], v_t, P_2[v_t, v_{t-1}], u_j\) is a cycle with chord \(v_{t-1}v_t\), a contradiction. Assume \(v_\ell \in N_{P_2}(v_{t-1})\) for some \(\ell' \leq t - 3\). Since \(v_tv_t \in E(H)\), we may assume \(\ell' < \ell\). Then \(P_2[v_\ell, v_t], v_t, u_s, P_2[v_\ell+1, v_{t-1}], v_\ell\) is a cycle with chord \(v_{t-1}v_t\), a contradiction. Assume \(N_{R}(v_{t-1}) \neq \emptyset\). Let \(w \in N_{R}(v_{t-1})\). Now we consider the path \(P'_2 = P_2[v_1, v_{t-1}], w\). Then \(w\) can be regarded as an endpoint of \(P'_2\). Since \(d_{P_1}(v_i) = 1\), we may assume \(d_{P_1}(w) = 1\). Let \(u_j \in N_{P_1}(w)\) for some \(1 \leq j \leq s\). We may assume \(j \leq i\). Then \(P_2[v_\ell, v_{t-1}], w, P_1[u_j, u_i], v_t, v_t\) is a cycle with chord \(v_{t-1}v_t\), a contradiction.

Suppose \(j \leq i - 2\). If \(d_H(u_h) = 2\) for some \(h \in \{j + 1, i - 1\}\), then \(X = \{u_1, u_h, u_s, v_1\}\) is the desired set. Thus \(d_H(u_h) \geq 3\) for each \(h \in \{j + 1, i - 1\}\). Now we claim \(N_{R}(u_h) \neq \emptyset\) for some \(h \in \{j + 1, i - 1\}\). Assume not. Note \(N_{P_2}(u_h) = \emptyset\), since \(P_1\) is a longest path in \(H\). Since \(H\) does not contain a chorded cycle, there exist edges \(u_{j+1}u_m\) with \(m > i\) and \(u_{i-1}u_{m'}\) with \(m' < j\). Then \(P_1[u_{m'}, u_j], v_{i-1}, P_2[v_{i-1}, v_t], u_i, P_1[u_i, u_m], u_{j+1}, P_1[u_{j+1}, u_{i-1}], u_{m'}\) is a cycle with chord \(u_ju_{j+1}\) (and \(u_{i-1}u_i\)), a contradiction. Thus the claim holds. We also note that if \(j \leq i - 3\), then we may assume \(N_{R}(u_{i-1}) \neq \emptyset\), otherwise, consider \(P^-[u_s, u_1]\). Let \(w_1 \in N_{R}(u_{i-1})\), and let \(P_3 = w_1, \ldots, w_p (p \geq 1)\) be a longest path in \(R\). Then, as in the above case where \(d_{P_1}(v_i) = 2\), there exists a chorded cycle in \(H\), a contradiction. 

\(\Box\)

Lemma 13 \([\text{[11]}]\). Let \(k \geq 2\) be an integer, and let \(G\) be a graph. Suppose \(G\) does not contain \(k\) vertex-disjoint chorded cycles. Let
\( \mathcal{C} = \{C_1, \ldots, C_{k-1}\} \) be a minimal set of \( k - 1 \) vertex-disjoint chorded cycles in \( G \), and let \( H = G - \mathcal{C} \) and \( X \subseteq V(H) \) with \( |X| = 4 \). Suppose \( H \) contains a Hamiltonian path. Then \( d_{C_i}(X) \leq 12 \) for each \( 1 \leq i \leq k - 1 \).

4 Proof of Theorem 4

Suppose \( G \) does not contain a chorded cycle.

Claim 1. \( G \) is connected.

Proof. Suppose not, then \( \text{comp}(G) \geq 2 \). Let \( G_1, G_2, \ldots, G_{\text{comp}(G)} \) be the components of \( G \).

First suppose \( \text{comp}(G) \geq 4 \). By Theorem 1, there exists \( x_i \in V(G_i) \) for each \( 1 \leq i \leq 4 \) such that \( d_{G_i}(x) \leq 2 \). Let \( X = \{x_1, x_2, x_3, x_4\} \). Then \( X \) is an independent set with \( d_G(X) \leq 8 \). This contradicts the \( \sigma_4(G) \) condition.

Next suppose \( \text{comp}(G) = 3 \). Let \( |G_1| \geq |G_2| \geq |G_3| \). Since \( |G| \geq 15 \) by the assumption, we have \( |G_1| \geq 5 \). If \( G_1 \) is complete, then \( G_1 \) contains a chorded cycle. Thus we may assume \( G_1 \) is not complete. By Theorem 2, there exist non-adjacent \( x_0, x_1 \in V(G_1) \) such that \( d_{G_1}([x_0, x_1]) \leq 4 \). Also, by Theorem 1, there exists \( x_i \in V(G_i) \) for each \( i \in \{2, 3\} \) such that \( d_{G_i}(x_i) \leq 2 \). Then \( X = \{x_0, x_1, x_2, x_3\} \) is an independent set with \( d_G(X) \leq 8 \), a contradiction.

Finally, suppose \( \text{comp}(G) = 2 \). Let \( |G_1| \geq |G_2| \). Since \( |G| \geq 15 \), \( |G_1| \geq 8 \). By Theorem 3 (Remark 1), \( G_1 \) contains an independent set \( X_0 \) of three vertices with \( d_{G_1}(X_0) \leq 6 \). Also, by Theorem 1, there exists \( x \in V(G_2) \) such that \( d_{G_2}(x) \leq 2 \). Then \( X = X_0 \cup \{x\} \) is an independent set with \( d_G(X) \leq 8 \), a contradiction.

Let \( P_1 = u_1, \ldots, u_s \) be a longest path in \( G \). Note \( s \geq 3 \), since \( |G| \geq 15 \) and \( G \) is connected by Claim 1.

Claim 2. \( G \) contains a Hamiltonian path.

Proof. Suppose not, then \( P_1 \) is not a Hamiltonian path in \( G \), and \( V(G - P_1) \neq \emptyset \). Let \( P_2 = v_1, \ldots, v_t (t \geq 1) \) be a longest path in
such that \(d_{P_1}(v_1) \leq d_{P_1}(v_t)\). By Lemma 12, there exists an independent set \(X\) of four vertices in \(G\) such that \(d_{G}(X) \leq 8\). This contradicts the \(\sigma_4(G)\) condition.

Since \(|G| \geq 15\), by Claim 2 and Lemma 10, there exists an independent set \(X\) of four vertices in \(G\) such that \(d_{G}(X) \leq 8\), a contradiction. This completes the proof of Theorem 4.

5 Proof of Theorem 5

By Theorem 4, we may assume \(k \geq 2\). Suppose Theorem 5 does not hold. Let \(G\) be an edge-maximal counter-example. If \(G\) is complete, then \(G\) contains \(k\) vertex-disjoint chorded cycles. Thus we may assume \(G\) is not complete. Let \(xy \notin E(G)\) for some \(x, y \in V(G)\), and define \(G' = G + xy\), the graph obtained from \(G\) by adding the edge \(xy\). By the edge-maximality of \(G\), \(G'\) is not a counter-example. Thus \(G'\) contains \(k\) vertex-disjoint chorded cycles \(C_1, \ldots, C_k\). Without loss of generality, we may assume \(xy \notin \bigcup_{i=1}^{k-1} E(C_i)\), that is, \(G\) contains \(k - 1\) vertex-disjoint chorded cycles. Over all sets of \(k - 1\) vertex-disjoint chorded cycles, choose \(C_1, \ldots, C_{k-1}\) with \(\mathcal{C} = \bigcup_{i=1}^{k-1} C_i\), \(H = G - \mathcal{C}\), and with \(P_1\) a longest path in \(H\), such that:

(A1) \(|\mathcal{C}|\) is as small as possible,

(A2) subject to (A1), \(\text{comp}(H)\) is as small as possible, and

(A3) subject to (A1) and (A2), \(|P_1|\) is as large as possible.

We may also assume \(H\) does not contain a chorded cycle, otherwise, \(G\) contains \(k\) vertex-disjoint chorded cycles, a contradiction.

Claim 1. \(H\) has an order at least 18.

Proof. Suppose to the contrary that \(|H| \leq 17\). Next suppose \(|C_i| \leq 11\) for each \(1 \leq i \leq k - 1\). Since \(|G| \geq 11k + 7\) by assumption, it follows that \(|H| \geq (11k + 7) - 11(k - 1) = 18\), a contradiction. Thus \(|C_i| \geq 12\) for some \(1 \leq i \leq k - 1\). Without loss of generality, we may assume \(C_1\) is a longest cycle in \(\mathcal{C}\). Then \(|C_1| \geq 12\). By Lemma 1, \(C_1\)
contains at most two chords, and if $C_1$ has two chords, then these chords must be crossing. For integers $t$ and $r$, let $|C_1| = 4t + r$, where $t \geq 3$ and $0 \leq r \leq 3$.

**Subclaim 1.1.** Let $t \geq 3$ be an integer. The cycle $C_1$ contains $t$ vertex-disjoint sets $X_1, \ldots, X_t$ of four independent vertices each in $G$ such that $d_{C_1}(\bigcup_{i=1}^t X_i) \leq 8t + 4$.

**Proof.** For any $4t$ vertices of $C_1$, their degree sum in $C_1$ is at most $4t \times 2 + 4 = 8t + 4$, since $C_1$ has at most two chords. Thus it only remains to show that $C_1$ contains $t$ vertex-disjoint sets of four independent vertices each. Recall $|C_1| = 4t + r \geq 4t$. Start anywhere on $C_1$ and label the first $4t$ vertices of $C_1$ with labels 1 through $t$ in order, starting over again with 1 after using label $t$. If $r \geq 1$, then label the remaining $r$ vertices of $C_1$ with the labels $t + 1, \ldots, t + r$. (See Fig. 2.) The labeling above yields $t$ vertex-disjoint sets of four vertices each, where all the vertices labeled with 1 are one set, all the vertices labeled with 2 are another set, and so on. Given this labeling, since $t \geq 3$, any vertex in $C_1$ has a different label than the vertex that precedes it on $C_1$ and the vertex that succeeds it on $C_1$. Let $C_0$ be the cycle obtained from $C_1$ by removing all chords. Then the vertices in each of the sets are independent in $C_0$. Thus the only way vertices in the same set are not independent in $C_1$ is if the endpoints of a chord of $C_1$ were given the same label. Note any vertex labeled $i$ is distance at least 3 in $C_0$ from any other vertex labeled $i$. Thus even if we exchange the label of $x$ in $C_0$ for the one of $x^-$ (or $x^+$), the vertices in each of the resulting $t$ sets are still independent in $C_0$.

**Case 1.** No chord of $C_1$ has endpoints with the same label.

Then there exist $t$ vertex-disjoint sets of four independent vertices each in $C_1$.

**Case 2.** Exactly one chord of $C_1$ has endpoints with the same label.

Recall $C_1$ contains at most two chords, and if $C_1$ contains two chords, then these chords must be crossing. Since $|C_1| \geq 12$, even if $C_1$ has two chords, each chord has an endpoint $x$ such that there
exists a vertex $x' \in \{x^-, x^+\}$ which is not an endpoint of the other chord. Choose such an endpoint $x$ of the chord whose endpoints were assigned the same label, and exchange the label of $x$ for the one of $x'$. The vertices in each of the resulting $t$ sets are independent in $C_1$, and now no chord of $C_1$ has endpoints with the same label. Thus there exist $t$ vertex-disjoint sets of four independent vertices each in $C_1$.

Case 3. Two chords of $C_1$ each have endpoints with the same label.

Then the two chords are crossing. Since endpoints of a chord have the same label in this case, recall these endpoints have distance at least 3. First suppose there exists an endpoint $x$ of one chord of $C_1$ which is adjacent to an endpoint $y (= x^+)$ of the other chord on $C_1$. (See Fig. 3(a).) Now we exchange the label of $x$ for the one of $y$. Then no chord of $C_1$ has endpoints with the same label, and the vertices in each of the resulting $t$ sets are independent in $C_1$. Thus there exist $t$ vertex-disjoint sets of four independent vertices each in $C_1$.

Next suppose no endpoint of one chord of $C_1$ is adjacent to an endpoint of the other chord on $C_1$. (See Fig. 3(b).) Let $x_1x_2, y_1y_2$ be the two distinct chords of $C_1$. Since the two chords are crossing, without loss of generality, we may assume $x_1, y_1, x_2, y_2$ are in that order on $C_1$. Now we exchange the labels of $x_1$ and $x_1^+$, and next the
ones of $y_2$ and $y_2^-$. Then no chord of $C_1$ has endpoints with the same label, and the vertices in each of the resulting $t$ sets are independent in $C_1$. Thus there exist $t$ vertex-disjoint sets of four independent vertices each in $C_1$. 

\[\begin{align*}
\cdot & \quad \cdot \\
\cdot & \quad y (x^+) \\
3 & \quad 2
\end{align*}\]

(a)

\[\begin{align*}
\cdot & \quad \cdot \\
\cdot & \quad x_1 \\
1 & \quad y_1 \\
\cdot & \quad \cdot \\
\cdot & \quad y_2 \\
1[3] & \quad 3[1] \\
\cdot & \quad \cdot \\
\cdot & \quad x_2 \\
2 & \quad 3
\end{align*}\]

(b)

Fig. 3. Examples: (a) – the labels of $x$ and $y$ are 2 and 3, (b) – the labels of $x_1$ and $y_2$ are 2 and 1. ([i] means $i$ is a new label for a vertex after the exchange.)

Since $|C_1| \geq 12$, $d_{C_1}(v) \leq 2$ for any $v \in V(H)$ by Lemma 2 and (A1). Thus since $|H| \leq 17$ by our assumption, it follows that $|E(H, C_1)| \leq 34$. Let $\mathcal{X} = \bigcup_{i=1}^t X_i$ be as in Subclaim 1.1. By the $\sigma_4(G)$ condition, $d_G(\mathcal{X}) \geq t(12k - 3)$. Suppose $k = 2$. Then $\mathcal{C}$ has only one cycle $C_1$. Since $k = 2$ and $t \geq 3$, $|E(C_1, H)| \geq d_H(\mathcal{X}) \geq t(12k - 3) - (8t + 4) = 13t - 4 \geq 35$, a contradiction. Thus $k \geq 3$. Then we have

\[
|E(\mathcal{X}, \mathcal{C} - C_1)| = d_G(\mathcal{X}) - d_{C_1}(\mathcal{X}) - d_H(\mathcal{X}) \\
\geq t(12k - 3) - (8t + 4) - 34 \\
= 12kt - 11t - 38,
\]

and since $t \geq 3$,

\[
12kt - 11t - 38 = 12t(k - 1) + t - 38 \geq 12t(k - 1) - 35 \\
> 12t(k - 1) - 12t \\
= 12t(k - 2).
\]
Thus $|E(\mathcal{X}, C')| > 12t$ for some $C'$ in $\mathcal{C} - C_1$, since $\mathcal{C} - C_1$ contains $k - 2$ vertex-disjoint chorded cycles. Let $h = \max \{d_{C'}(v) | v \in \mathcal{X} \}$. Let $v^*$ be a vertex of $\mathcal{X}$ such that $d_{C'}(v^*) = h$. Since $|E(\mathcal{X}, C')| > 12t$, if $h \leq 3$, then $|E(\mathcal{X}, C')| \leq 3 \times 4t = 12t$, a contradiction. Thus we may assume $h \geq 4$. By the maximality of $C_1$, $|C'| \leq |C_1| = 4t + r$. It follows that $h = d_{C'}(v^*) \leq |C'| \leq 4t + r$. Recall $t \geq 3$ and $0 \leq r \leq 3$. Then

$$|E(\mathcal{X} - \{v^*\}, C')| \geq (12t + 1) - d_{C'}(v^*) \geq (12t + 1) - (4t + r)$$
$$= 8t - r + 1 \geq 22. \quad (1)$$

Since $h = d_{C'}(v^*) \geq 4$, let $v_1, v_2, v_3, v_4$ be neighbors of $v^*$ in that order on $C'$. Note that $v_1, v_2, v_3, v_4$ partition $C'$ into four intervals $C'[v_i, v_{i+1}]$ for each $1 \leq i \leq 4$, where $v_5 = v_1$. By (1), there exist at least 22 edges from $C_1 - v^*$ to $C'$. Thus some interval $C'[v_i, v_{i+1}]$ contains at least six of these edges. Without loss of generality, we may assume this interval is $C'[v_4, v_1]$. Then by Lemma 4, $\langle (C_1 - v^*) \cup C'[v_4, v_1] \rangle$ contains a chorded cycle not containing at least one vertex of $\langle (C_1 - v^*) \cup C'[v_4, v_1] \rangle$.

Also, $v^*, C'[v_1, v_3], v^*$ is a cycle with chord $v^*v_2$, and it uses no vertices from $C'[v_4, v_1]$. Thus we have two shorter vertex-disjoint chorded cycles in $\langle C_1 \cup C' \rangle$, contradicting (A1). Hence Claim 1 holds. \hfill \Box

**Claim 2.** $H$ is connected.

**Proof.** Suppose not, then $\text{comp}(H) \geq 2$. Let $H_1, H_2, \ldots, H_{\text{comp}(H)}$ be the components of $H$. First we prove the following subclaim.

**Subclaim 2.1.** Suppose $X$ is an independent set of four vertices in $H$ such that $d_H(X) \leq 8$. Then there exists some $C$ in $\mathcal{C}$ such that the degree sequences from four vertices of $X$ to $C$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$. Furthermore, then $|C| = 4$.

**Proof.** By the $\sigma_4(G)$ condition, $d_{\mathcal{G}}(X) \geq (12k - 3) - 8 = 12k - 11 > 12(k - 1)$. Thus there exists some $C$ in $\mathcal{C}$ such that $d_C(X) \geq 13$. 


By Lemma 2, $d_C(x) \leq 4$ for any $x \in X$. Now we consider degree sequences defined in Section 1 (Introduction) from four vertices of $X$ to $C$. Recall that when we write $(d_1, d_2, d_3, d_4)$, we assume $d_C(x_j) = d_j$ for each $1 \leq j \leq 4$, since it is sufficient to consider the case of equality. It follows that the degree sequences from four vertices of $X$ to $C$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$. Since each degree sequence contains a vertex with degree 4 in $C$, we have $|C| = 4$ by Lemma 2. Thus the subclaim holds.

Now we consider the following three cases based on $\text{comp}(H)$.

**Case 1.** Suppose $\text{comp}(H) \geq 4$.

By Theorem 1, there exists $x_i \in V(H_1)$ for each $1 \leq i \leq 4$ such that $d_{H_1}(x_i) \leq 2$. Let $X = \{x_1, x_2, x_3, x_4\}$. Then $X$ is an independent set and $d_H(X) \leq 8$. By Subclaim 2.1, the degree sequences from four vertices of $X$ to some $C$ in $\mathcal{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$ and $|C| = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$. Without loss of generality, we may assume $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3) \geq d_C(x_4)$. Then $d_C(x_1) = 4$. Since $|C| = 4$, for each degree sequence, $x_2, x_3, x_4$ must all have a common neighbor in $C$, say $v_1$. Since $d_C(x_1) = 4$, $C' = x_1, v_2, v_3, v_4, x_1$ is a 4-cycle with chord $x_1v_3$. If $x_1$ is not a cut-vertex of $H_1$, then $H_1 - x_1$ is connected. Replacing $C$ in $\mathcal{C}$ by $C'$, we consider the new $H'$. Then $\text{comp}(H') \leq \text{comp}(H) - 2$. This contradicts (A2). Thus we may assume $x_1$ is a cut-vertex of $H_1$. Since $d_{H_1}(x_1) \leq 2$, $d_{H_1}(x_1) = 2$. Thus $\text{comp}(H_1 - x_1) = 2$, and $\text{comp}(H') \leq \text{comp}(H) - 1$ for the new $H'$. This contradicts (A2).

**Case 2.** Suppose $\text{comp}(H) = 3$.

Without loss of generality, we may assume $|H_1| \geq |H_2| \geq |H_3|$. Since $|H| \geq 18$ by Claim 1, we have $|H_1| \geq 6$. Let $P_1 = u_1, \ldots, u_s$ be a longest path in $H_1$. Note $s \geq 3$. By Theorem 1, there exists $x_j \in V(H_j)$ for each $j \in \{2, 3\}$ such that $d_{H_j}(x_j) \leq 2$.

First suppose $u_1u_s \in E(G)$. Then $P_1[u_1, u_s], u_1$ is a Hamiltonian cycle in $H_1$, otherwise, since $H_1$ is connected, there exists a longer path than $P_1$, a contradiction. Since $H_1$ does not contain a chorded cycle, we have $u_1u_3 \notin E(H_1)$. Note $d_{H_1}(u_i) = 2$ for each $i \in \{1, 3\}$. 


Let $X = \{u_1, u_3, x_2, x_3\}$. Then $X$ is an independent set and $d_H(X) \leq 8$. By Subclaim 2.1, the degree sequences from four vertices of $X$ to some $C$ in $\mathcal{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$ and $|C| = 4$.

Let $C = v_1, v_2, v_3, v_4, v_1$. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_3)$. Then $d_C(u_1) \geq 3$ and $N_C(u_3) \cap N_C(x_2) \cap N_C(x_3) \neq \emptyset$ by the degree sequences. Without loss of generality, we may assume $v_1 \in N_C(u_3) \cap N_C(x_2) \cap N_C(x_3)$. Suppose $d_C(u_1) = 4$. Then $C' = u_1, v_2, v_3, u_4$ is a 4-cycle with chord $u_1v_3$. Since $H_1$ contains a Hamiltonian cycle, $u_1$ is not a cut-vertex of $H_1$. Thus $H_1 - u_1$ is connected. Replacing $C$ in $\mathcal{C}$ by $C'$, we consider the new $H'$. Then $\text{comp}(H') \leq \text{comp}(H) - 2 = 3 - 2 = 1$. This contradicts (A2). Thus $d_C(u_1) = 3$ since $d_C(u_1) \geq 3$. Then the degree sequence is $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$.

In either case, it suffices to consider $d_C(u_1) = 3$, $d_C(u_3) = 2$ and $d_C(x_2) = 3$ and $d_C(x_3) = 4$. Without loss of generality, we may assume $v_j \in N_C(u_1)$ for each $1 \leq j \leq 3$. If $v_4 \in N_C(x_2) \cap N_C(x_3)$ then $C' = u_1, v_1, v_2, v_3, u_1$ is a 4-cycle with chord $u_1v_2$. Further, replacing $C$ with $C'$ we again reduce the number of components in $H$, a contradiction. Thus, we may assume $N_C(u_1) = N_C(x_2)$. Also, note that $C$ has a chord. Suppose $v_1v_3 \in E(G)$. Then $C' = u_1, v_1, v_3, u_4, u_3, u_1$ is a 4-cycle with chord $v_1v_3$. Since $d_C(x_3) = 4$, $v_3 \in N_C(x_3)$. Thus, we can again reduce the number of components in $H$, a contradiction. A similar argument applies if $v_2v_4 \in E(G)$.

Next suppose $u_1u_s \notin E(G)$. Let $X = \{u_1, u_s, x_2, x_3\}$. Since $H_1$ does not contain a chorded cycle, $d_{H_1}(u_i) \leq 2$ for each $i \in \{1, s\}$. Then $X$ is an independent set and $d_H(X) \leq 8$. Replacing $u_3$ by $u_s$ in the above case where $u_1u_s \in E(G)$, we get a similar contradiction.

**Case 3.** Suppose $\text{comp}(H) = 2$.

Let $|H_1| \geq |H_2|$. Since $|H| \geq 18$ by Claim 1, $|H_1| \geq 9$. Let $P_i = u_1, \ldots, u_s$ be a longest path in $H_1$. Note $s \geq 3$. By Theorem 1, there exists $x_2 \in V(H_2)$ such that $d_{H_2}(x_2) \leq 2$.

First suppose $u_1u_s \in E(H_1)$. Note $P_i[u_1, u_s], u_1$ is a Hamiltonian cycle in $H_1$. Then $X_0 = \{u_1, u_3, u_5\}$ is an independent set and $d_{H_1}(X_0) = 6$, and $X = X_0 \cup \{x_2\}$ is an independent set and $d_H(X) \leq 8$. By Subclaim 2.1, the degree sequences from four vertices of $X$ to
some $C$ in $\mathcal{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$, and $|C| = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$. Since $X_0$ is on the Hamiltonian cycle, we may assume $d_C(u_1) = \max\{d_C(u) | u \in \{u_1, u_3, u_5\}\}$. Then $d_C(u_1) \geq 3$ by the degree sequences. Suppose $d_C(u_1) = 4$. Since $N_C(u_3) \cap N_C(x_2) \neq \emptyset$ by the degree sequences, without loss of generality, we may assume $v_4 \in N_C(u_3) \cap N_C(x_2)$. Since $d_C(u_1) = 4$, $v_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Then $C' = u_1, v_1, v_2, v_3, u_1$ is a 4-cycle with chord $u_1v_2$. Since $H_1$ contains a Hamiltonian cycle, $u_1$ is not a cut-vertex of $H_1$. Thus $H_1 - u_1$ is connected. Replacing $C$ in $\mathcal{C}$ by $C'$, we consider the new $H'$. Then $\text{comp}(H') \leq \text{comp}(H) - 1 = 2 - 1 = 1$ for the new $H'$. This contradicts (A2). Now suppose $d_C(u_1) = 3$. Then by the maximality of $d_C(u_1)$, we have only to consider the case where $d_C(u_i) = 3$ for each $i \in \{1, 3, 5\}$, and $d_C(x_2) = 4$. Let $v_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Then we may assume $N_C(u_1) = N_C(u_3) = N_C(u_5)$, otherwise, we get a contradiction by the same arguments as the case where $d_C(u_1) = 4$. Note $C$ has a chord. Suppose $v_1v_3 \in E(G)$. Then $C' = u_1, v_1, v_4, v_3, u_1$ is a 4-cycle with chord $v_1v_3$. Since $d_C(x_2) = 4$, $v_2 \in N_C(u_3) \cap N_C(x_2)$. Then $\text{comp}(H') \leq \text{comp}(H) - 1 = 2 - 1 = 1$ for the new $H'$, a contradiction. Suppose $v_2v_4 \in E(G)$. Then $C' = u_1, v_1, v_4, v_2, u_1$ is a 4-cycle with chord $v_1v_2$. Since $d_C(x_2) = 4$, $v_3 \in N_C(u_3) \cap N_C(x_2)$. Then $\text{comp}(H') \leq \text{comp}(H) - 1 = 2 - 1 = 1$ for the new $H'$, a contradiction.

Next suppose $u_1u_s \notin E(H_1)$. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_s)$. Assume $P_1$ is a Hamiltonian path in $H_1$. Note $s \geq 9$ since $|H_1| \geq 9$. Since $P_1$ is a Hamiltonian path in $H_1$, note $d_{P_1}(u) = d_{H_1}(u)$ for any $u \in V(P_1)$. We also note $d_{P_1}(u_i) \leq 2$ for each $i \in \{1, s\}$. Suppose $d_{P_1}(u_1) = 1$. By Lemma 7, $d_{H_1}(u_i) = 2$ for some $3 \leq i \leq 5$. Since $s \geq 9$, $X_0 = \{u_1, u_i, u_s\}$ is an independent set and $d_{H_1}(X_0) \leq 6$. Thus $X = X_0 \cup \{x_2\}$ is an independent set and $d_{H}(X) \leq 8$. Then we get a contradiction by the same arguments as the case where $u_1u_s \in E(G)$. Next suppose $d_{P_1}(u_1) = 2$. Now assume $u_1u_3 \in E(H_1)$. By Lemma 7, $d_{H_1}(u_i) = 2$ for some $4 \leq i \leq 6$. Since $s \geq 9$, $X_0 = \{u_1, u_i, u_s\}$ is an independent set and $d_{H_1}(X_0) \leq 6$, and we get a contradiction by considering $X = X_0 \cup \{x_2\}$ similar to the case where $u_1u_s \in E(H_1)$. Thus $u_1u_3 \notin E(H_1)$, that is, $u_1u_i \in E(H_1)$ for some $4 \leq i \leq s - 1$. By Lemma 6, $d_{H_1}(u_{i-1}) = 2$. Since $s \geq 9$, $X_0 = \{u_1, u_{i-1}, u_s\}$ is an independent set and $d_{H_1}(X_0) \leq 6$,
and we get a contradiction by considering $X = X_0 \cup \{x_2\}$.

Assume $P_1$ is not a Hamiltonian path in $H_1$. Then $V(H_1 - P_1) \neq \emptyset$. Let $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) be a longest path in $H_1 - P_1$. Without loss of generality, we may assume $d_{H_1}(v_1) \leq d_{H_1}(v_t)$. If $u_1u_s \in E(H_1)$, then since there exists a longer path than $P_1$, we may assume $u_1u_s \notin E(H_1)$. Also we may assume $d_{H_1}(v_1) \leq 2$, otherwise, since $d_{P_1}(v_i) \geq 1$ for each $i \in \{1, t\}$ by Lemma 11 (iii) and (iv), there exists a cycle in $\langle P_1 \cup P_2 \rangle$ with chord adjacent to $v_1$, a contradiction. Thus $X_0 = \{u_1, u_s, v_1\}$ is an independent set and $d_{H_1}(X_0) \leq 6$. Then $X = X_0 \cup \{x_2\}$ is an independent set and $d_H(X) \leq 8$. By Subclaim 2.1, the degree sequences from four vertices of $X$ to some $C$ in $\mathcal{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$, and $|C| = 4$. Let $C = w_1, w_2, w_3, w_4, w_1$. Since $d_C(u_1) \geq d_C(u_s)$ by our assumption, $d_C(u_1) \geq 3$ by the degree sequences. First suppose $d_C(u_1) = 4$. Since $N_C(v_1) \cap N_C(x_2) \neq \emptyset$ by the degree sequences, without loss of generality, we may assume $w_3 \in N_C(v_1) \cap N_C(x_2)$. Since $d_C(u_1) = 4$, $w_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Then $C' = u_1, w_1, w_2, w_3, u_1$ is a 4-cycle with chord $u_1w_2$. Since $u_1$ is an endpoint of the longest path $P_1$, $u_1$ is not a cut-vertex of $H_1$. Thus $H_1 - u_1$ is connected. Then $\text{comp}(H') \leq \text{comp}(H) - 1 = 2 - 1 = 1$ for the new $H'$. This contradicts (A2). Suppose $d_C(u_1) = 3$. Then we may assume the degree sequence is $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$.

Then it suffices to assume that $d_C(u_1) = 3$, $d_C(u_s) = 2$, and \{d_C(v_1), d_C(x_2)\} = \{3, 4\}. Without loss of generality, we may assume $w_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Suppose $d_C(v_1) = 3$ and $d_C(x_2) = 4$. Then we may assume $N_C(v_1) = N_C(v_1)$, otherwise, we get a contradiction by the same arguments as the case where $d_C(u_1) = 4$. Note that $C$ has a chord. Suppose $w_1w_3 \in E(G)$. Then $C' = u_1, w_1, w_3, u_1$ is a 4-cycle with chord $w_1w_3$. Since $d_C(x_2) = 4$, $w_2 \in N_C(v_1) \cap N_C(x_2)$. Then $\text{comp}(H') \leq \text{comp}(H) - 1 = 2 - 1 = 1$ for the new $H'$, a contradiction. Suppose $w_2w_4 \in E(G)$. Then $C' = u_1, w_1, w_2, u_1$ is a 4-cycle with chord $w_1w_2$. Since $d_C(x_2) = 4$, $w_3 \in N_C(v_1) \cap N_C(x_2)$. Then $\text{comp}(H') \leq \text{comp}(H) - 1 = 2 - 1 = 1$ for the new $H'$, a contradiction. If $d_C(v_1) = 4$ and $d_C(x_2) = 3$, then we get a contradiction in a similar manner.

$\square$
Claim 3. $H$ contains a Hamiltonian path.

Proof. Suppose not, and let $P_1 = u_1, \ldots, u_s$ be a longest path in $H$. Note $s \geq 3$ since $|H| \geq 18$ and $H$ is connected by Claim 2. Let $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) be a longest path in $G - P_1$ such that $d_{P_2}(v_i) \leq d_{P_2}(v_t)$. By Lemma 12, there exists an independent set $X$ of four vertices in $H$ such that $\{u_1, u_s, v_1\} \subseteq X$ and $d_H(X) \leq 8$.

Then the degree sequences from four vertices of $X$ to some $C$ in $\mathcal{C}$ are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$, and $|C| = 4$. Let $C = \{x_1, x_2, x_3, x_4, x_5\}$. We may assume $u_1 u_s \notin E(H)$, otherwise, a path longer than $P_1$ exists, a contradiction. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_s)$. By the degree sequences, we have $d_C(u_1) \geq 3$.

Suppose $d_C(u_1) = 4$. Since $N_C(u_s) \cap N_C(v_1) \neq \emptyset$ by the degree sequences, without loss of generality, we may assume $x_4 \in N_C(u_s) \cap N_C(v_1)$. Since $d_C(u_1) = 4$, $u_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord $u_1 x_2$. Since $u_1$ is an endpoint of the longest path $P_1$, $u_1$ is not a cut-vertex of $H$. Thus $H - u_1$ is connected. Replacing $C$ in $\mathcal{C}$ by $C'$, we consider the new $H'$. Then $P_1[u_2, u_s, x_4, P_2[v_1, v_t]$ is a longer path than $P_1$ in $H'$. This contradicts (A3).

Suppose $d_C(u_1) = 3$. Then we may assume the degree sequence is $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$. First assume the degree sequence is $(4, 4, 3, 2)$. Since $d_C(u_1) \geq d_C(u_s)$, we have $d_C(u_1) = 3$, $d_C(u_s) = 2$ and $d_C(v_1) = 4$. Without loss of generality, we may assume $x_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord $u_1 x_2$. Note $u_1$ is not a cut-vertex of $H$. If $x_4 \not\in N_C(u_s)$, then since $d_C(v_1) = 4$, there exists a longer path than $P_1$ in the new $H'$, a contradiction. Thus we may assume $x_4 \not\in N_C(u_s)$. Note $C$ has a chord. Suppose $x_1 x_2 \in E(G)$. Assume $x_2 \in N_C(u_s)$. Then $C' = u_1, x_3, x_4, x_1, u_1$ is a 4-cycle with chord $x_1 x_3$. Since $d_C(v_1) = 4$, $x_2 \not\in N_C(u_s) \cap N_C(v_1)$, and there exists a longer path than $P_1$ in the new $H'$, a contradiction. Thus $x_2 \not\in N_C(u_s)$. Since $d_C(u_s) = 2$, $x_1, x_3 \in N_C(u_s)$. Then $C' = u_s, x_3, x_4, x_1, u_s$ is a 4-cycle with chord $x_1 x_3$. Note $u_s$ is not a cut-vertex of $H$. Since $d_C(v_1) = 4$, $x_2 \not\in N_C(u_1) \cap N_C(v_1)$. Then $P_1^{-}[u_{s-1}, u_1, x_2, P_2[v_1, v_t]$ is a longer path than $P_1$ in the new $H'$, a contradiction. Suppose $x_2 x_4 \in E(G)$.
Assume $x_3 \in N_C(u_s)$. Then $C' = u_1, x_1, x_4, x_2, u_1$ is a 4-cycle with chord $x_1x_2$. Since $d_C(v_1) = 4$, $x_3 \in N_C(u_s) \cap N_C(v_1)$. Then there exists a longer path than $P_1$ in the new $H'$, a contradiction. Thus $x_3 \notin N_C(u_s)$. By symmetry, $x_1 \notin N_C(u_s)$. Thus $d_C(u_s) \leq 1$. This contradicts $d_C(u_s) = 2$.

Next assume the degree sequence is $(4, 3, 3, 3)$. In this case, we have only to consider the degree sequence $(3, 3, 3)$ for $\{u_1, u_s, v_1\}$. Then $d_C(u_1) = d_C(u_s) = d_C(v_1) = 3$. Thus $|N_C(u_s) \cap N_C(v_1)| \geq 2$. Let $x_i \in N_C(u_1)$ for each $1 \leq i \leq 3$. Suppose $x_1x_3 \in E(G)$. If $x_i \in N_C(u_s) \cap N_C(v_1)$ for some $i \in \{2, 4\}$, then there exists a longer path than $P_1$, a contradiction. Thus $x_1, x_3 \in N_C(u_s) \cap N_C(v_1)$. Suppose $x_4 \in N_C(u_s)$ and $x_2 \in N_C(v_1)$. Then $C' = u_s, x_4, x_1, x_3, u_s$ is a 4-cycle with chord $x_3x_4$, and $P_1^{-1}[u_{s-1}, u_1], x_2, P_2[v_1, v_1]$ is a longer path than $P_1$ in the new $H'$, a contradiction. Suppose $x_2 \in N_C(u_s)$ and $x_4 \in N_C(v_1)$. Let $w \in X - \{u_1, u_s, v_1\}$. Then $d_C(w) = 4$ by our assumption of the degree sequence $(3, 3, 3)$. Assume $w \in V(P_1)$. Then $P_1[u_1, u_s], x_2, u_1$ is a cycle with chord $wx_2$, and $v_1, x_1, x_4, x_3, v_1$ is the other cycle with chord $x_1x_3$. Thus we have two distinct chorded cycles in $H \cup C'$, and $G$ contains $k$ vertex-disjoint chorded cycles, a contradiction. Assume $w \notin V(P_1)$. Then $C' = u_s, x_3, x_4, x_1, u_s$ is a 4-cycle with chord $x_1x_3$. Since $d_C(w) = 4$, $w, x_2, P_1[u_1, u_{s-1}]$ is a longer path than $P_1$ in the new $H'$, a contradiction. Suppose $x_2x_4 \in E(G)$. Note $|N_C(u_s) \cap N_C(v_1)| \geq 2$. If $x_i \in N_C(u_s) \cap N_C(v_1)$ for some $i \in \{1, 3, 4\}$, then there exists a longer path than $P_1$, a contradiction. Thus $|N_C(u_s) \cap N_C(v_1)| \leq 1$, a contradiction.

By Claims 1, 3 and Lemma 10, $H$ contains an independent set $X$ of four vertices such that $d_H(X) \leq 8$. By Claim 3 and Lemma 13,

$$d_G(X) = d_{\mathcal{G}}(X) + d_H(X) \leq 12(k - 1) + 8 = 12k - 4.$$  

This contradicts the $\sigma_4(G)$ condition. This completes the proof of Theorem 5.

\textbf{Acknowledgments.} The first author is supported by the Heilbrun Distinguished Emeritus Fellowship from Emory University. The second author is supported by JSPS KAKENHI Grant Number JP19K03610.
References


