DEGREE SUM AND VERTEX DOMINATING PATHS

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Abstract. A *vertex dominating path* in a graph is a path P such that every vertex outside P has a neighbor on P. In 1988 H. Broersma [HB88] stated a result implying that every n-vertex k-connected graph G such that $\sigma_{(k+2)}(G) \geq n - 2k - 1$ contains a vertex dominating path. We provide a short, self-contained proof of this result and further show that every n-vertex k-connected graph such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ contains a vertex dominating path of length at most $(20k)|T|$, where T is a minimum dominating set of vertices. An immediate corollary of this result is that every such graph contains a vertex dominating path with length bounded above by a logarithmic function of the order of the graph. To derive this result, we prove that every n-vertex k-connected graph with $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ contains a path of length at most $20k|T|$, through any set of T vertices where $|T| \leq n/900k^4$.

1. Introduction

Interest in dominating cycles and paths of various sorts began as a natural relaxation of hamiltonian cycle and path problems and moved in a number of directions: edge dominating cycles (paths), vertex dominating cycles (paths), longest cycles (paths) that dominate in some manner and so forth.

In particular, a paper by Bondy and Fan ([BF87]) proving a conjecture of Clark, Coburn and Erdős [CCE], gave a condition for degree sums of sets of $k+1$ independent vertices in k-connected graphs that imply the existence of a vertex dominating cycle. Shortly thereafter Broersma ([HB88]) produced a general result providing generalized degree sum conditions in k-connected graphs forcing all vertices to be within a fixed distance of a cycle. Furthermore, at the end of the paper, an analogue for paths was stated. There has continued to be investigation into vertex dominating cycle structures, though it has tended to focus on long cycles. (See [MOS], [SY].)

On the other hand, a vertex dominating path may also be viewed as a spanning tree of a particular type, sometimes called a caterpillar. There has been much recent work on conditions implying particular structural properties in spanning trees. See for example [FKKLR], [CFHJL], or the recent survey by Ozeki and Yamashita ([OY11]) on spanning trees.

A converging of these two streams of research occurred in a recent paper by Faudree, Gould, Jacobson, and West ([FGJW]) which contains several theorems relating minimum degree and vertex dominating paths (or spanning caterpillars). Motivated by [FGJW], this paper contains

Date: January 12, 2018.
*This work was supported by the Heilbrun Distinguished Emeritus Fellowship from the Emeritus College of Emory University.
**This work was partially funded by NSA Young Investigator Grant H98230-15-1-0258.

results relating degree sum conditions and dominating paths. Included is a short, self-contained proof of a result originally stated in [HB88] as well as a theorem and corollary that answers a question from [FGJW].

All graphs are finite and simple. Notation and terminology generally follows West [DW96]. A set $X \subset V(G)$ *dominates* the graph G if every vertex of $G - X$ has a neighbor in X. Observe that this definition means the set X dominates the vertex set but not necessarily the edge set of G. We will often say a path P or a cycle C dominates G if $V(P)$ or $V(C)$ dominates $V(G)$.

Given a graph G and integer $k \geq 2$, we denote by $\sigma_k(G)$ the minimum degree-sum of independent sets of k vertices. Observe that, by a natural extension of this definition, $\sigma_1(G) = \delta(G)$. Given a set of vertices $X = \{x_1, x_2, \cdots, x_m\} \subseteq V(G)$, the notation $N[X] = X \cup \{v \in V(G) \mid \exists x \in X, vx \in E(G)\}$, often called the *closed neighborhood of X.*

The primary result in this paper is the theorem stated below. It gives sufficient connectivity and σ_2 conditions to ensure that there is a short path in G containing any given subset of vertices T such that $|T|$ is a linear function of n. It requires G have sufficiently large order and "short" in this case means a constant multiple of $|T|$.

Theorem 1.1. *Let $k \geq 1$ be an integer. Let G be a k-connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ where $f(k)$ is a sufficiently large constant depending only on k, and let $T \subseteq V(G)$ such that $|T| \leq n/900k^4$. Then there exists a path in G on at most $(20k)|T|$ vertices containing all the vertices of T.*

The proof of this theorem is given in Section 4, after several preliminary lemmas.

2. Degree Sum and the Existence of a Dominating Path

In 1985, Clark, Colbourn, and Erdős [CCE] conjectured that every k-connected graph with minimum degree at least $\frac{n}{k+1} + f(k)$ has a vertex dominating cycle. In [BF87], Bondy and Fan proved the conjecture holds and made another similar conjecture replacing degree sum with distance m neighborhoods and replacing dominating cycles with m-dominating cycles (where a cycle is m-dominating if every vertex is at most distance m from the cycle.) In [HB88], Broersma generalized the classic Erdős-Chvátal condition for hamiltonicity and, as one part of one corollary, settled the conjecture of Bondy and Fan. At the end of the paper by Broersma, several general results concerning paths, all of which are analogues of the cycle results proved earlier, are stated. These earlier proofs are intricate, nontrivial, and carefully linked. Below is one corollary. Note that a Δ_λ*-traceable graph* is one in which there exists a path such that all vertices are a distance less than λ from the path and a set of *mutually r-distant vertices* is a set $S \subseteq V(G)$ such that for every $u, v \in S$, $u \neq v$, the distance from u to v is at least r.

Corollary 2.1. [HB88] *Let G be an n-vertex, k-connected graph $(k \geq 1)$ and let $\lambda \geq 2$. If the degree sum of any $k+2$ mutually $(2\lambda - 1)$-distant vertices is at least $n - 2k - 1 - (\lambda - 2)k(k+2)$, then G is Δ_λ-traceable.*

We offer a self-contained proof of the special case of Corollary 2.1 when $\lambda = 2$ and the path is vertex-dominating. Note that when $\lambda = 2$, the hypothesis in the corollary above concerns degree sums of mutually 3-distant vertices and the hypothesis in the theorem below uses a σ_{k+2} condition which applies to *all* nonadjacent pairs of vertices. Since any set of 3-distant vertices would certainly be nonadjacent, the Theorem below appears to have stronger hypotheses than the corollary above. In fact, the proof of the Theorem only requires the condition on 3-distant vertices, but we state the stronger condition as it is more common in recent literature.

Theorem 2.1. *Every n-vertex k-connected graph with $\sigma_{(k+2)}(G) \geq n - 2k - 1$ contains a vertex-dominating path.*

Proof. Let G satisfy the hypothesis of the theorem and proceed by contradiction. Let P be a path in G such that $|N[P]|$ is maximized. That is, the total number of vertices on the path and dominated by the path is maximized. Further, among all paths dominating a maximum number of vertices, choose P to be as short as possible. Label the vertices of P as: $x = x_1, x_2, \cdots, x_t = y$. If all the neighbors of x (or y) were dominated by the vertices of $P - x$ (or $P - y$), then a shorter path dominating the same number of vertices is possible. Thus there exist vertices x' (and y') such that $N(x') \cap V(P) = \{x\}$ (and $N(y') \cap V(P) = \{y\}$). In this case, we will say x' is *uniquely dominated* by x or that x *uniquely dominates* x'. Clearly this relationship depends upon the choice of P, but we are assuming the choice of P is fixed.

Since G has no vertex dominating path, there exists a vertex, z, not dominated by P. Since G is k-connected, there exist k paths from z to P that are vertex disjoint other than at initial vertex z. Pick these k paths to be as short as possible and label the terminal vertices $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ in order as they appear on P from x to y.

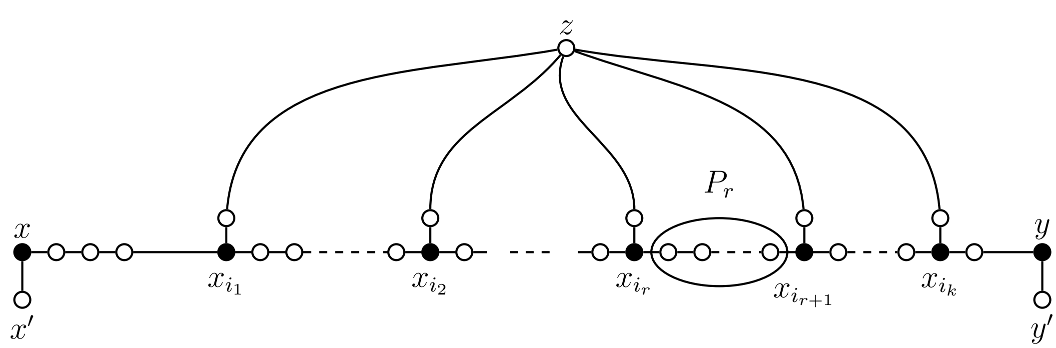


FIGURE 1. This figure illustrates the paths from vertex z to P, a shortest path dominating a maximum number of vertices. Note the circled section denotes P_r, the section of the path P strictly between the endpoints of two consecutive paths from z, x_{i_r} and $x_{i_{r+1}}$.

Let $P_r = P(x_{i_r}, x_{i_{r+1}})$ denote the subpath of P strictly between two consecutive endpoints of paths from z. Note the index r ranges from $r = 1$ to $r = k - 1$.(See Figure 1.)

A vertex in a subpath P_r is called *moveable* if it and all of its neighbors in $G - P$ have adjacencies on $P - P_r$. (That is, a vertex is moveable if it and all of its neighbors are dominated by vertices of P outside P_r and thus could be moved.) Note that for each r, the vertex that follows x_{i_r} on the path P, namely x_{i_r+1}, would be moveable unless x_{i_r+1} is the unique neighbor on P for some vertex in $G - P$.

If there exists a single subpath such that all vertices on it are moveable, then a path dominating more vertices can be obtained by replacing this subpath with a path through z. Thus, for each of the $k-1$ subpaths, there exists at least one vertex that is not moveable. Label as u_r the first vertex in P_r that is not moveable as P_r is traversed from x_{i_r} to $x_{i_{r+1}}$.

Let Q_r consist of all the vertices on P_r between x_{i_r} and u_r. (That is, Q_r consists of moveable vertices.) Observe that, for $r_1 \neq r_2$, there can be no edges between vertices in Q_{r_1} and those in Q_{r_2}. Further, vertices in Q_{r_1} and vertices in Q_{r_2} can have no common neighbors in $V - V(P)$. If any such edges or paths existed, choosing the "first" one (that is, the one with smallest index on P) would produce a path that dominates all the vertices that P dominates and z. The importance of this observation is the conclusion that all vertices of Q_{r_1} are not only moveable, but are moveable to regions of P *other than* Q_{r_2} as is the case for the vertices dominated by Q_{r_1}. (See Figure 2.)

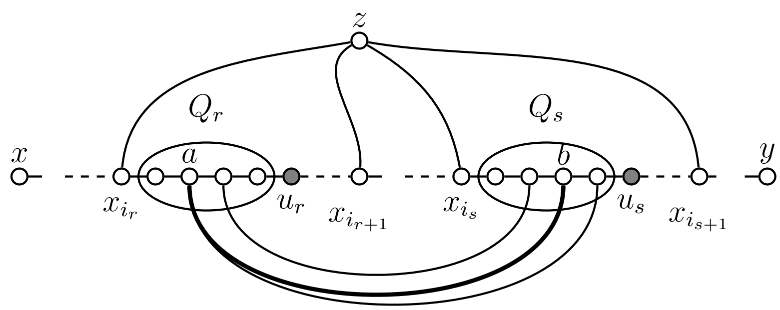


FIGURE 2. For each subpath of P, u_i is the first *nonmoveable* vertex and, so, those in Q_i are all moveable. Observe that any edges between vertices in distinct Q_i's results in a path dominating more vertices by using the edge between vertices of smallest indices (or, alternatively, left-most vertices). New path: x to x_{i_r} to z to x_{i_s} to a to b to y. Note that all vertices of Q_r and Q_s not on the new path are dominated elsewhere.

For each r, the fact that u_r is not moveable implies either u_r dominates a vertex, u_r^*, that is not dominated by any vertex outside P_r (i.e. $N_P(u_r^*) \subseteq V(P_r)$), or u_r itself is not dominated outside of P_r (i.e. $N_P(u_r) \subseteq V(P_r)$.) Observe that the fact that u_r is the first vertex that is not moveable, implies that neither u_r nor u_r^* can have any adjacencies on any of the paths P_i or the extremal choice of P is contradicted.

Now define a set of vertices S to contain x', y', and z. Furthermore, for every subpath, add either u_r^* or u_r to S, using whichever one has no neighbors on $P - P_r$.

By definition, $|S| = k+2$, S is independent, and no pair of vertices in S can have a common neighbor on P. Moreover, if any pair of vertices in S has any common neighbor outside of P a path dominating more vertices can be found. Specifically, this path can be built to include z, along with the two involved vertices of S and relies on the fact that vertices of Q_{r_1} and Q_{r_2} can be moved if needed. Thus, for every $a, b \in S$ such that $a \neq b$, $N(a) \cap N(b) = \emptyset$. Observe that none of the vertices in S are adjacent to $\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$ since z is, by definition, a distance at least 2 from the path and the remaining vertices are chosen precisely such that their neighborhoods on P are restricted to x, y, or P_r for some r. Thus, we have $\sigma_{k+2}(G) \leq \sum_{a \in S} d(a) \leq n - 2k - 2$, a contradiction. □

The following example shows that the preceding theorem is best possible.

Example 2.1. *Construct an n-vertex k-connected graph G as follows. Begin with a complete graph on k vertices, K_k, and $k+2$ independent vertices, $v_1, v_2, \cdots, v_{k+2}$. Partition the remaining $n-2k-2$ vertices into $k+2$ complete graphs such that the orders of the graphs are as equal as possible and label them $B_1, B_2, \cdots, B_{k+2}$. For each i add all edges between B_i and $K_k \cup \{v_i\}$. This graph has no dominating path and $\sigma_{k+2}(G) = n-2k-2$. Note that $n \geq k^2+4k+2$.*

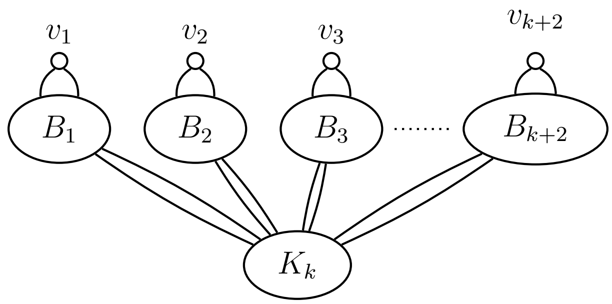


FIGURE 3. Example 2.1

The following corollary follows immediately from the statement of the theorem:

Corollary 2.2. *Let $1 \leq r \leq k+2$ be an integer. Every n-vertex k-connected graph with $\sigma_r(G) \geq r\left(\frac{n-2k-1}{k+2}\right)$ contains a vertex-dominating path.*

The following corollary follows immediately from the proof of the theorem:

Corollary 2.3. *If $\alpha(G) \leq \kappa(G)+1$, then G contains a vertex dominating path.*

The graph from Example 2.1 without the vertices v_i shows that this corollary is sharp.

3. Preliminary Lemmas

In order to streamline the proof of the main theorem in Section 4, several preliminary results are presented. Note that in all of the lemmas below $k \in \mathbb{Z}^+$. We begin with a well-known result by Dirac:

Lemma 3.1. [GD52] *If G is k-connected and $X \subseteq V(G)$ such that $|X| \leq k$, then there exists a cycle in G containing all the vertices of X.*

The following lemma asserts the existence of small dominating sets relative to a minimum degree condition. It is similar to some old, well known results (see eg. [AS, VA74, CP75]) and is used to prove an analogous result for degree sum conditions, Lemma 3.3.

Lemma 3.2. *Every n vertex graph G with minimum degree $\delta \geq \beta n$ where $0 < \beta < 1$ contains a dominating set $X \subseteq V(G)$ such that $|X| \leq \lceil \log_{1/(1-\beta)} n \rceil$.*

Proof. Let G be an n vertex graph with minimum degree $\delta \geq \beta n$ where $0 < \beta < 1$. The proof will proceed by iteratively constructing a dominating set of vertices using no more than $\lceil \log_{1/(1-\beta)} n \rceil$ vertices.

Begin by choosing an arbitrary vertex x_1. Let $X_1 = \{x_1\}$ and let $S_1 = V(G) - N[X_1]$. So S_1 consists of the set of vertices not dominated by x_1. Given the iteratively constructed set $X_i = \{x_1, x_2, \cdots, x_i\}$ and $S_i = V(G) - N[X_i]$, we construct X_{i+1} by adding any vertex x_{i+1} such that $|N(x_{i+1}) \cap S_i| \geq \beta|S_i|$.

We must first show such a vertex as x_{i+1} exists. For a given i, let $m = |N[X_i]|$. Thus, $|S_i| = n - m$. Proceeding by contradiction, we will assume that no vertex v of G has the property that $|N(v) \cap S_i| \geq \beta|S_i|$. Since all edges incident with vertices of S_i either come from vertices in S_i or vertices of $N[X_i] - X_i$, we can count the degree sum of vertices in S_i as follows:

$$\begin{aligned}\beta n \cdot (n-m) &\leq \sum_{v \in S_i} d(v) \\ &< \beta(n-m)\cdot(n-m) + \beta(n-m)\cdot(m-i) \\ &= \beta(n-m)(n-i) \\ &< \beta(n-m)n,\end{aligned}$$

a contradiction.

We claim that in $r = \lceil \log_{1/(1-\beta)} n \rceil$ iterations, the set $|S_r| < 1$ and so X dominates $V(G)$.

By construction $|S_{i+1}| < (1-\beta)|S_i| < (1-\beta)^{i+1}n$. For $r > \log_{1/(1-\beta)} n$, $|S_r| < 1$. Thus, the dominating set X requires at most $r = \lceil \log_{1/(1-\beta)} n \rceil$ vertices. □

The next lemma, an analogue of Lemma 3.2, uses the same proof technique.

Lemma 3.3. *Every n vertex graph G with $\sigma_2(G) \geq 2\beta n$ where $0 < \beta < 1$ contains a dominating set $X \subseteq V(G)$ such that $|X| \leq \lceil \log_{1/(1-\beta)} n \rceil$.*

Proof. Let G be an n vertex graph with $\sigma_2(G) \geq 2\beta n$ where $0 < \beta < 1$. The proof will proceed by iteratively constructing a dominating set of vertices using no more than $\lceil \log_{1/(1-\beta)} n \rceil$ vertices.

If $\delta(G) \geq \beta n$, then apply Lemma 3.2. Otherwise, choose a pair of nonadjacent vertices, x_1 and x_2 such that $d(x_1) < \beta n$. Let $X_2 = \{x_1, x_2\}$ and define $S_2 = V(G) - N[X_2]$, the set of vertices not dominated by x_1 or x_2. Observe that for every $v \in S_2$, $d(v) > \beta n$ because v and x_1 are nonadjacent. Also, note that $|S_2| < (1-2\beta)n < (1-\beta)^2 n$. Now, noting that the minimum degree in S_2 is at least βn, the sets $X_i = \{x_1, x_2, \cdots, x_i\}$ and $S_i = V(G) - N[X_i]$ can be constructed as before.

The proof now follows precisely the same argument and the same arithmetic as Lemma 3.2. □

The next lemma provides conditions under which the existence of some cycle through a specified set of vertices implies the existence of a *small* cycle through the specified set.

Lemma 3.4. *Let G be an n-vertex graph with $\sigma_2(G) \geq \frac{2n}{k+2}$. Let $X \subseteq V(G)$ such that $|X| \leq n/10k^2$. If there exists a cycle in G containing all vertices of X, then, for n sufficiently large, a smallest cycle containing all vertices of X has at most $(3k+5)|X|$ vertices.*

Proof. Let G and X satisfy the hypothesis of the Lemma and let C be a smallest cycle of G containing X. Proceed by contradiction and assume $|V(C)| > (3k+5)|X|$. Then there exists a segment of C between consecutive vertices of X with at least $3k+5$ vertices strictly between them, say $x_1 = v_0, v_1, v_2, \cdots, v_q = x_2$, where $q \geq 3k+6$, $x_1, x_2 \in X$ and $v_1, v_2, \cdots, v_{q-1} \notin X$. Let $S = \{v_0, v_3, v_6, \cdots, v_{3k+6}\}$.

Because C was chosen to be as small as possible, the set S is independent and no two vertices of S can have a common neighbor in $G - C$. Let $b = n - |V(C)|$.

Taking indices of v_i modulo $3k+6$, the σ_2-condition implies

$$\frac{1}{2}\sum_{i=0}^{k+2}(d(v_{3i}) + d(v_{3(i+1)})) \geq \frac{1}{2}(k+3)\left(\frac{2n}{k+2}\right) > n,$$

Thus, the number of chords in C from vertices of S is at least $\frac{(k+3)n}{k+2} - b - 2(k+3)$. Observe that all of these chords have their other endpoint outside the segment of C containing S. Let c count the number of vertices at the other end of the chords from S. Now, $b+c+3k+7 \leq n$. After allowing one chord to each of the c vertices, the number of "excess" chords is at least $\frac{(k+3)n}{k+2} - b - 2k - 6 - c > \frac{n}{k+2}$.

Now, we establish an upper bound on the number excess chords from S. Let t denote the number of vertices in a particular segment of C with at least one excess chord. Observe that no pair of chords from S to this segment can cross due to the choice of C as smallest possible. (See Figure 4.) Thus, consecutive pairs of the t vertices, taken in the order they appear on C, can share at most one neighbor in S. Thus, these t vertices can accept at most $k+3+(t-1)$ chords in total and, therefore, at most $k+2$ excess chords. Thus, the total number of excess chords is at most $(k+2)|X| < ((k+2)n)/(10k^2)$.

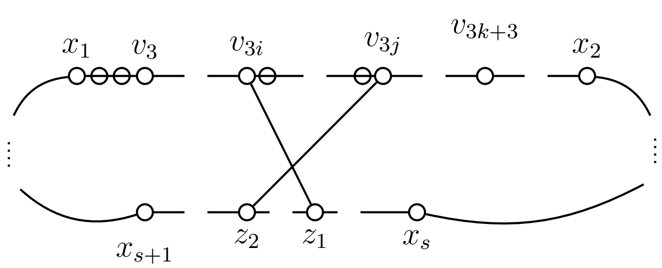


FIGURE 4. A single pair of crossing edges results in a smaller cycle. Follow $x_1 = v_0$ to v_{3i}, down to z_1, around to v_{3j} via x_s, down to z_2 and back to x_1.

But now we have a contradiction since the lower bound on the excess chords, $n/(k+2)$, is larger than the upper bound $(k+2)n/(10k^2)$ for all $k \in \mathbb{Z}^+$. Thus, C can have at most $(3k+5)|X|$ vertices. □

This next lemma gives conditions under which there exists a short path between two specified vertices avoiding a given set of vertices.

Lemma 3.5. *Let G be an n-vertex graph such that $\sigma_2(G) \geq \frac{2n}{k+2}$. Let $Y \subseteq V(G)$ such that $|Y| \leq n/10k^2$ and $u, v \in V(G) - Y$. If there exists a u,v-path in $G - Y$, then, for n sufficiently large, there exists a Y-avoiding u,v-path using at most $3k+6$ vertices.*

Proof. Let G, Y, u, and v satisfy the hypotheses of the Lemma. Proceed by contradiction and assume that the shortest Y-avoiding u,v-path contains at least $3k+7$ vertices, labeled $u = w_0, w_1, \cdots, w_q = v$, where $q \geq 3k+6$. Thus the set $\{w_0, w_3, \cdots, w_{3k+6}\}$ is not only an independent set of vertices, but has the property that any pair of vertices from this set have disjoint neighborhoods in $G - Y$. It follows that,

$$\begin{aligned} 2\left(n + \frac{n}{k+2}\right) = (k+3)\left(\frac{2n}{k+2}\right) &\leq \sum_{i=0}^{k+2} \left(d(w_{3i}) + d(w_{3(i+1)})\right) \\ &\leq 2\left(n - |Y| + (k+3)|Y|\right) \\ &\leq 2\left(n + \frac{(k+2)n}{10k^2}\right), \end{aligned} \tag{1}$$

where $w_{3k+9} = w_0$. Again, we have a contradiction since $n/(k+2) > (k+2)n/(10k^2)$. Thus, a smallest Y-avoiding u,v-path has at most $3k+6$ vertices. □

Lemma 3.6. *Let G be an n-vertex graph such that $\sigma_2(G) \geq \frac{2n}{k+2}$. Let $T \subseteq V(G)$ such that $|T| \leq n/(10k^2)$. If $\kappa(G) \geq (3k+6)|T|$, then, for n sufficiently large, there exists a cycle (or path) in G on at most $(3k+6)|T|$ vertices containing all the vertices of T in any order.*

Proof. Arbitrarily order the vertices of T: $x_1, x_2, \cdots, x_t$. Since $G - \{x_3, x_4, \cdots, x_t\}$ is connected, Lemma 3.5 implies there exists an x_1, x_2-path on at most $9k$ vertices avoiding $T - \{x_1, x_2\}$. Call it $P_{1,2}$. Inductively, extend this path to include the first r vertices of T. Call this path $P_{1,r}$ and assume it contains at most $(3k+6)(r-1)$ vertices. Since $\kappa(G - P_{1,r}) \geq (3k+6)(t-r+1)$, there will always exist an x_r, x_{r+1}-path avoiding $P_{1,r}$ provided $r \leq t$. Then, Lemma 3.5 implies a shortest x_r, x_{r+1}-path adds at most $(3k+6)$ additional vertices. Since $P_{1,t}$ uses at most $9k(t-1)$ vertices, $G - P_{1,t}$ is still connected. Thus, it is possible to find an x_r, x_1-path avoiding $P_{1,t}$ and Lemma 3.5 implies it adds at most $(3k+6)$ additional vertices. □

4. Proof of the Main Theorem

We restate the main theorem for ease of reference.

Theorem 1.1 *Let $k \geq 1$ be an integer. Let G be a k-connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ where $f(k)$ is a sufficiently large constant depending only on k, and let $T \subseteq V(G)$ such that $|T| \leq n/900k^4$. Then there exists a path in G on at most $(20k)|T|$ vertices containing all the vertices of T.*

Remark: The proof gives an explicit bound on $f(k)$ as a recursively defined function of k.

Proof. Let G and T satisfy the hypotheses of the Theorem and let $t = |T|$. Define a recursive sequence: $b_0 = 10k^2$ and, for $i \geq 1$, $b_i = 10k^2\left(\frac{k+2}{2}\right)^2 + \sum_{j=0}^{i-1} b_j$. Let $f(k) = 2b_{k+1}$. We remark that, for now, the precise form of $f(k)$ is unimportant – it suffices to think about it as simply a large constant. The reason for the precise form will become clear when it is needed in Case 3 below.

Assume G is t-connected. Lemma 3.1 implies there exists a cycle containing all the vertices of T, and Lemma 3.4 implies the smallest such cycle contains at most $(3k+6)t$ vertices. Thus, there exists a path containing all vertices of T using at most $(3k+6)t$ vertices, and the Theorem follows. Thus, we can assume G is k-connected and $k < t$.

Now, we will iteratively find minimum cut sets, denoted by S_i's. We will collect all the vertices of these iteratively selected cut sets into a set denoted by $\mathcal{S}$. The connected components of $G - \mathcal{S}$ will be denoted by G_j's. The indices of the S_i's and the G_j's are not important except that we will eventually argue that there are at most $k+2$ of the G_j's. Thus, we allow arbitrary reordering of the indices as needed. (See Table 1 for an illustration of this algorithm.)

For iteration one, we let S_0 be a minimum cut set of G; so $|S_0| < t$. Let $\mathcal{S} = S_0$ and $G_1, G_2, \cdots, G_l$ be the connected components of $G - \mathcal{S}$. If each G_j is complete or has connectivity at least $30kt$, we stop.

Otherwise, we proceed to iteration two where, for each component G_j in $G - \mathcal{S}$ that is not complete and has connectivity less than $30kt$, we find a minimum cut set S_i. We add all the vertices from these new cut-sets to $\mathcal{S}$. If each component of $G - \mathcal{S}$ is either complete or has connectivity at least $30kt$, we stop.

In general, on the ith iteration, we stop if all components of $G - \mathcal{S}$ are either complete or have connectivity at least $30kt$.

We claim the process stops with at most $k+2$ connected components, G_j. Observe that the first iteration produces at least two components and each subsequent iteration produces at least one additional component. Thus, the ith iteration must end in a collection of cut sets producing at least $i+1$ connected components.

Proceed by contradiction and find the first iteration, i_0, such that $G - \mathcal{S}$ contains more than $k+2$ connected components. From the previous observation, we know that $i_0 \leq k+2$. In fact, we know that at most $k+2$ cut sets in total were deleted from G. Since, every cut set has order less than $30kt$, at the end of iteration i_0, $|\mathcal{S}| < (k+2)(30kt)$.

Let $v_j \in G_j$ for all j. Now, applying the degree sum condition to consecutive pairs of vertices v_j (which lie in different components and hence are non-adjacent), we have

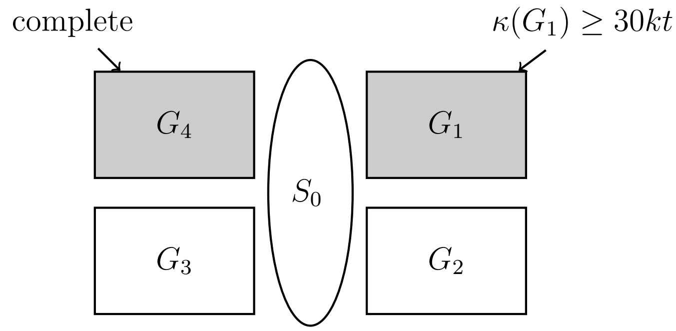
$$\sum_j |G_j| > \frac{1}{2}\sum_j (d(v_j) + d(v_{j+1}) - 2|\mathcal{S}|) \geq \frac{k+3}{2}\left(\frac{2n}{k+2} + f(k) - 2|\mathcal{S}|\right), \tag{2}$$

(where indices of v_j are taken modulo the number of components in $G - \mathcal{S}$.) Thus, using (2), we produce the contradiction:

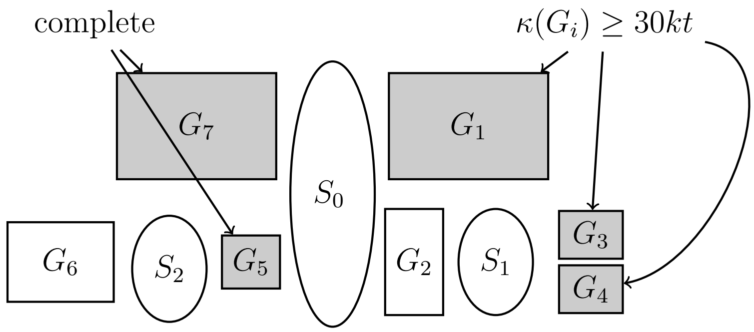
$$n = |\mathcal{S}| + \sum |G_i| > |\mathcal{S}| + \left(\frac{k+3}{k+2}\right)n + \frac{(k+3)f(k)}{2} - (k+3)|\mathcal{S}| > \frac{k+3}{k+2}n - (k+2)^2(30kt) > n, \tag{3}$$

for n sufficiently large and $t < \frac{n}{900k^4}$. Thus, we can assume that the iterative selection of cut sets terminates in at most $k+1$ iterations and with at most $k+2$ connected components.

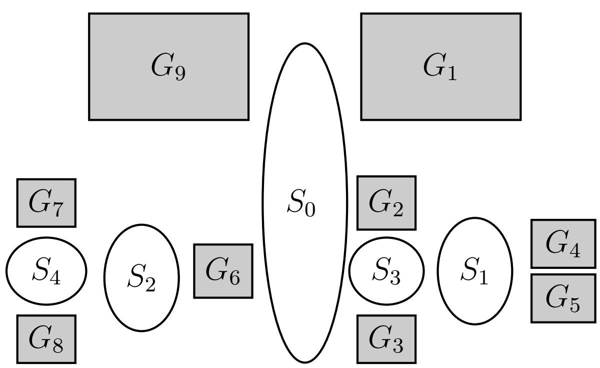
Table 1. Cut-Set Selection Algorithm



Iteration 1: A minimum cut set S_0 results in four connected components, G_1, G_2, G_3, and G_4. So $\mathcal{S} = S_0$ and $k \leq s < t$.



Iteration 2: Minimum cut sets S_1 and S_2 are found in noncomplete components of $G - \mathcal{S}$ with connectivity less than $30kt$. Now, $\mathcal{S} = S_0 \cup S_1 \cup S_2$, $s < t + 2(30kt)$ and $G - \mathcal{S}$ results in seven connected components.



Iteration 3: Minimum cut sets S_3 and S_4 are found in noncomplete components of $G - \mathcal{S}$ with connectivity less than $30kt$. Now, $\mathcal{S} = \cup_{i=0}^{4} S_i$, $s < t + 4(30kt)$ and $G - \mathcal{S}$ results in 9 connected components. The algorithm would terminate with all components either complete or with connectivity at least $30kt$.

Since $\mathcal{S}$ is well-defined, let $s = |\mathcal{S}|$ and note that $s < 30kt \cdot$ (the number of components $- 1) \leq 30k(k+1)t \leq \frac{(k+1)n}{30k^3}$.

Next, we establish the following claims which will allow the application of Lemmas 3.4, 3.5, and 3.6 to each component G_j.

Claim 1: For every $\mathcal{S}$ and resulting collection of G_j's, $\sigma_2(G_j) > \frac{2|V(G_j)|}{k+2}$.

Let $L = \{v \in V(G) \,|\, d(v) < \frac{1}{2}\sigma_2(G)\}$ and $H = V(G) - L$ (where L stands for *low-degree vertices* and H for *high-degree vertices*). Observe that the graph induced by L is complete. Thus, at most one component of $G - \mathcal{S}$ can contain any vertices from L and every non-complete component must contain vertices from H.

If $G - \mathcal{S}$ has at least three components, then for every non-complete component, say G_{j_1}, there exists a different component, say G_{j_2}, containing a vertex, $v \in G_{j_2} \cap H$. Since $N(v) \cap G_{j_1} = \emptyset$, we know $|V(G_{j_1})| \le n - d(v) \le n - \frac{\sigma_2(G)}{2}$, or, equivalently, $\frac{\sigma_2(G)}{2} \le n - |V(G_{j_1})|$.

Now, for n sufficiently large,

$$\begin{aligned}\sigma_2(G_{j_1}) &\geq \frac{2n}{k+2} + f(k) - 2s = \frac{2|V(G_{j_1})| + 2(n - |V(G_{j_1})|)}{k+2} + f(k) - 2s \\ &> \frac{2|V(G_{j_1})|}{k+2} + \frac{2(\sigma_2(G)/2)}{k+2} - 2s \geq \frac{2|V(G_{j_1})|}{k+2} + \frac{2n}{(k+2)^2} - 2s \\ &> \frac{2|V(G_{j_1})|}{k+2}\end{aligned} \tag{4}$$

since $s \le \frac{(k+1)n}{30k^3}$.

If $G - \mathcal{S}$ contains exactly two components, the argument and arithmetic above applies, unless one of the components, say G_1, contains only vertices from L. Thus, $|G_1| < \frac{n}{k+1} + f(k)/2$ and is complete. Moreover, $\mathcal{S} = S_0$ where $|S_0| = s < t$. If G_2 is complete, then the lemma holds.

If G_2 is not complete, then let $u \in V(G_1)$. Now, for n sufficiently large,

$$\begin{aligned}\sigma_2(G_2) &> 2[\sigma_2(G) - d(u) - |S|] > 2\left[\frac{2n}{k+2} - |V(G_1)| - 2|S|\right] \\ &= \frac{2|V(G_2)|}{k+2} + 2\left[\frac{n}{k+2} + \frac{|V(G_1) \cup S|}{k+2} - |V(G_1)| - 2|S|\right] \\ &\geq \frac{2|V(G_2)|}{k+2} + \frac{2n}{k+2}\left[1 - \frac{k+1}{k+2} - \frac{2k+1}{900k^4}\right] \\ &> \frac{2|V(G_2)|}{k+2}\end{aligned} \tag{5}$$

and Claim 1 follows. □

Claim 2: Either a component G_i of $G - \mathcal{S}$ is complete or $n_i = |V(G_i)| \ge \frac{n}{k+2}$.

If G_i is not complete, pick a pair of nonadjacent vertices in G_i. Then

$$|V(G_i)| \geq \sigma_2(G) - 2s > 2n\left(\frac{1}{k+2} - \frac{k+1}{30k^3}\right) > \frac{n}{k+2},$$

for n sufficiently large. □

Before explicitly constructing a path containing T, some additional notation will be introduced. Let $T_{\mathcal{S}} = T \cap \mathcal{S}$. Define T_L, T_H, $T_{L\cap\mathcal{S}}$, and $T_{H\cap\mathcal{S}}$ analogously.

Case 1: Assume $G - \mathcal{S}$ has exactly r connected components where $2 \le r \le k$.

First observe that it is sufficient to construct a short cycle containing all of T_H because the vertices in T_L form a clique and any shortest path from T_L to the cycle could trivially be extended to a path through all of T. Thus, we show how to construct such a cycle. Second, observe that any component of $G - \mathcal{S}$ of order less than k must be contained in L and thus there is at most one of these. Finally, note that since the number of components is at most k, the number of cut-sets producing $\mathcal{S}$ is at most $k-1$. Hence, $s \leq 30k(k-1)t$.

Because the steps used in Case 1 will be essentially the same in Cases 2 and 3, the steps are given names for ease of reference later. To aid the reader, pictures of the various steps are shown in Table 2.

[**Construct new graph G' and cycle C'.**] Construct a new graph G' by adding a new vertex u_i adjacent to all the vertices of G_i, for every component G_i containing vertices from T_H. Since all such G_i have at least k vertices and at most k new vertices were added, the new graph G' is still k-connected and therefore contains a cycle through all the u_i's. Choose a shortest such cycle. By Lemma 3.4, this shortest cycle contains at most $r(3k+5)$ vertices and call this cycle C'. Note that the cycle C' is useful as a way to 'navigate' $\mathcal{S}$, whose structure we have little control over. Dirac's theorem gives a way to find a rough cycle through the components, which we will now modify by routing it through the points of T as desired.

For ease of reference, we will split the cycle, C' into $2r' \leq 2r$ segments: $P'_1, Q'_1, P'_2, Q'_2, \cdots, P'_{r'}, Q'_{r'}$ as follows. For each i, P'_i will be the segment of the cycle between two consecutive vertices of $\mathcal{S}$ that contains u_i. That is, $P'_i = w_i, y_1, \cdots, u_i, \cdots, y_l, z_i$, where $w_i, z_i \in \mathcal{S}$ and $y_1, \cdots, y_l \in V(G_i)$. The segment between P'_i and P'_{i+1} will form Q'_i (where indices are taken modulo r'.) In particular, Q'_i is a path from z_i to w_{i+1} and, if $z_i = w_{i+1}$, then Q'_i is a single vertex. Clearly, $|\cup_i V(Q'_i)| < r(3k+5)$.

[**Identify T_i's.**] For every $x \in T_{\mathcal{S}\cap H}$, there exists some i such that x has at least $\frac{\sigma_2(G)-2s}{2k} \geq \frac{n}{k}\left(\frac{1}{k+2} - \frac{k+1}{30k^3}\right)$ neighbors in G_i. Thus, as $t \leq n/(900k^4)$, for every such x, it is possible to associate a pair of neighbors in some G_i unique to x. If $x \in T_{\mathcal{S}\cap H} - V(\cup_i Q'_i)$, identify such a pair and label them $v_{x,1}$ and $v_{x,2}$. In this case, we say the vertex x *belongs to* component G_i.

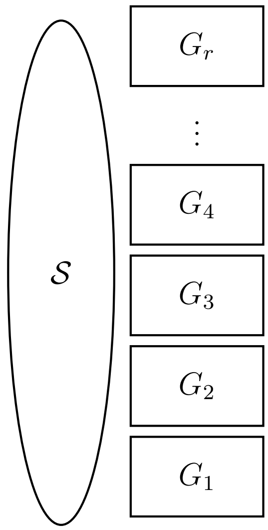
For each component G_i, we define T_i to be the union of all vertices in $(T_H \cap V(G_i)) - (\cup_i Q'_i)$ along with all associated pairs, $v_{x,1}$ and $v_{x,2}$, chosen in G_i. Observe that $|\cup_i T_i| \leq 2t$, and that this upper bound would be achieved only if $T = T_{H\cap\mathcal{S}}$ and $T \cap (\cup_i Q'_i) = \emptyset$. Trivially, $|T_i| \leq 2t$.

[**Replace each P'_i with new paths P_i containing the vertices of T_i in a convenient order and leaving $\cup_i Q'_i$ intact.**] Specifically, we want to construct new w_i, z_i-paths, P_i, with the following three properties: (1) all internal vertices are contained in G_i and vertices of T that belong to G_i, (2) all vertices of T_i lie on the path, and (3) any associated pairs ($v_{x,1}$ and $v_{x,2}$) occur on P_i as a $P_3 = v_{x,1}, x, v_{x,2}$.

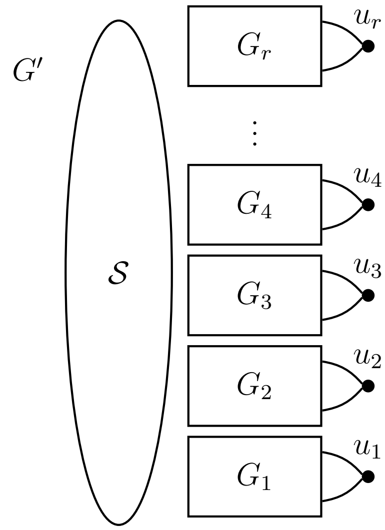
If G_i is complete, then we trivially construct a w_i, z_i-path using y_i, y_l, and vertices of T_i in any convenient order and $|V(P_i)| \leq 4 + 3|T_i|/2$. If G_i is not complete, then we would like to apply Lemma 3.6.

From Claim 1 and Claim 2, we know $\sigma_2(G_i) \geq 2n_i/(k+2)$ and $n_i \geq n/(k+2)$. Thus, $|T_i| + 2 \leq 2t + 2 \leq n_i/(10k^2)$. Finally since $\kappa(G_i) \geq 30kt$, we know $\kappa(G_i - \cup_i V(Q'_i)) > 30kt - r(3k+5) \geq (3k+6)(2t+2)$.

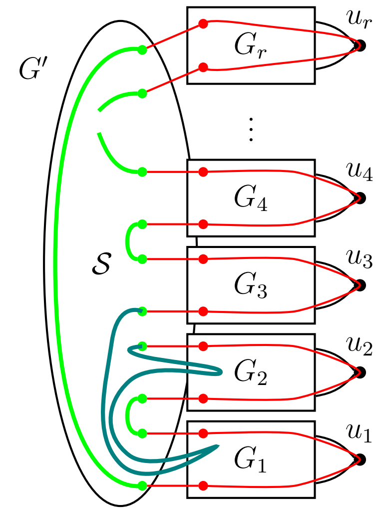
TABLE 2. Illustration of the Proof of Theorem 1.1



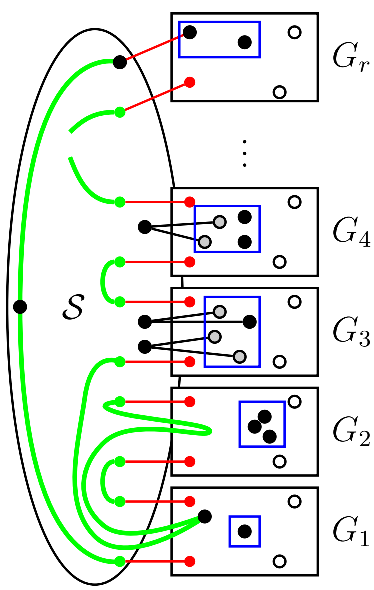
This shows $\mathcal{S}$ and the components of $G - \mathcal{S}$



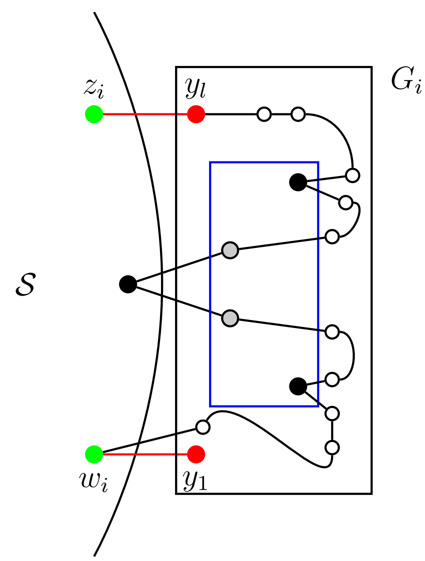
This shows G' with added vertices u_i.



This shows C' (in red, green, and teal) in G'. Note that the the Q_i's are depicted in green (and teal) which is the portion of C' left intact. The teal section (Q_2') illustrates how complicated these connector sections may be. They may include vertices from G outside $\mathcal{S}$ and possibly vertices from T.



In the diagram above, black vertices are in T, gray vertices are designated neighbors of vertices of $T_{\mathcal{S}}$ not on any green path. For each G_i, the set T_i is in a blue box. Observe that vertices of T on any Q_j' (in green) are not included in any T_i.



This diagram illustrates how the red portion of C' is replaced with a path containing the vertices of T_i. As before, vertices of T are black, associated neighbors of vertices that belong to G_i are in gray. Observe that y_1 and y_l may or may not be on this new path. It is enough to know that z_i and w_i have distinct neighbors in G_i.

Thus, Lemma 3.6 applies which means that a $\cup Q_i$-avoiding path P_i can be constructed in G_i through all the vertices of T_i in any order containing at most $(3k+6)(|T_i|+2)$ vertices. Because the order is flexible, we can always include associated pairs consecutively on a 3-path of the form $v_{x,1}, x, v_{x,2}$.

Observe that $|V(P_i)| \leq (3k+6)(|T_i|+2)+2$ and $|\cup_i (V(P_i))| \leq (3k+6)(2t+2k)+2k < 19kt$.

[**Form a new cycle** C **containing** T_H.] Form a new cycle $C = P_1 Q'_1 P_2 Q'_2 \cdots P_r Q'_r$. We know $|V(C)| \leq |\cup_i V(P_i)| + |\cup_i V(Q'_i)| < 19kt + 8k^2 < 20kt$. Observe that $T_H \subseteq V(C)$, and thus, Case 1 is proved.

Case 2: Assume $G - \mathcal{S}$ has $k+1 \geq 3$ connected components. Note that in this case, Dirac's Theorem cannot guarantee a cycle through all of the G_i containing vertices of T_H. Instead we find a cycle through k of the G_i and link the last one at the end creating a dominating path.

We will begin as in Case 1. As before, construct G', which may require adding $k+1$ vertices. Construct C' through u_1, $u_2, \cdots u_k$ and define P'_i and Q'_i as in Case 1. As before, $|V(C')| \leq 8k^2$ and hence $|\cup_i V(Q'_i)| < 8k^2$.

In Case 2, we will show that $\delta(G) \geq \frac{n}{k^2}$, and therefore for every vertex in $T_{\mathcal{S}}$ there exists some i such that the vertex could be assigned a unique pair of neighbors in G_i (called $v_{1,x}$, $v_{2,x}$). Furthermore, our proof of this claim will also hold in Case 3 when there are $k+2$ components.

Proceed by contradiction and assume that $\delta(G) < \frac{n}{k^2}$. Let $D = \{v \in V(G) \mid d(v) = \delta\}$. Clearly $D \subseteq L$ and we know at most one component G_i of $G - \mathcal{S}$ can contain vertices of L. Thus, all but at most one component must contain a vertex of degree at least $\sigma_2/2$ and consequently must contain at least $\sigma_2/2 - s$ vertices.

Observe that at most δ vertices can be adjacent to every vertex in D since otherwise the degree of vertices in D would be too large. But, $2\left(\frac{\sigma_2}{2} - s\right) > \delta$ which implies that it is not possible to have two components of $G - \mathcal{S}$ such that every vertex is adjacent to every vertex of D. Thus, all but at most one of the $k+1$ (or $k+2$) components of $G - \mathcal{S}$ have at least one vertex of degree at least $\sigma_2(G) - \delta(G)$.

Choose vertex x such that $d(x) \geq \sigma_2(G) - \delta(G)$ and among all vertices with this property, choose x to be a member of a component of $G - \mathcal{S}$ of smallest cardinality. Using the bound on the degree of x and its location in G, we know $\sigma_2(G) - \delta(G) \leq d(x) < \frac{n-s}{k} + s$. Hence, $\delta(G) > \frac{n}{k^2}$ and the claim holds.

So, unlike Case 1, for every $x \in T_{\mathcal{S}}$ (not just those in $T_{\mathcal{S} \cap H}$), there exists some G_i such that x can be assigned a unique pair of neighbors in G_i (called $v_{1,x}$, $v_{2,x}$) and we say x belongs to G_i. Thus, in Case 2, if $x \in T_{\mathcal{S}} - \cup_i V(Q'_i)$, we associate such a pair.

Similar to Case 1, for every $i \in \{1, 2, \cdots, k+1\}$, a set T_i will be identified, but in this case it will not be restricted to T_H. Specifically, T_i is the union of the vertices in $(T \cap V(G_i)) - (\cup_i Q'_i)$ along with all associated pairs, $v_{x,1}$ and $v_{x,2}$, chosen in G_i. Now every vertex in T either lies on some Q'_i, is contained in some T_i, or has a pair of neighbors in some T_i. Let $t_{k+1} = |T_{k+1}|$ and let $t_0 = |\cup_{i=1}^{k} T_i|$.

By replacing each P_i' with a P_i as in Case 1, we form a cycle C containing all vertices of T except those in $G_{k+1} - \cup_i V(Q_i')$. As before, $|V(C)| \leq |\cup_i V(P_i)| + |\cup_i V(Q_i')| < 19kt_0 + 8k^2$.

Observe that $t_{k+1} \leq 2t$ and, either G_{k+1} is complete or $\kappa(G_{k+1} - \cup_i Q_i') > 30kt - 8k^2 > 9k(2t)$. Thus, by Lemma 3.6 $G_{k+1} - \cup_i Q_i'$ contains a cycle through all of T_{k+1} in any order we choose. Let C_{k+1} be a smallest such cycle and observe that $|V(C_{k+1})| \leq 9kt_{k+1}$.

Since G is connected, there must exist a path from C to C_{k+1}. Now these two cycles and a shortest path between them contains at most $19kt_0 + 8k^2 + 9kt_{k+1} + 9k \leq 20kt$ vertices and trivially contains a path through all of T. Thus, the theorem holds for Case 2.

Case 3: Assume $G - \mathcal{S}$ has $k+2$ connected components. Again, Dirac's Theorem is not sufficient to guarantee a cycle through all of the G_i's. In this final case, we prove the existence of a sufficiently "fat" cut set allowing a pair of components to be combined via a cycle and added at the end.

As in Case 2, we are guaranteed that every vertex $v \in T_{\mathcal{S}}$ can be assigned a unique pair of neighbors in some G_i. Furthermore, in this case, no component of $G - \mathcal{S}$ can have order less than k since otherwise the σ_2 condition would fail to be satisfied by vertices in the smallest two components.

Claim 3: There exists some pair of the connected components, say G_{k+1} and G_{k+2}, and some cut set, S_i, such that there exist $10k^2$ vertex disjoint P_3's from G_{k+1} to G_{k+2} through S_i.

We will proceed by contradiction and let $S_0, S_1, \cdots, S_r$ be the cut sets in the order in which they were chosen and such that $G - \cup_i S_i$ has connected components $G_1, G_2, \cdots, G_{k+2}$.

Recall, now, the definition of $f(k)$ and the underlying recurrence relation: We let $b_0 = 10k^2$ and $b_i = 10k^2 \left(\frac{k+2}{2}\right)^2 + \sum_{j=0}^{i-1} b_j$, and $f(k) = 2b_{k+1}$.

Given an iteratively selected cut set $S_i \subset \mathcal{S}$, we define the *children of* S_i to be those connected components of $G - \mathcal{S}$ with neighbors in S_i. If $|S_r| \geq b_0 = 10k^2$, any two children of S_r will suffice, and Claim 3 holds.

So, assume $|S_r| < 10k^2$. Furthermore, assume there exists an $i_0 < r$, such that for all $i > i_0$, $|S_i| < b_{r-i}$ but $|S_{i_0}| \geq b_{r-i_0} = 10k^2 \left(\frac{k+2}{2}\right)^2 + \sum_{j=0}^{r-i_0} b_j$.

Observe that, in the matchings between S_{i_0} and its children, we can delete from S_{i_0} all vertices on matching edges to $\{S_{i_0+1}, \cdots, S_r\}$ and still have $10k^2 \left(\frac{k+2}{2}\right)^2$ matching edges left. Now all remaining matching edges must go to vertices in $\cup G_j$. As there are less than $\left(\frac{k+2}{2}\right)^2$ different pairs of G_j's in all, at least one pair of children of S_{i_0} has at least $10k^2$ P_3's through S_{i_0}.

On the other hand, if no such S_{i_0} exists (that is, if $|S_i| < b_{r-i}$ for all i), then $s < f(k)/2$, by definition of $f(k)$. Yet, by considering two vertices v and w in the two smallest components of $G - \{S_1, S_2, \cdots, S_r\}$, we observe that

$$\frac{2n}{k+2} + f(k) \leq d(v) + d(w) \leq 2\left(\frac{n-s}{k+2} + s - 1\right) < \frac{2n}{k+2} + 2s$$

which implies $s > f(k)/2$. So Claim 3 holds.

Label the cut sets S_i and components of $G - \cup S_i$ such that G_{k+2} and G_{k+1} have at least $10k^2$ P_3's through a cut set S_r. Now, we repeat the arguments from Cases 1 and 2. Specifically, construct G' by adding as many as $k+2$ new vertices. Construct cycle C' through $u_1, u_2, \cdots, u_k$ using at most $8k^2$ vertices. Define subpaths P'_i and Q'_i as before.

Observe that since the cycle C uses at most $8k^2$ vertices, there still exist at least $10k^2 - 8k^2 = 2k^2 \geq 2$ paths on three vertices between G_{k+1} and G_{k+2} through S_r. Label these two paths Q_{k+1} and Q_{k+2} and call the middle vertices on these paths m_1 and m_2.

For every $x \in (T_{\mathcal{S}}) - (\cup_i Q'_i) - \{m_1, m_2\}$, associate a unique pair of neighbors in some G_i, called $v_{1,x}$ and $v_{2,x}$. For each $i \in \{1, 2, \cdots, k+2\}$, identify T_i as in Case 2. Let $t_{k+2} = |T_{k+2}|$, $t_{k+1} = |T_{k+1}|$, and $t_0 = | \cup_{i=1}^k T_i|$. Thus, $t_0 + t_{k+1} + t_{k+2} \leq 2t$. Replace each P'_i with a P_i as in previous cases to form the cycle C on at most $19kt_0 + 8k^2$ vertices.

Finally, construct path P_{k+1} from m_1 to m_2 in G_{k+1} such that all internal vertices are contained in $V(G_{k+1})$ and vertices of T that belong to G_i, all vertices of T_i lie on the path, and such that any associated pairs ($v_{x,1}$ and $v_{x,2}$) occur on P_{k+1} as a $P_3 = v_{x,1}, x, v_{x,2}$. Find the analogous path P_{k+2} in G_{k+2} and vertices that belong to G_{k+2}. Together $P_{k+1}Q_{k+1}P_{k+2}Q_{k+2}$ form a cycle, C_1 on at most $9k(t_{k+1} + t_{k+2} + 4) + 2$ vertices.

Any shortest path between C and C_1 contains our desired path and uses at most:

$$|V(C)| + |V(C_1)| + 9k \leq 19kt_0 + 8k^2 + 9k(t_{k+1} + t_{k+2} + 4) + 2 + 9k < 20kt$$

concluding Case 3.

Thus, in all three cases, a path on at most $20kt$ vertices can be found through the set T, and the Theorem holds. ☐

Corollary 4.1. *Let $k \geq 1$ be an integer. Let G be a k-connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ where $f(k)$ is a sufficiently large constant depending only on k. Then there exists a vertex dominating path in G on at most $O(\ln n)$ vertices.*

Proof. Any G satisfying the hypotheses of the Corollary contains a dominating set of vertices of order at most $O(\ln n)$ using Lemma 3.3. ☐

5. Concluding Remarks

Observe that Example 2.1 shows that Corollary 4.1 is close to sharp. That is, the example has no dominating path and $\sigma_2(G) = \frac{2n-4(k+1)}{k+2}$.

We conjecture that this is the extremal example.

Conjecture 5.1. *Let $k \geq 1$ be an integer. Let G be a k-connected graph on n vertices such that $\sigma_2(G) > \frac{2n-4(k+1)}{k+2}$. Then there exists a vertex dominating path in G on at most $O(\ln n)$ vertices.*

In fact, using the same techniques as in Theorem 1.1, it is possible to prove the cycle analogues of Theorem 1.1 and Corollary 4.1, stated below.

Theorem 5.1. *If G is a k-connected graph on n vertices and $\sigma_2(G) \geq \frac{2n}{k+1} + f(k)$ and $T \subseteq V(G)$ such that $|T| = o(n)$, then there exists a cycle through T on at most $O(|T|)$ vertices.*

Corollary 5.1. *Let $k \geq 1$ be an integer. Let G be a k-connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+1} + f(k)$ where $f(k)$ is a recursively defined function of k. Then there exists a vertex dominating cycle in G on at most $O(\ln n)$ vertices.*

6. Acknowledgement

This work is dedicated to RJF and was started before his untimely death. The memory of his tireless energy and generosity is motivation to all who knew him.

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