On Fan Saturated Graphs

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Abstract

Given a graph H, we say that a graph G is *H-saturated* if it does not contain H as a subgraph, but the addition of any edge $e \notin E(G)$ results in at least one copy of H as a subgraph. Let F_t be the graph consisting of t edge-disjoint triangles that intersect at a single vertex v. We investigate the set of all m such that there exists an n vertex, m edge F_t-saturated graph, for $t \geq 2$. This set is called the saturation spectrum of F_t.

1 Introduction

A graph G is *H-saturated* if, given a graph H, G does not contain a copy of H as a subgraph, but the addition of any edge $e \notin E(G)$ creates at least one copy of H within G. The study of saturated graphs has seen a recent surge in popularity. The question of the minimum number of edges in an H-saturated graph on n vertices, known as the *saturation number* and denoted $\text{sat}(n, H)$, has been addressed for many different types of graphs. The saturation number contrasts the popular question of the maximum number of edges possible in a graph G on n vertices that does not contain a copy of H, known as the *extremal number* (or *Turán number*) and denoted $\text{ex}(n, H)$. In one sense, determining the extremal number and determining the saturation number are dual problems. The *saturation spectrum* of the family of H-saturated graphs on n vertices is the set of all possible sizes ($|E(G)|$) of an H-saturated graph G.

We use the common notation $V(G)$ and $E(G)$ for the vertex and edges sets of G, K_n for the complete graph of order n, and $\bar{G}$ for the complement of G. We also use C_n for a cycle on n vertices, $\delta(G)$ for the minimum degree of G, $\text{diam}(G)$ for the diameter of the graph G, and $\text{dist}(u, v)$ for the distance between vertices u and v in G. We use $N(x)$ for the set of neighbors of the vertex x. For a set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ we denote the graph induced by these vertices as either $< S >$ or $< v_1, v_2, \ldots, v_k >$. We use $G + H$ for the join of graphs G and H. Given consecutive integers $x, x+1, \ldots, x+k$, we call this collection of integers an interval, and denote it $[x, x+k]$. For terms not defined here see [5].

The idea of saturation spectrum has been explored for a few graphs. The saturation spectrum for K_3-saturated graphs was found in 1995 by Barefoot, Casey, Fisher, Fraughnaugh, and Harary [4]. In [2], Amin, Faudree, and Gould found the spectrum for K_4-saturated graphs and in [3] Amin, Faudree, Gould and Sidorowicz found the spectrum for $K_t, t \geq 4$. Continuing this idea, Gould, Tang, Wei, and Zhang

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addressed the saturation spectrum of small paths [9], while in [8], the spectrum for complete graphs minus an edge was studied.

The *t-fan* (sometimes called the *friendship graph*), F_t ($t \geq 2$), is the graph consisting of t edge-disjoint triangles that intersect at a single vertex v. P. Erdős (personal communication) suggested the problem of determining the extremal number of F_t (see [6]), while the saturation number was determined in [7]. These results are presented in Section 2, where we also develop several lemmas. In Section 3 we study F_2-saturated graphs. In Sections 4 and 5 we study the saturation spectrums of F_3 and F_4, respectively. In Section 6 we generalize two constructions for F_4-saturated graphs to F_t-saturated graphs ($t \geq 5$).

2 Saturation and extremal numbers for F_t

In this Section we establish boundaries on the saturation spectrum for F_t and several useful lemmas.

Remark 1: Before we begin, note that in the constructions done in the following sections, we rely heavily on a result due to Abbott, Hanson, and Sauer [1]. Let $\beta(G)$ denote the edge independence number of G and $\Delta(G)$ the maximum degree of G. They defined

$$f(\beta, \Delta) = \max \{ \ |E(G)| \ : \ \beta(G) \leq \beta, \Delta(G) \leq \Delta\}.$$

In particular, they showed that

$$f(t-1, t-1) = \begin{cases} t^2 - t & \text{if } t \text{ is odd} \\ t^2 - \frac{3}{2}t & \text{if } t \text{ is even.} \end{cases}$$

In the constructions to come, the special graphs inserted in our constructions usually have $f(t-1,t-1)$ edges and are $(t-1)$-regular or nearly regular depending on the parities of n and t. Further, these graphs have edge independence number $t-1$. This is useful because upon inserting any other edge, either t independent edges are produced, or a vertex of degree t is produced. Either situation allows the construction of the desired F_t, using one or more vertices from a neighboring set or sets.

Note that the extremal number for F_t-saturated graphs is given in the following theorem from [6]. This result also uses the Abbott, Hanson, and Sauer [1] result.

Theorem 1. [6] *For every $t \geq 1$, and for every $n \geq 50t^2$, if a graph G on n vertices has more than*

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} t^2 - t & \text{if } t \text{ is odd} \\ t^2 - \frac{3}{2}t & \text{if } t \text{ is even} \end{cases}$$

edges, then G contains a copy of the t-fan, F_t. Furthermore, the number of edges is best possible.

Now for $p \geq 3$, let $F_{t,p,s}$ denote the graph comprised of t copies of K_p intersecting on a common K_s. Clearly, $F_{t,3,1} = F_t$. The saturation number for $F_{t,p,s}$ was determined in [7].

Theorem 2. [7] *Let $p \geq 3$ and $t \geq 2$ and $p-2 \geq s \geq 1$. Then for n sufficiently large,*

$$\text{sat}(n, F_{t,p,s}) = (p-2)(n-p+2) + \binom{p-2}{2} + (t-1)\binom{p-s+1}{2}.$$

In particular, the graph $K_{p-2}+((t-1)K_{p-s+1}\cup\overline{K}_{n-(p-2)-(t-1)(p-s+1)})$ *is* $F_{t,p,s}$*-saturated with the minimum number of edges.*

Clearly, for $t \geq 2$ and $n \geq 3t-2$, $\text{sat}(n, F_t) = n + 3t - 4$.

Having established the boundaries for the saturation spectrum of F_t, we begin our exploration of the saturation spectrum with a few useful lemmas. We can see that an F_t-saturated graph with a cut vertex achieves the minimum number of edges (called a *saturation graph*). A graph achieving the maximum number of edges is called an *extremal graph.*

Lemma 1. *If G is an F_t-saturated ($t \geq 2$) graph with $n \geq 5$ vertices, then* $\text{diam}(G) = 2$*.*

Proof. First suppose that an F_t-saturated graph G is not connected. Then inserting an edge between two components of G cannot create a copy of F_t, a contradiction. Hence, G cannot be F_t-saturated. Thus, we may suppose that G is a connected F_t-saturated graph.

Suppose that $\text{diam}(G) \geq 3$. Then for some $u, v \in V(G)$, there is no path from u to v of length at most two. Since G is F_t-saturated, adding the edge uv must create a copy of F_t, so it creates the triangle $\{u, v, w\}$ for some $w \in V(G)$. Then $uw \in E(G)$ and $vw \in E(G)$ and there is a path of length two from u to v through w, which is a contradiction. Thus, $\text{diam}(G) = 2$ if G is an F_t-saturated graph. □

The following lemma is from [4].

Lemma 2. *If G is a 2-connected graph of order n with* $\text{diam}(G) = 2$*, then* $|E(G)| \geq 2n-5$.

The next Lemma is easily seen from Theorem 2.

Lemma 3. *For $t \geq 2$ and $n \geq 3t-2$, the graph*

$$G_t{}^* = K_1 + ((t-1)K_3 \cup \overline{K}_{n-1-3(t-1)})$$

is a 1-connected saturation graph for F_t.

See Figure 1 for a saturation graph for F_4.

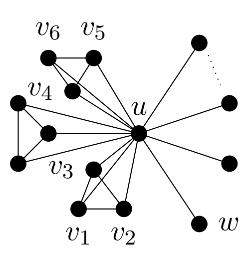


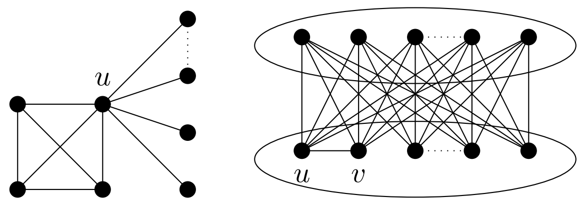
Figure 1: A saturation graph for F_4.

3 Saturation graphs for F_2

From Theorem 2, $\text{sat}(n, F_2) = n + 2$ and is realized as the size ($|E(G^*{}_2)|$) of the graph $G^*{}_2$ consisting of a K_4 with $n-4$ pendant edges on one vertex u of the K_4 (see Figure 2(a)). Now consider G, a 1-connected

F_2 saturated graph with cut vertex x. As $\operatorname{diam}(G) = 2$, every other vertex of G is adjacent to x. The only way to insert four or more edges into $N(x)$ without creating two or more independent edges is as a star. But if this star does not span $N(x)$, yet another edge could be inserted without creating a copy of F_2. Thus, the star must span $N(x)$. In this case, G is 2-connected, a contradiction. Hence, there are no 1-connected F_2-saturated graphs with more than $\operatorname{sat}(n, F_2)$ edges.

At the high end of the spectrum, from Theorem 1, the extremal number for F_2 is given by $\operatorname{ex}(n, F_2) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$ for $n \geq 5$ and is realized as the size of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ with any additional edge $e = uv$ for $u, v \in V(G)$ (see Figure 2(b)). The graph $B_p^+ = K_{p,n-p} + e$ $(2 \leq p \leq n/2)$ is F_2-saturated as vertices u and v are contained in at least two triangles that intersect only at e. Hence, adding any other edge creates an additional triangle intersecting with one of the triangles containing e at exactly one vertex, thus forming a copy of F_2.



(a) (b)

Figure 2: (a) ${G^*}_2$ with size $n - 1 + 3 = n + 2$; (b) ${B^+}_p = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} + uv$.

The following lemmas establish the lower bound on the saturation spectrum for 2-connected F_2-saturated graphs.

Lemma 4. *Let G be an F_2-saturated graph with $\delta(G) \geq 3$ on $n \geq 10$ vertices. Then $|E(G)| \geq 2n - 4$.*

Proof: Let G be an F_2-saturated graph with $\delta(G) \geq 3$. Then, by Lemma 1, $\operatorname{diam}(G) = 2$. Note that if $\delta(G) \geq 4$, then $|E(G)| \geq 2n > 2n - 4$ and we are done. Hence, assume there is a vertex u in G adjacent to exactly three other vertices of G, say x, y and z. Let $X = \{x, y, z\}$ and let $A = V(G) - \{u, x, y, z\}$. Since $\operatorname{diam}(G) = 2$, every vertex in A is adjacent to at least one of the vertices in X. Let A_1 be the set of vertices in A that are adjacent to exactly one vertex of X, let A_2 be the vertices in A adjacent to exactly two vertices of X and let A_3 be the vertices in A adjacent to all vertices of X. The minimum degree condition implies that each $v \in A_1$ must be adjacent to at least two other vertices in A and each $w \in A_2$ must be adjacent to at least one other vertex in A. Hence, we have a minimum size as follows:

$$\begin{aligned} |E(G)| &\geq 3 + |A_1| + 2|A_2| + 3|A_3| + \left\lceil \tfrac{2|A_1| + |A_2|}{2} \right\rceil \\ &= 3 + 2|A_1| + 2|A_2| + \left\lceil \tfrac{|A_2|}{2} \right\rceil + 3|A_3| \\ &= 3 + 2(n - |A_3| - 4) + \left\lceil \tfrac{|A_2|}{2} \right\rceil + 3|A_3| \\ &= 2n - 5 + \left\lceil \tfrac{|A_2|}{2} \right\rceil + |A_3|. \end{aligned}$$

If either A_2 or A_3 is non-empty, we are done. Thus, assume that $|A_2| = |A_3| = 0$. Then $|E(G)| \geq 2n - 5$ and it remains to show that there is at least one additional edge in G.

If at least one of the edges xy, yz, xz is in $E(G)$, we are done. Assume that xy, yz, and xz are not edges of G. Since $\delta(G) = 3$, there must be at least two vertices of A_1 adjacent to x, two vertices of A_1 adjacent to y and two vertices of A_1 adjacent to z. Also, to maintain diam$(G) = 2$, each vertex of A_1 adjacent to x must be adjacent to at least one vertex adjacent to y and at least one vertex adjacent to z. Similarly, each vertex adjacent to y must be adjacent to at least one vertex adjacent to x and one vertex adjacent to z and each vertex adjacent to z must be adjacent to at least one vertex adjacent to x and one vertex adjacent to y. This requirement allows the minimum possible size to remain at $|E(G)| \geq 2n - 5$ as it requires at least $|A_1|$ edges amongst the vertices of A_1. However, this graph is not F_2-saturated, as it is possible to add xy without creating a copy of F_2, so there must be at least one additional edge. This completes the proof of the lemma. □

Lemma 5. *Let G be a 2-connected F_2-saturated graph on $n \geq 10$ vertices. Then $|E(G)| \geq 2n - 4$.*

Proof. Let G be a 2-connected F_2-saturated graph with m edges and $n \geq 10$ vertices. Since G is F_2-saturated, diam$(G) = 2$ by Lemma 1. It follows from Lemma 4, that if $\delta(G) \geq 3$, then $m \geq 2n-4$. Clearly, $\delta(G) \geq 2$, so suppose $\delta(G) = 2$ and let $\deg(z) = 2$ for some $z \in V(G)$. Let z be adjacent to $x, y \in V(G)$ and partition the remaining vertices of G into three sets A, B, C with every $u \in A$ adjacent only to x, every $v \in B$ adjacent to x and y, and every $w \in C$ adjacent only to y as in Figure 3. For convenience suppose that $V(A) = \{a_1, a_2, \ldots, a_{|A|}\}$, $V(B) = \{b_1, b_2, \ldots, b_{|B|}\}$, and $V(C) = \{c_1, c_2, \ldots, c_{|C|}\}$. Since G is 2-connected, A and B cannot both be empty, as this would make y a cut vertex. Similarly, both C and B cannot be empty.

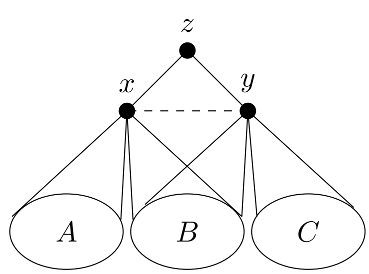


Figure 3: Basic structure of 2-connected F_2-saturated graphs with $\delta(G) = 2$.

Case 1: Suppose that $xy \in E(G)$.

First supppose that $B = \emptyset$. Then both A and C must be nonempty, or a cut vertex would result, a contradiction to 2-connectivity. Also note that $E(A) = E(C) = \emptyset$ or an F_2 would exist in G, a contradiction. Note that for two positive integers r and s with $r + s = w$, $rs \geq w - 1$.

Now, $n = 3 + |A| + |C|$. Since diam$(G) = 2$, each vertex of A must be adjacent to each vertex of C. Thus, since $|A||C| \geq n - 4$ we see that

$$m \geq 3 + |A| + |C| + |A||C| \geq n + n - 4 = 2n - 4.$$

Next suppose that $B \neq \emptyset$. Note that there can be no edges from A to B or from C to B or a copy of F_2 would exist in G. Now A and C must also be nonempty for otherwise G would be 1-connected, a contradiction. Again, $E(G) = E(C) = \emptyset$ or an F_2 exists in G. Thus, again, all edges from A to C must exist or G would not have diameter two. Note that $n = 3 + |A| + |B| + |C|$, and $|A|$ and $|C|$ are both positive integers. Thus,

$$\begin{aligned} m &= 3 + |A| + 2|B| + |C| + |A||C| \\ &\geq n + |B| + (n - |B| - 4) = 2n - 4. \end{aligned}$$

Case 2: Suppose $xy \notin E(G)$

In this case the sets A and C may not contain two or more independent edges. Thus, there are only three possibilities for edges in A or C: no edges, edges that form a single triangle, or edges that form a single star. We now consider subcases based on these possibilities.

Subcase 2.1: Suppose that $B = \emptyset, A \neq \emptyset, C \neq \emptyset$.

First suppose that A and C contain no edges. If $a_1 \in A$, then $< z, x, a_1 >= P_3$, but none of the these three vertices lie on a triangle. Thus, no matter what edges lie between the sets A and C, inserting the edge za_1 into G cannot form a copy of F_2. Hence, G is not F_2-saturated. We conclude that at least one of A and C must contain edges.

Note that a similar argument applies if A (or C) contain vertices not in a triangle or star, say $a_1 \in A$ and suppose C (A) contains no edges. Then a_1, x and z lie on no triangles, hence inserting the edge za_1 would not produce a copy of F_2, a contradiction. Thus, we may assume that if one of A or C contains no edges, then the other set is spanned by either a triangle or star.

Suppose, without loss of generality, that A is spanned by a triangle with vertices a_1, a_2, a_3, and C contains no edges. If $c \in C$, for the edge zc to produce a copy of F_2 when inserted, c must be adjacent to two adjacent vertices of A, say a_1, a_2. But then $< x, a_1, a_2, a_3, c >= F_2$, a contradiction. A similar argument holds if A is spanned by a star. Thus, we conclude that A and C must both contain either a triangle or star.

Suppose A contains either a triangle or a star and also contains vertices not in the triangle or star. Say $a \in A$ is such a vertex. Then for the addition of the edge za to produce a copy of F_2, a must be adjacent to both end vertices of an edge in C. Say a is adjacent to both c_1 and c_2. If c_1 and c_2 are in a triangle, then F_2 exists in G using y, c and the triangle, a contradiction. If c_1 and c_2 are in a star of order at least three, a copy of F_2 also exists. If the star in C has order two, then the only way inserting the edge za produces a copy of F_2 is if a is adjacent to both end vertices of the one edge in C. A similar argument applies to any vertex of C not incident to the edge. But now, as $n = 3 + |A| + |C|$ we have

$$\begin{aligned} m &\geq 2 + |A| + |C| + 2 + 2(|C| - 2) + 2(|A| - 2) \\ &= (2 + |A| + |C|) + 2(|A| + |C|) - 6 \\ &= (n - 1) + 2(n - 3) - 6 = 3n - 13. \end{aligned}$$

But, $3n - 13 \geq 2n - 4$ when $n \geq 9$. Hence, we conclude that the triangle or star must span the set they are in.

Subcase 2.1.1: Suppose $B = \emptyset$ and both A and C are spanned by a triangle.

Then G woud only contain nine vertices and $n \geq 10$. Hence, this Subcase cannot happen.

Subcase 2.1.2: Suppose $B = \emptyset$ and both A and C are spanned by a star.

Let the star in A be centered at a_1 with edges to $a_2, a_3, \ldots, a_{|A|}$ and the star in C be centered at c_1 with edges to $c_2, c_3, \ldots, c_{|C|}$. Since $\delta(G) \geq 2$, and diam$(G) = 2$, there must be edges between A and C.

First suppose that $a_1c_1 \in E(G)$. Then each of $a_2, a_3, \ldots, a_{|A|}$ must be nonadjacent to c_1 or a copy of F_2 would exist in G. By a similar argument, $c_2, c_3, \ldots, c_{|C|}$ are nonadjacent to a_1. Now each a_i, $i \geq 2$, must be adjacent to each c_j, $j \geq 2$. Now suppose that $a_1c_1 \notin E(G)$. Then a_1 must be adjacent to $c_2, c_3, \ldots, c_{|C|}$ and c_1 must be adjacent to $a_2, a_3, \ldots, a_{|A|}$ or the diameter of G would exceed two. This is clearly the minimum number of edges that achieves both the minimum degree and diameter conditions. Now as $n = 3+|A|+|C|$, we see that

$$\begin{aligned} m &= 2+|A|+|C|+2(|A|-1)+2(|C|-1) \\ &= (n-1)+2(|A|+|C|)-4 \\ &= (n-1)+2(n-3)-4 = 3n-11. \end{aligned}$$

But $3n-11 \geq 2n-4$ when $n \geq 7$.

Subcase 2.1.3: Suppose $B = \emptyset$ and A is spanned by a triangle and C is spanned by a star

As $n \geq 10$, we see that $|C| \geq 4$. Since diam$(G) = 2$, there must be edges from A to C. In fact, each vertex of the triangle in A must have at least one edge to C, or the distance to y would exceed two. Note that the center of the star in C, say c_1, cannot be adjacent to two of the vertices of A or an F_2 would exist in G. Also, no vertex of A is adjacent to both c_1 and another vertex of C, say c_2, or a copy of F_2 would exist in G. If say $a_1c_1 \in E(G)$, then both a_2 and a_3 must be adjacent to each of $c_2, c_3, \ldots, c_{|C|}$ in order to have the diam$(G) = 2$. If none of the vertices of A are adjacent to c_1, then each must be adjacent to all the other vertices of C. Thus, the edge count is minimized when c_1 has a single adjacency to A. Now we see that $n = 3+3+|C|$ hence,

$$\begin{aligned} m &\geq 2+3+|C|+3+(|C|-1)+1+2(|C|-1) \\ &= 2+n+3|C|-2 = n+3(n-6) = 4n-18. \end{aligned}$$

Further, $4n-18 \geq 2n-4$ when $n \geq 7$. Clearly, a similar argument holds if A is spanned by a star and C is spanned by a triangle.

Subcase 2.2: Suppose $B \neq \emptyset$ and A and C are spanned by stars.

Now $E(B) = \emptyset$ or an F_2 would exist in G. Suppose the star in A is centered at a_1 with edges to $a_2, a_3, \ldots, a_{|A|}$ and the star in C is centered at c_1 with edges to $c_2, \ldots, c_{|C|}$. As in Subcase 2.1.2, the minimum edge count is realized when a_1 is adjacent to $c_2, c_3, \ldots, c_{|C|}$ and c_1 is adjacent to $a_2, a_3, \ldots, a_{|A|}$. As $n = 3+|A|+|B|+|C|$ we have

$$\begin{aligned} m &\geq 2+|A|+2|B|+|C|+2(|A|-1)+2(|C|-1) \\ &= (2+|A|+|B|+|C|)+(|B|+|A|+|C|)+|A|+|C|-4. \\ &= (n-1)+(n-3)+|A|+|C|-4 = 2n-4+|A|+|C|-4. \end{aligned}$$

If $|A|+|C| \geq 4$ we are done. So assume, without loss of generality, that $|A| = 1$ and $|C| = 2$. In this case, a_1 can send an edge to say c_1 and the diameter and minimum degree conditons are satisfied. But a_1 can also send edges to every vertex of B and no copy of F_2 is formed. In fact, a vertex of C could also send edges to B as long as no vertex of B has an adjacent in both A and C. Now $n = 6+|B|$. Then

$$\begin{aligned} m &\geq 2+1+2|B|+2+|B|+1+1 \\ &= (5+|B|)+2|B|+2 \\ &= (n-1)+2|B|+2 = (n-1)+2(n-6)+2 = 3n-11. \end{aligned}$$

But $3n-11 \geq 2n-4$ when $n \geq 7$. A similar argument holds if $|A| = 2$ and $|C| = 1$ or if $|A| = |C| = 1$.

Subcase 2.2.1: Suppose $B \neq \emptyset$ and A and C are each spanned by a triangle.

Now $n = 9+|B|$. There can be no edges from A or C to B or a copy of F_2 would exist. By 2-connectivity there are edges from A to C. But no vertex of A (or C) can have two or more edges to C (A) or again a copy of F_2 would exist. But diam$(G) = 2$ implies each of a_1, a_2, a_3 has an edge to C. Hence, there is a matching between A and C. Thus,

$$m \geq 2+3++2|B|+3+3+3+3 = 2|B|+17 = 2n-1.$$

Subcase 2.2.2: Suppose $B \neq \emptyset$, A is spanned by a triangle, C is spanned by a star.

There are no edges from A to B or a copy of F_2 would exist. Hence, the fact that $\mathrm{diam}(G) = 2$ implies there are edges between A and C. Each of a_1, a_2, a_3 must have edges to C or the distance to y would be greater than two. Note that c_1 cannot be be adjacent to two or more vertices of A or an F_2 would exist in G. Also, no a_i is adjacent to both c_1 and some other c_i, $i \geq 2$ or again, a copy of F_2 would exist in G. If c_1 is adjacent to no vertices of A, then each of a_1, a_2, a_3 is adjacent to each of $c_2, c_3, \ldots, c_{|C|}$. If say $a_1c_1 \in E(G)$, then a_2 and a_3 must each be adjacent to each of $c_2, c_3, \ldots, c_{|C|}$. This minimizes the edge count. Here $n = 6 + |B| + |C|$. Thus,

$$\begin{aligned} m &\geq 2 + 3 + 2|B| + |C| + 3 + (|C| - 1) + 1 + 2(|C| - 1) \\ &= (5 + |B| + |C|) + +|B| + 4 + 3|C| - 3 \\ &= (n-1) + (|B| + |C| + 4) + 2|C| - 3 \\ &= (n-1) + (n-2) + 2|C| - 3 = 2n - 3 + 2|C| - 3. \end{aligned}$$

But as $|C| \geq 1$ we see that $2n - 3 + 2|C| - 3 \geq 2n - 4$. Clearly a similar argument holds if the roles of A and C are reversed.

Subcase 2.2.3: Suppose $B \neq \emptyset$, $E(A) = E(C) = \emptyset$.

There are no paths of the form a, b, c with $a \in A$, $b \in B$, and $C \in C$ or a copy of F_2 would exist in G. Thus, all edges must be present between A and C. In addition, there can be an edge from A (or C) to B. As $n = 3 + |A| + |B| + |C|$, we have

$$\begin{aligned} m &\geq 2 + |A| + 2|B| + |C| + |A||C| + 1 \\ &= (2 + |A| + |B| + |C|) + |B| + (n - |B| - 4) + 1 \\ &= (n-1) + (n-3) = 2n - 4. \end{aligned}$$

Subcase 2.2.4: Suppose $B \neq \emptyset$, $E(A) = \emptyset$, and C is spanned by a triangle.

Again there are no edges from C to B. As before, no vertex of A has two edges to C or a copy of F_2 would exist. But, since $\mathrm{diam}(G) = 2$, each vertex of A must have an edge to C or the distance to y would exceed two. As $n = 6 + |A| + |B|$ we have that

$$\begin{aligned} m &\geq 2 + |A| + |2|B| + 3 + 3 + |A| \\ &= (|A| + |B| + 5) + (|B| + |A| + 3) \\ &= (n-1) + (n-3) = 2n - 4. \end{aligned}$$

By symmetry, the result also holds if A is spanned by a triangle and $E(C) = \emptyset$.

Subcase 2.2.5: Suppose $B \neq \emptyset$, $E(A) = \emptyset$, and C is spanned by a star.

Now there are no paths from A to C through B or an F_2 would exist in G. By the diameter and degree conditions, each vertex of A has at least one edge to C. If each vertex of A is adjacent to c_1, the center of the star, that conditon is satisfied with a minimum number of edges. Further, each vertex of A can be adjacent to the same vertex of B without creating a copy of F_2. Now $n = 3 + |A| + |B| + |C|$, so that

$$\begin{aligned} m &\geq 2 + |A| + 2|B| + |C| + (|C| - 1) + |A| + |A| \\ &= (2 + |A| + |B| + |C|) + (|B| + |C| + |A|) + |A| - 1 \\ &\geq (n-1) + (n-3) = 2n - 4. \end{aligned}$$

Subcase 2.3: Suppose $B \neq \emptyset$, $C \neq \emptyset$ and $A = \emptyset$.

As G is 2-connected, there must be edges from C to B. If the edge $c_1b_1 \in E(G)$, then there can be no edges in C that are not incident with c_1 or a copy of F_2 would exist in G. Thus, C must contain a triangle or a star. But if say c_1, c_2, c_3 induce a traingle in C, then a copy of F_2 exists in G using b_1, y, c_1, c_2, and c_3. Thus we may assume C contains a star and as before, this star spans C.

But $\text{dist}(x, c_i) > 2$ for all $i \geq 2$. Thus, each of $c_2, c_3, \ldots, c_{|C|}$ has an edge to B. Each such edge must also be to b_1. Now $n = 3 + |B| + |C|$ and so

$$\begin{aligned} m &\geq 2 + 2|B| + |C| + (|C| - 1) + |C| \\ &= (2 + |B| + |C|) + (|B| + |C|) + |C| - 1 \\ &= (n-1) + (n-3) + |C| - 1 \geq 2n - 4. \end{aligned}$$

Clearly, a similar arguement holds if $C = \emptyset$ and $A \neq \emptyset$.. This completes the proof of the Lemma. □

We are now ready to consider the spectrum of F_2. We have already established the saturation number and Turán number for F_2 and the fact $K_{p,n-p}$ with one extra edge is also F_2-saturated and has size $p(n-p)+1$. Lemma 3 establishes $\text{sat}(n, F_2) = n + 2$. Lemma 5 and our observation on 1-connected F_2-saturated graphs establishes the fact there are no F_2-saturated graphs with sizes in the interval $[n+3, 2n-5]$.

Next, expand the graph C_5 such that each vertex of C_5 becomes a set of independent vertices with adjacencies according to the original C_5, that is, where an edge xy becomes a $K_{s,t}$, when $x \in V(C_5)$ expands to a set of s vertices and $y \in V(C_5)$ expands to a set of t vertices. We say that the graph $C_5[A, B, C, D, E]$ is an expanded C_5 with each vertex set A, B, C, D, E an independent set. Let $|A| = a$, $|B| = b$, $|C| = c$, $|D| = d$, and fix $|E| = 1$.

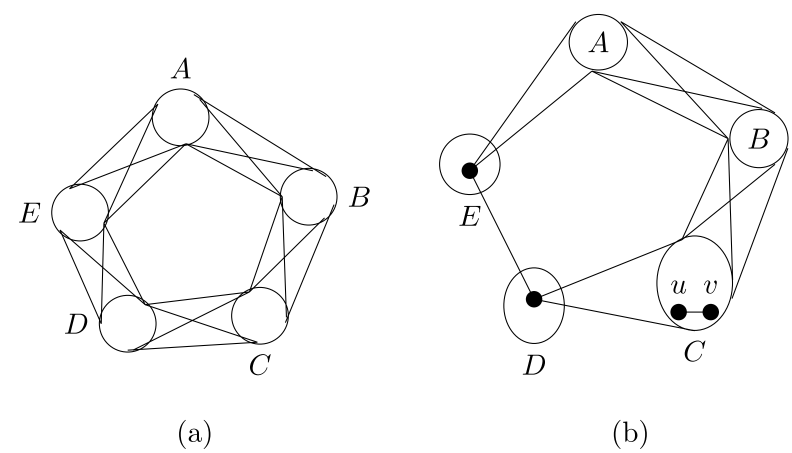


Figure 4: (a) The expanded C_5.; (b) F_2-saturated graph G_2.

The graph in Figure 4(b), which we denote as G_2, is a copy of $C_5[A, B, C, D, E]$, with $|E| = 1$ and exactly one additional edge $e = uv$ for some $u, v \in V(C)$. The graph G_2 has order n and is F_2-saturated with $a = n - b - c - d - 1 \geq 1$ provided $b \geq 1$, $c \geq 2$, $d \geq 2$, and $|E| = 1$. To see that this graph is saturated, we note that since one edge $e = uv$ is added in C, each vertex $b_i \in B$ is in a triangle $< u, v, b_i >$, each triangle sharing the edge e. Then an additional edge a_1a_2 within A would create a copy of F_2 with the triangle $< a_1, a_2, b_i >$ and $< u, v, b_i >$ for some $b_i \in B$. An additional edge b_1b_2 in B would create a copy of F_2 with the triangle $< b_1, b_2, u >$ and $< u, v, d_1 >$ for $d_1 \in D$. Also, adding an edge d_1d_2 in D would create a copy of F_2 with the triangle $< d_1, d_2, u >$ and $< u, v, b_i >$ for $b_i \in B$. Adding an independent edge in C clearly creates an F_2, while adding an edge incident to uv also creates an F_2 using a vertex from B and a vertex from D. Adding an edge from B to D, say b_1d_1, creates an F_2 with triangles $< b_1, d_1, u >$ and

$< d_2, u, v >$. Finally, adding an edge between sets A and D or B and E or A and C is easily seen to create an F_2. Thus, G is F_2-saturated with size $|E(G)| = m$ given by the products of the orders of consecutive vertex sets such that:

$$\begin{aligned} m &= (n-b-c-d-1)b + bc + cd + d + (n-b-c-d-1) + 1 \\ &= bn - b^2 - bd - 2b + cd + n - c \\ &= (n-b)(b+1) - b(d+1) + c(d-1). \end{aligned}$$

Then for $d = 2$, $m = (n-b)(b+1) - 3b + c$. Hence, for fixed values of b, when c increases by 1, as vertices are moved from A to C, m increases by 1. Initially, since $a \geq 1$, $|E| = 1$, and $d = 2$, to maintain the required number of vertices in each set of G, we must have $c \in [2, n-b-4]$. Thus, for a fixed value of b, and letting c take on each value in $[2, n-b-4]$, we can create an F_2-saturated graph having size m for each m in the interval

$$[(n-b)(b+1) - 3b + 2, (n-b)(b+2) - 3b - 4].$$

If we let $c = n-b-4$, and fix n, we have $m = bn + 2n - b^2 - 5b - 4$, which, as a function of b, is maximized when $b = \lfloor \frac{n-5}{2} \rfloor$. The function calculating the size increases until $|B|$ and $|C|$ are approximately the same before decreasing, hence the construction only produces unique sizes for $b \in [1, \lfloor \frac{n-5}{2} \rfloor]$. Now, for $b = 1$, we obtain the interval of m values $[2n-3, 3n-10]$. For $b = 2$ we obtain the interval $[3n-10, 4n-18]$ and continuing to increase b in this manner to its maximum value, we obtain the set of intervals

$$[2n-3, 3n-10], [3n-10, 4n-18], [4n-19, 5n-28], [5n-30, 6n-40], \cdots,$$

$$\left[\left\lceil \frac{n+5}{2} \right\rceil (\left\lfloor \frac{n-5}{2} \right\rfloor + 1) - 3\left\lfloor \frac{n-5}{2} \right\rfloor + 2,\ \left\lceil \frac{n+5}{2} \right\rceil \left\lfloor \frac{n-5}{2} \right\rfloor + 3\left\lfloor \frac{n-5}{2} \right\rfloor + 4\right].$$

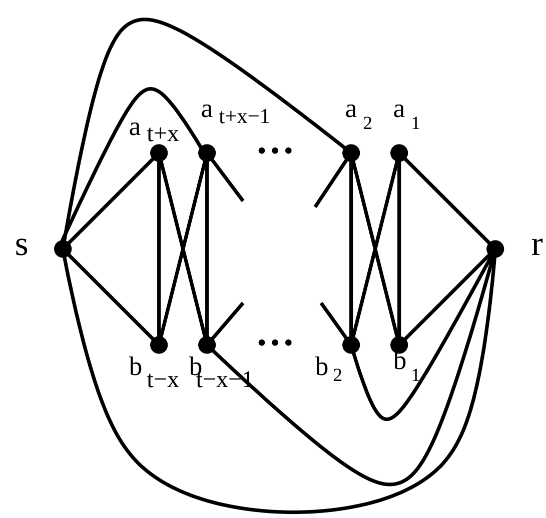


Figure 6. E_2, a 4-partite F_2-saturated family of graphs.

The upper endpoint of the interval evaluated at b minus the lower endpoint at $b+1$ plus one equals the number of values common to the consecutive intervals at b and at $(b+1)$. Here we have

$$[(n-b)(b+2) - 3b - 4] - [(n-b-1)(b+2) - 3(b+1) + 2] + 1 = b.$$

As $b \geq 1$, the intervals overlap, so their union produces one interval of sizes for F_2-saturated graphs.

We now provide another class of graphs that provide some additional values of the spectrum. Consider the graph obtained by taking a copy of $K_{t+x,t-x}$ $(x \geq 1)$ with partite sets consisting of $a_1, a_2, \ldots, a_{t+x}$ and $b_1, b_2, \ldots, b_{t-x}$ along with two additional vertices r and s. Let vertex r be adjacent to $b_1, b_2, \ldots, b_{t-x-1}$ and a_1. Let vertex s be adjacent to $a_{t+x}, a_{t+x-1}, \ldots, a_2$ and b_{t-x}. Further, add the edge rs (See Figure 5). Then this graph E_2 is F_2-saturated and has order $2(t+1)$ and size $t^2 - x^2 + 2t - 1$. Thus, $t = (n-2)/2$ and so E_2 has size $\frac{n^2}{4} - n - x^2 + 1$.

We summarize the results of this section in the following Theorem.

Theorem 3. *There exists an F_2-saturated graph G on $n \geq 10$ vertices and m edges for $m = n+2$, or $2n-3 \leq m \leq \lceil \frac{n+5}{2} \rceil \lfloor \frac{n-5}{2} \rfloor + 3\lfloor \frac{n-5}{2} \rfloor + 4$, or $m = p(n-p)+1$, the size of the complete bipartite graph ${B_p}^+$, or $m = \frac{n^2}{4} - n - x^2 + 1$, $(x \geq 1)$ the size of the graph E_2. Further, there are no F_2-saturated graphs with size in $[n+3, 2n-5]$.*

Question 1. *Does Theorem 3 include all the values of the saturation spectrum for F_2?*

4 Constructing F_3-saturated Graphs

We know that $\text{sat}(n, F_3) = n + 5$ and in [6] it was shown that $\text{ex}(n, F_3) = \lfloor \frac{n^2}{4} \rfloor + 6$. Complete bipartie graphs $K_{p,n-p}$ $(1 \leq p \leq n-1)$ with two edge disjoint triangles added (either to one partite set or one triangle in each set) will also be F_3-saturated. The extremal graph occurs when this graph is a balanced complete bipartite graph. Lemmas 4, 5 show that there are no 2-connected F_3-saturated graphs with size m for $n+6 \leq m \leq 2n-5$. However, if we insert the edges of a K_5 in the neighborhood of a star $K_{1,n-1}$ we obtain a new 1-connected F_3-saturated graph with size $n - 1 + 10 = n + 9$.

We can construct F_3-saturated graphs in a manner similar to our construction of F_2-saturated graphs, with a modified $C_5[A, B, C, D, E]$ denoted G_3. However, in place of the edge $uv \in E(C)$ from the G_2-construction, we need a C_4, as it is 2-regular and has two vertex disjoint edges inducing a copy of F_2 (see Remark 1). The graph G_3 is F_3-saturated when $a \geq 1, b \geq 2$, $d \geq 2$, (so that $b + d \geq 4 = t + 1$ when $t = 3$, needed when looking for F_t) and $c \geq 4$. We again fix $|E| = 1$ and $d = 2$ in G_3.

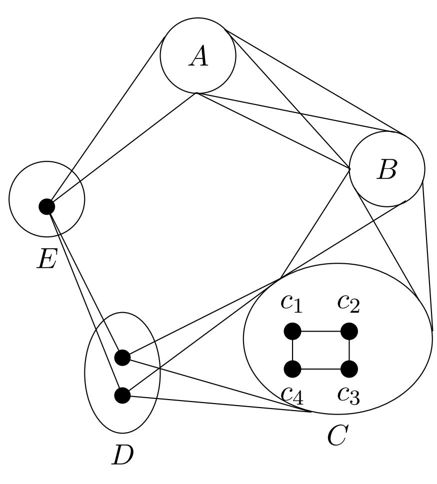


Figure 6: Construction of G_3 for F_3-saturated graphs.

Note that each vertex $b_i \in B$ is in two edge disjoint triangles for example, triangles $< c_1, c_2, b_i >$ and $< c_3, c_4, b_i >$. Then an additional edge $a_1 a_2$ within A would create a copy of F_3 with the third triangle

$< a_1, a_2, b >$. Since each vertex in the C_4 is the shared vertex of an induced copy of F_2, an additional edge in B, say b_1b_2, would create a copy of F_3 with triangles $< b_1, b_2, c_1 >$, $< c_1, c_2, d_1 >$ and $< c_1, c_4, d_2 >$ for $d_1, d_2 \in D$. As $b \geq 2$, adding an edge in D similarly creates a copy of F_3. Adding an independent edge in C clearly creates an F_3 with center $b_i \in B$, while adding an edge incident to the C_4, say c_1c_5, also creates a copy of F_3 using vertices from both B and D. Adding a chord to the cycle in C, say c_1c_3, creates an F_3 with triangles $< d_1, c_1, c_2 >$, $< d_2, c_1, c_4 >$ and $< b_i, c_1, c_3 >$ for any $b_i \in B$. Adding an edge from C to A, say c_ka_i, creates a triangle $< a_i, b_j, c_k >$ for $a_i \in A, b_j \in B$ and $c_k \in C$, and if $c_k \notin \{c_1, c_2, c_3, c_4\}$, we have a copy of F_3 with three edge disjoint triangles sharing b_j while $c_k \in \{c_1, c_2, c_3, c_4\}$ creates a copy of F_3 with triangles sharing c_k. Similarly, adding an edge from C to E produces a copy of F_3. Adding an edge from B to D, say b_1d_1 produces an F_3 with triangles $< b_1, c_1, d_1 >$, $< d_2, c_1, c_4 >$, and $< b_2, c_1, c_2 >$. Adding an edge between B and E or A and D is easily seen to create a copy of F_3. Thus, the graph G_3 is F_3-saturated with size m given by the products of the orders of consecutive vertex sets as follows:

$$\begin{aligned} m &= (n-b-c-3)b + bc + 2c + 2 + (n-b-c-3) + 4 \\ &= bn - b^2 - 4b + n + c + 3 \\ &= (n-b)(b+1) - 3b + c + 3. \end{aligned}$$

Hence, using G_3, for fixed values of $b \geq 2$, when c increases by 1, as vertices are moved from A to C, m increases by 1. To maintain the required number of vertices in each set of G_3, we must have $c \in [4, n-b-4]$. For a fixed value of b and letting c range over all values in $[4, n-b-4]$, we can create F_3-saturated graphs with sizes for all possible integers in the interval

$$[(n-b)(b+1) - 3b + 7, (n-b)(b+2) - 3b - 1].$$

If we let $c = n - b - 4$ for fixed n, then we have $m = bn + 2n - b^2 - 5b - 2$, which, as a function of b, is maximized when $b = \lfloor \frac{n-5}{2} \rfloor$.

Fix $|E| = 1$ with $a = n - 9$, $b \geq 2$, $d = 2$ and move vertices from A to C such that $|C|$ increases by 1. Then, in a manner similar to that of the previous section, for each fixed value of $b \geq 2$ we have an F_3-saturated graph with size for each value in the interval below corresponding to that value of b. These intervals are

$$[3n-5, 4n-15], [4n-14, 5n-25], [5n-25, 6n-37],$$

$$[6n-38, 7n-51], [7n-53, 8n-67], [8n-70, 9n-85], \cdots$$

$$\cdots, \left[\left\lceil \frac{n+5}{2} \right\rceil \left\lfloor \frac{n-5}{2} \right\rfloor + \left\lceil \frac{n+5}{2} \right\rceil - 3 \lfloor n-52 \rfloor + 7, \left\lceil \frac{n+5}{2} \right\rceil \left\lfloor \frac{n-5}{2} \right\rfloor + 2 \left\lceil \frac{n+5}{2} \right\rceil - 3 \left\lfloor \frac{n-5}{2} \right\rfloor - 1 \right].$$

Now $[(n-b)(b+2) - 3b - 1] - [(n-b-1)(b+2) - 3(b+1) + 7] + 1 = b - 2$ counts the number of terms that overlap between intervals evaluated at b and $b+1$. Since the first two interval do not overlap, but have consecutive ending and starting values, and the remaining consecutive pairs of intervals have a positive number of terms that overlap, the union of the above intervals is itself and interval.

Alternately, modifying G_3 slightly with $b = 1$ and $d = 3$, (so that $b + d = 4$), then adding an edge from B to D, say b_1d_1, also creates a copy of F_3 with triangles $< b_1, c_1, d_1 >$, $< d_2, c_1, c_4 >$, and $< d_3, c_1, c_2 >$. This modified graph is F_3-saturated. Further, when $b = 1$ and $d = 3$, then $m = 2n + 2c - 3$. Thus, transfering one vertex from A to C (with $c \geq 4$), increases m by 2. Thus, as c increases from 4 to $n - 6$, we obtain the sizes $2n + 5$, $2n + 7$, $\ldots$, $4n - 15$.

The graph in Figure 6 has size $3n - 6$ and is clearly F_3-saturated for $n \geq 7$, as adding any edge will create a triangle that is edge disjoint from the two edge disjoint triangles sharing v. The graph C_4 with two adjacent vertices of the C_4 joined to all vertices of the graph $[K_3 \cup \overline{K}_{n-7}]$ is also F_3-saturated and has size $2n-1$. Similarly, the graph $K_2 + [K_3 \cup \overline{K}_{n-5}]$ is F_3-saturated with size $2n$. Also, the graph P_4 with two end vertices joined to all vertices of the graph $(K_4 \cup \overline{K}_{n-8})$ is F_3-saturated and has size $2n+1$.

For possible values near the extremal number of F_3, consider the family E_3, constructed by adding to the graph E_2 from the previous section the edges ra_2, sb_{t-x-1}, a_1a_2, b_1b_2, $a_{t+x}a_{t+x-1}$, and $b_{t-x}b_{t-x-1}$ and then removing the edges rb_{t-x-1} and sa_2. For $t \geq x+2 \geq 6$, the graph E_3 is F_3-saturated and has size $m = t^2 - x^2 + 2t + 5$ and order $n = 2(t+1)$, hence, $m = \frac{n^2}{4} - x^2 + 4$. We now summarize what we know about the existence of m edge, n vertex F_3-saturated graphs in the following theorem.

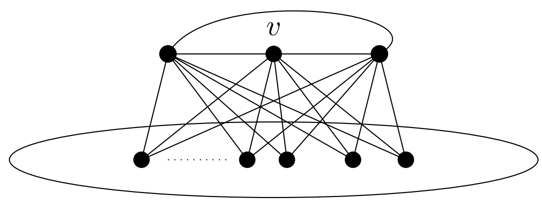


Figure 7: An F_3-saturated graph with $m = 3n-6$.

Theorem 4. *There exists an F_3-saturated graph G of order n with m edges for $m = n+5$, $n+9$, $2n-1, 2n, 2n+1$ and $2n+5, 2n+7, \ldots, 4n-15$. Also, for each m where $3n-6 \leq m \leq \lceil \frac{n+5}{2} \rceil \left(\lfloor \lceil \frac{n-5}{2} \rceil + 2 \right) - 3\lfloor \frac{n-5}{2} \rfloor - 1$. Further, $m = xn - x^2 + 6$, the size of a complete bipartite graph $K_{x,n-x}$ $(0 \leq x \leq n/2)$ with two edge disjoint triangles added, or $m = \frac{n^2}{4} - x^2 + 4$, for $x = 0, 1, \ldots, \frac{n}{4}$, the size of E_3. Further, there are no 2-connected F_3-saturated graphs with size in $[n+6, 2n-5]$.*

Question 2: *Do F_3-saturated graphs on n vertices with m edges exist for other m in the interval $[n+6, 2n-5]$? Also, are there F_3-saturated graphs with sizes $2n+6, 2n+8, \ldots, 3n-8$?*

5 Constructing F_4-saturated Graphs

We know from Lemma 3 that sat$(n, F_4) = n + 8$ and if we insert the edges of a K_7 into a $K_{1,n-1}$ we obtain a 1-connected F_4-saturated graph with $n+20$ edges. From Theorem 1 we have that ex$(n, F_4) = \lfloor \frac{n^2}{4} \rfloor + 10$. Lemma 5 implies that there are no 2-connected F_4-saturated graphs of size m for $n+9 \leq m \leq 2n-5$. Also, complete bipartie graphs with the proper 10 additional edges (for example, a C_7 with three independent chords) are also F_4-saturated.

We can again construct F_4-saturated graphs with a modified $C_5[A, B, C, D, E]$ denoted G_4. In place of the edge uv added to get G_2, we add in C, a chorded C_6 with chords such that the degree of each vertex within the cycle is three. This chorded cycle has three independent edges inducing a copy of F_3 with sets B and D and each vertex of the cycle has degree three. This graph is F_4-saturated when $a \geq 1, b \geq 3$, $d \geq 2$ (so that $b + d \geq 5$) and $c \geq 6$.

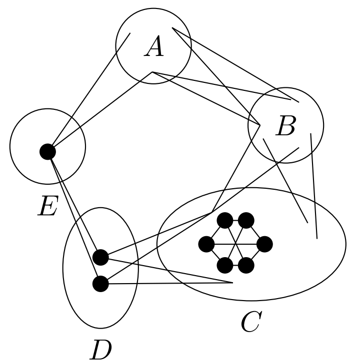


Figure 8: Construction of G_4, for F_4-saturated graphs.

For $d = 2$ and $|E| = 1$, an argument similar to that of the previous section shows that G_4 is F_4-saturated, with size m given by the products of the orders of consecutive vertex sets as follows:

$$m = (n - b - c - 3)b + bc + 2c + 2 + (n - b - c - 3) + 9 = (n - b)(b + 1) - 3b + c + 8.$$

Hence, for fixed values of b, when c increases by 1, as vertices are moved from A to C, the size increases by 1. To maintain the required number of vertices in each set of G_4, we must have $c \in [6, n - b - 4]$. In a manner similar to that in the previous section, for a fixed value of b, we can construct an F_4-saturated graph having size in the interval

$$[(n - b)(b + 1) - 3b + 14, (n - b)(b + 2) - 3b + 4].$$

If we let $c = n - b - 4$ for fixed n, then we have $m = bn + 2n - b^2 - 5b + 4$, which, as a function of b, is maximized when $b = \lfloor \frac{n-5}{2} \rfloor$. The maximum size is achieved when the orders of B and C are as balanced as possible.

If we let $|A| = n-b-9$ and move vertices from A to C such that $|C|$ increases by 1, we have F_4-saturated graphs with sizes in the intervals

$$\begin{gathered}[4n - 7, 5n - 20], [5n - 18, 6n - 32], [6n - 31, 7n - 46],\\ [7n - 46, 8n - 62], [8n - 63, 9n - 80], [9n - 82, 10n - 100], \cdots\\ \cdots, \left[\left\lceil \frac{n+5}{2} \right\rceil \left\lfloor \frac{n-5}{2} \right\rfloor + \left\lceil \frac{n+5}{2} \right\rceil - 3 \left\lfloor \frac{n-5}{2} \right\rfloor + 14, \left\lceil \frac{n+5}{2} \right\rceil \left\lfloor \frac{n-5}{2} \right\rfloor + 2 \left\lfloor \frac{n+5}{2} \right\rceil - 3 \left\lfloor \frac{n-5}{2} \right\rfloor + 4\right].\end{gathered}$$

We can partially extend the possible sizes using a similar construction for F_4-saturated graphs by altering the chorded cycle in G_4 as seen in Figure 9.

In the construction of G'_4 for F_4-saturated graphs shown in Figure 9 in place of the the chorded C_6, we have a chorded C_7 with chords such that the degree of a vertex within the cycle is three for each vertex in $V(C_7) - v$. This chorded cycle has three independent edges inducing a copy of F_3 with sets B and D and each vertex of the cycle except v is the shared vertex of an induced copy of F_3. This graph is F_4-saturated when $a \geq 1, b \geq 2, d = 3$ and $c \geq 7$ with size:

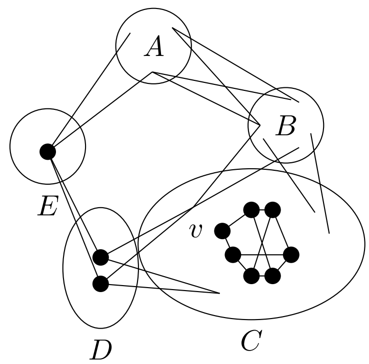


Figure 9: The graph G'_4, an altered G_4 construction for F_4-saturated graphs.

$$\begin{aligned} m &= (n-b-c-4)b + bc + 3c + 3 + (n-b-c-4) + 10 \\ &= (n-b)(b+1) - 4b + 2c + 9. \end{aligned}$$

So when $b = 2, d = 3$ and $c = 7$, $m = 3n - 9$. Increasing c repeatedly by one up to $n - 7$ produces the values: $3n - 7, 3n - 5, 3n - 3, \dots, 5n - 19$.

Finally, consider the graph $2K_2 + [K_4 \cup \overline{K}_{n-8}]$. This graph is F_4-saturated with $4n - 8$ edges. We summarize the results of this section in the following theorem.

Theorem 5. *There exists an F_4-saturated graph G on n vertices and m edges if $m = n + 8$, or $n + 20$, or $3n - 9, 3n - 7, 3n - 5, \dots, 5n - 19$, or for each m where $4n - 8 \leq m \leq \lceil \frac{n+5}{2} \rceil \lfloor \frac{n-5}{2} \rfloor + 2 \lceil \frac{n+5}{2} \rceil - 3 \lfloor \frac{n-5}{2} \rfloor + 4$, or $m = xn - x^2 + 10$, the size of a complete bipartite graph $K_{x,n-x}$ with the proper 10 additional edges. There are no 2-connected F_4-saturated graphs with size in the interval $[n + 9, 2n - 5]$.*

6 Constructing F_t-saturated Graphs, $t \geq 5$

In this section we determine some sizes for F_t-saturated graphs where $t \geq 5$. We know that for $t \geq 2$, $\text{sat}(n, F_t) = n + 3t - 4$. If we insert the edges of a K_{2t-1} into a copy of $K_{1,n-1}$ we obtain a F_t-saturated graph with size $n - 1 + (2t - 1)(t - 1)$.

We generalize the two constructions for F_4-saturated graphs to construct F_t-saturated graphs. The graph G_2 that is F_2-saturated can be made into an F_t-saturated G_t ($t \geq 5$) by replacing the edge $uv \in E(C)$ with a chorded cycle $\hat{C}$ on $2t - 2$ vertices. The chords of the cycle $\hat{C}$ must be distributed amongst the vertices such that each vertex in $\hat{C}$ has degree $t - 1$ in $\hat{C}$. Since $2t - 2$ is even, this can always be done. One way to distribute the chords when t is odd is seen in Figure 10(b) for $t = 5$. In the cycle $\hat{C}$ we label the vertices clockwise $v_1, v_2, ..., v_{2t-2}$. When t is odd, we add the edge v_iv_j, if the distance between v_i and v_j is exactly k where $k = 3, 5, ..., t - 2$ and, when t is even, $k = 3, 5, \dots, t - 1$. In this way, each vertex in $\hat{C}$ is adjacent to $t - 1$ other vertices of $\hat{C}$ so each $u \in \hat{C}$ is in exactly $t - 1$ edge disjoint triangles $\{uv, uw, vw\}$ where $v \in \hat{C}$ and w is a vertex in B or D.

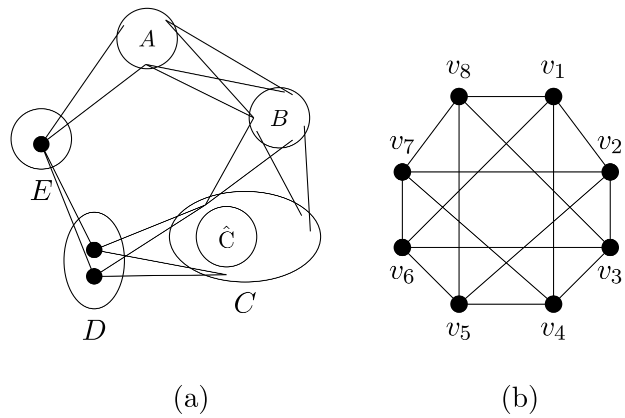


Figure 10: (a) The graph G_t; (b) Example of $\hat{C}$ for $t = 5$.

The graph G_t is F_t−saturated for $a \geq 1$, $b \geq t-1$, $d = 2$ (hence $b + d \geq t + 1$), $|E| = 1$ and $c \geq 2t - 2$. The argument that G_t is F_t-saturated follows exactly those of the previous sections.

In general, G_t will have $m = (n-b)(b+1) - 3b + c + (t-1)^2 - 1$ edges. For fixed values of b, when c increases by 1, as vertices are moved from A to C, the size increases by 1. To maintain the required number of vertices in each set of G_t, we must have $c \in [2t-2, n-b-4]$. For a fixed value of b and n large enough, we can create an F_t-saturated graph having size in

$$\left[(n-b)(b+1) - 3b + t^2 - 2, (n-b)(b+1) + n - 4b + (t-1)^2 - 5\right].$$

If we let $c = n - b - 4$ for fixed n, then we have $m = bn + 2n - b^2 - 5b + t^2 - 2t - 4$, which, as a function of b, is maximized when $b = \lfloor \frac{n-5}{2} \rfloor$. Thus, the graphs from the construction have distinct sizes for each $b \in [t-1, \lfloor \frac{n-5}{2} \rfloor]$. Then the smallest size for an F_t-saturated G_t on $n \geq 3t$ vertices is given when $b = t - 1$ and $c = 2t - 2$ and is $m = (n-t+1)(t) - 3(t-1) + 2t - 2 + (t-1)^2 - 1 = t(n-2) + 1$.

If we let $|A| = n - b - c - 3$, fix b, and move vertices from A to C such that $|C|$ increases by 1, we have F_t-saturated graphs with sizes in the following set of intervals which we denote as $I_{n,t}$:

$$\begin{gathered}[nt - 2t + 1, nt - 5t + n], [nt + n - 4t - 2, nt + 2n - 7t - 4], \cdots \\ [2nt - 3n - 3t^2 + 8t - 2, 2nt - 2n - 3t^2 + 4t], [2nt - 2n - 3t^2 + 4t + 1, 2nt - n - 3t^2 + 2], \\ [2nt - n - 3t^2 + 2, 2nt - 3t^2 - 4t + 2], \cdots, \\ \left[\ \left\lceil \tfrac{n+5}{2} \right\rceil \left\lfloor \tfrac{n-5}{2} \right\rfloor + \left\lceil \tfrac{n+5}{2} \right\rceil - 3 \left\lfloor \tfrac{n-5}{2} \right\rfloor + t^2 - 2, \left\lceil \tfrac{n+5}{2} \right\rceil \left\lfloor \tfrac{n-5}{2} \right\rfloor + \left\lceil \tfrac{n+5}{2} \right\rceil + n - 4 \left\lfloor \tfrac{n-5}{2} \right\rfloor + (t-1)^2 - 5\ \right].\end{gathered}$$

Note that there is a gap between the intial intervals of length $2t - b - 3$, which is the distance between the end of an interval and the beginning of the next consecutive interval. However, once $b \geq 2t - 3$ the intervals begin to overlap. To partially fill this gap we use a modification of the previous construction.

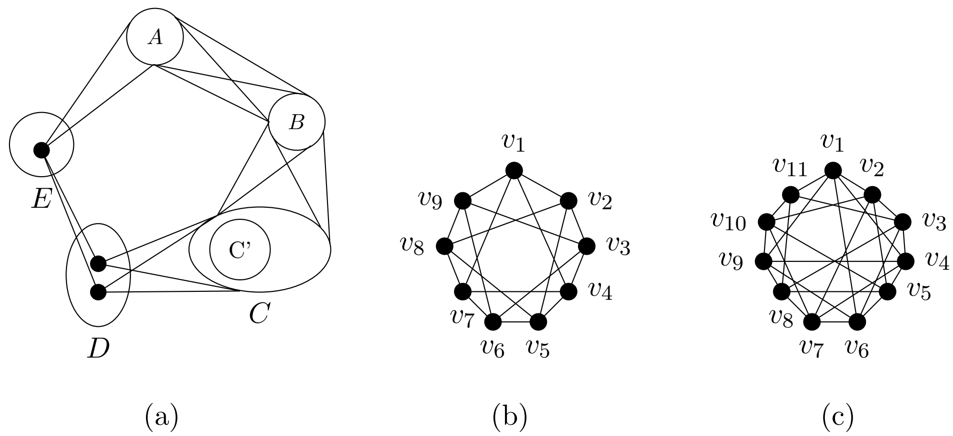


Figure 11: (a) Construction;(b) Example C' for $t = 5$; (c) Example C' for $t = 6$.

We form a new graph ${G'}_t$ from G_t by replacing $\hat{C}$ with a new cycle C'. The chorded cycle C' has order $2t-1$. If t is even, we distribute the chords of the cycle C' amongst the vertices such that all but one vertex, say v, in C' is adjacent to exactly $t-1$ other vertices in C' and v is adjacent to exactly $t-2$ vertices in C'. If t is odd, we distribute the chords of the cycle C' amongst the vertices so that all vertices are adjacent to exactly $t-1$ other vertices in C'. To do this we make v_iv_j and edge for v_j at distance $3, 5, \ldots, t-2$ from v_i. In this case, C' is $(t-1)$-regular.

That this graph is F_t-saturated is shown in the same way as has been done in the previous sections.

In general ${G'}_t$ has size m' where

$$m' = (n-b)(b+1) - 3b + c - 1 + \begin{cases} \frac{(2t^2-3t+1)}{2} & \text{t odd,} \\ \frac{(2t^2-3t)}{2} & \text{t even.} \end{cases}$$

We summarize the F_t case in the following Theorem.

Theorem 6. *There us an F_t-saturated graph $(t \geq 5)$ of size m if $m = n+3t-4$ or $m = n-1+(2t-1)(t-1)$, or m lies in one of the intervals in $I_{n,t}$ or $m = m'$.*

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References

[1] Abbott, H.L.; Hanson, D.; Sauer, N. Intersection theorems for systems of sets. *J. Combin. Theory Ser. A*, 12(1972), 381-389.

[2] Amin, K.; Faudree, J.R.; Gould, R. J. The edge spectrum of K_4-saturated graphs. *J. Combin.Math. Combin. Comput.* 81(2012), 233–242.

[3] Amin, K.; Faudree, J.R.; Gould, R. J.; Sidorowicz, E. On the non-$(p-1)$-partite K_p-free graphs. *Discuss. Math. Graph Theory* 33(2013), no. 1, 9–23.

[4] Barefoot, C.; Casey, K.; Fisher, D.; Fraughnaugh, K.; Harary, F. Size in maximal triangle-free graphs and minimal graphs of diameter 2. *Discrete Math.* 138(1995), no. 1-3, 93–99.

[5] Chartrand, G.; Lesniak, L. **Graphs & Digraphs** 4th Ed., Chapman Hall/CRC, Boca Raton, FL, 2005.

[6] Erdős, P.; Furedi, Z.; Gould, R. J.; Gunderson, D. S. Extremal graphs for intersecting triangles. *J Combin. Theory Ser. B* 64(1995), no. 1, 89-100.

[7] Faudree, R., Ferrara, M., Gould, R., and Jacobson, M. tK_p-saturated graphs of minimum size. *Discrete Math.* 309(2009), no. 19, 5870-5876.

[8] Fuller, J; Gould, R. J. On $(K_t - e)$-saturated graphs. *Graphs & Combin.* 34(2018), no. 1, 85-95.

[9] Gould, R. J.; Tang, W.; Wei, E.; Zhang, C-Q. The edge spectrum of the saturation number for small paths. *Discrete Math.* 312(2012), no. 17, 2682–2689.