On Independent Triples and Vertex-Disjoint Chorded Cycles in Graphs

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Abstract

Let G be a graph, and let $\sigma_3(G)$ be the minimum degree sum of three independent vertices of G. We prove that if G is a graph of order at least $8k+5$ and $\sigma_3(G) \geq 9k-2$ with $k \geq 1$, then G contains k vertex-disjoint chorded cycles. We also show that the degree sum condition on $\sigma_3(G)$ is sharp.

Keywords: Vertex-disjoint chorded cycles, Minimum degree sum, Degree sequence.

1 Introduction

The study of cycles in graphs is a rich and important area. One question of particular interest is to find conditions that guarantee the existence of k vertex-disjoint cycles. In 1963, Corrádi and Hajnal [3] proved that if $|G| \geq 3k$ and the minimum degree $\delta(G) \geq 2k$, then G contains k vertex-disjoint cycles. For an integer $t \geq 1$, let

$$\sigma_t(G) = \min \left\{ \sum_{v \in X} d_G(v) \,|\, X \text{ is an independent vertex set of } G \text{ with } |X| = t. \right\},$$

and $\sigma_t(G) = \infty$ when the independence number $\alpha(G) < t$. Enomoto [4] and Wang [11] independently extended the Corrádi and Hajnal result showing, if $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$, then G contains k vertex-disjoint cycles. Fujita et al. [6] proved that if $|G| \geq 3k+2$ and $\sigma_3(G) \geq 6k-2$, then G contains k vertex-disjoint cycles, and in [9], this result was extended to $\sigma_4(G) \geq 8k-3$.

A *chord* of a cycle is an edge between two non-adjacent vertices of the cycle, and a *chorded cycle* is a cycle with at least one chord. In 2008, Finkel improved Corrádi and Hajnal's result for chorded cycles.

Theorem 1.1. (Finkel [5]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint chorded cycles.*

In 2010, Chiba et al. proved Theorem 1.2 which is a stronger result than Theorem 1.1, since $\sigma_2(G) \geq 2\delta(G)$.

Theorem 1.2. (Chiba, Fujita, Gao, Li [1]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\sigma_2(G) \geq 6k-1$, then G contains k vertex-disjoint chorded cycles.*

In this paper, we consider a similar extension for chorded cycles, as Fujita et al. [6] proved the existence of k vertex-disjoint cycles under the condition $\sigma_3(G)$. In particular, we first show the following.

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Theorem 1.3. *If G is a graph of order at least 7 and $\sigma_3(G) \geq 7$, then G contains a chorded cycle.*

Remark 1. We define the following graphs: $G_1 = K_2 \cup K_2$, $G_2 = K_2 \cup K_3$, and $G_3 = K_3 \cup K_3$, where $H_1 \cup H_2$ denotes the union of two disjoint graphs H_1 and H_2. Then for each $1 \leq i \leq 3$, G_i satisfies the $\sigma_3(G)$ condition of Theorem 1.3, since the independence number $\alpha(G_i) = 2$. However, G_i for each $1 \leq i \leq 3$ does not contain a chorded cycle. Thus $|G| \geq 7$ is necessary.

Our main result is the following theorem.

Theorem 1.4. *Let $k \geq 1$ be an integer. If G is a graph of order at least $8k+5$ and $\sigma_3(G) \geq 9k-2$, then G contains k vertex-disjoint chorded cycles.*

Remark 2. Theorem 1.4 is sharp with respect to the degree sum condition. Consider the complete bipartite graph $G = K_{3k-1,n-3k+1}$, where large $n = |G|$. Then $\sigma_3(G) = 3(3k-1) = 9k-3$. However, G does not contain k vertex-disjoint chorded cycles, since any chorded cycle must contain at least 3 vertices from each partite set. Thus $\sigma_3(G) \geq 9k-2$ is necessary. Also, since $\sigma_3(G) \geq 3\sigma_2(G)/2$, when the order of G is sufficiently large, Theorem 1.4 is a stronger result than Theorem 1.2.

For other related results on vertex-disjoint chorded cycles in graphs and bipartite graphs, we refer the reader to see [2, 7, 10].

In this paper, all graphs are simple. Let G be a graph, H a subgraph of G and $X \subseteq V(G)$. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$. For $u \in V(G)$, we denote $N_H(u) = N_G(u) \cap V(H)$ and $d_H(u) = |N_H(u)|$. Also we denote $d_H(X) = \sum_{u \in X} d_H(u)$. If $H = G$, then $d_G(X) = d_H(X)$. The subgraph of G induced by X is denoted by $\langle X \rangle$. Let $G - X = \langle V(G) - X \rangle$ and $G - H = \langle V(G) - V(H) \rangle$. If $X = \{x\}$, then we write $G - x$ for $G - X$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For a graph G, comp(G) is the number of components of G. If G is one vertex, that is, $V(G) = \{x\}$, then we simply write x instead of G. For an integer $r \geq 1$ and two vertex-disjoint subgraphs A, B of G, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from A to B such that $d_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. In this paper, since it is sufficient to consider the case of equality in the above inequality, when we write $(d_1, d_2, \ldots, d_r)$, we assume $d_B(v_i) = d_i$ for each $1 \leq i \leq r$. For two disjoint $X, Y \subseteq V(G)$, $E(X, Y)$ denotes the set of edges of G connecting a vertex in X and a vertex in Y. Let Q be a path or a cycle with a given orientation and $x \in V(Q)$. Then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of Q from x to y (including x and y) in the given direction. The reverse sequence of $Q[x, y]$ is denoted by $Q^-[y, x]$. We also write $Q(x, y] = Q[x^+, y]$, $Q[x, y) = Q[x, y^-]$ and $Q(x, y) = Q[x^+, y^-]$. If Q is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t(, x_1)$, then we assume an orientation of Q is given from x_1 to x_t. If P is a path connecting x and y of $V(G)$, then we denote the path P as $P[x, y]$. A cycle of length ℓ is called a *ℓ-cycle*. For terminology and notation not defined here, see [8].

2 Preliminaries

Definition 2.1. Suppose $C_1, \ldots, C_r$ are r vertex-disjoint chorded cycles in a graph G. We say $\{C_1, \ldots, C_r\}$ is *minimal* if G does not contain r vertex-disjoint chorded cycles $C'_1, \ldots, C'_r$ such that $|\cup_{i=1}^r V(C'_i)| < |\cup_{i=1}^r V(C_i)|$.

Definition 2.2. Let $C = v_1, \ldots, v_t, v_1$ be a cycle with chord v_iv_j, $i < j$. We say a chord $vv' \neq v_iv_j$ is *parallel* to v_iv_j if either $v, v' \in C[v_i, v_j]$ or $v, v' \in C[v_j, v_i]$. Note if two distinct chords share an endpoint, then they are parallel. We say two distinct chords are *crossing* if they are not parallel.

Definition 2.3. Let u_iv_j and $u_\ell v_m$ be two distinct edges between two vertex-disjoint paths $P_1 = u_1, \ldots, u_s$ and $P_2 = v_1, \ldots, v_t$. We say u_iv_j and $u_\ell v_m$ are *parallel* if either $i \leq \ell$ and $j \leq m$, or $\ell \leq i$ and $m \leq j$. Note if two distinct edges between P_1 and P_2 share an endpoint, then they are parallel. We say two distinct edges between two vertex-disjoint paths are *crossing* if they are not parallel.

Definition 2.4. Let v_iv_j and $v_\ell v_m$ be two distinct edges between vertices of a path $P = v_1, \ldots, v_t$, with $j \geq i + 2$ and $m \geq \ell + 2$. We say v_iv_j and $v_\ell v_m$ are *nested* if either $i \leq \ell < m \leq j$ or $\ell \leq i < j \leq m$.

Definition 2.5. Let $P = v_1, \ldots, v_t$ be a path. We say a vertex v_i on P has a *left edge* if there exists an edge v_iv_j for some $j < i - 1$. We also say v_i has a *right edge* if there exists an edge v_iv_j for some $j > i + 1$.

3 Lemmas

Lemma 3.1. *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. For any $1 \leq i \leq r$, C_i cannot have two or more parallel chords.*

Proof. This follows easily from the minimality of $\mathscr{C}$. □

Lemma 3.2. *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. If $|C_i| \geq 7$ for some $1 \leq i \leq r$, then C_i has at most two chords. Furthermore, if C_i has two chords, then these chords must be crossing.*

Proof. Let $|C_i| \geq 7$ for some $1 \leq i \leq r$. Suppose C_i contains at least three chords. By Lemma 3.1, no two of them can be parallel. Thus they are all mutually crossing. Label the endpoints of these three chords $v_1, v_2, \ldots, v_6$ in that order on C_i. Since the chords are mutually crossing, the three chords are given by v_1v_4, v_2v_5, v_3v_6. These six endpoints partition C_i into six intervals $C_i[v_j, v_{j+1}), 1 \leq j \leq 6$, where $v_7 = v_1$. Since $|C_i| \geq 7$, some interval contains at least one vertex of C_i which is not an endpoint of the three chords. Without loss of generality, we may assume $C_i[v_1, v_2)$ contains some vertex of C_i other than v_1. Then $C_i[v_2, v_4], v_1, C_i^-[v_1, v_5], v_2$ is a shorter cycle with chord v_3v_6. Thus C_i has at most two chords. If the C_i has two chords, then these chords must be crossing by Lemma 3.1. □

Lemma 3.3. *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. Then $d_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G) - \cup_{i=1}^r V(C_i)$. Furthermore, for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$, if $d_C(x) = 4$, then $|C| = 4$, and if $d_C(x) = 3$, then $|C| \leq 6$.*

Proof. Suppose $d_C(x) \geq 5$ for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$. Let $v_j \in N_C(x)$ with $1 \leq j \leq 5$, and let $v_1, v_2, \ldots, v_5$ be in that order on C. Then $x, C[v_1, v_3], x$ is a shorter cycle with chord xv_2, contradicting the minimality of $\mathscr{C}$. Thus $d_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G) - \cup_{i=1}^r V(C_i)$.

Next suppose $d_C(x) = 4$ for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$. Let $v_i \in N_C(x)$ with $1 \le i \le 4$, and let v_1, v_2, v_3, v_4 be in that order on C. Let $X = \{v_1, v_2, v_3, v_4\}$. These neighbors define four intervals $C[v_i, v_{i+1}), 1 \le i \le 4$, where $v_5 = v_1$. Assume $|C| \ge 5$. Then a vertex of $C - X$ lies in one of the intervals. Without loss of generality, we may assume there exists a vertex of $C - X$ in $C[v_1, v_2)$. Then $x, C[v_2, v_4], x$ is a shorter cycle with chord xv_3, contradicting the minimality of $\mathscr{C}$. Thus $|C| = 4$.

Finally, suppose $d_C(x) = 3$ for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$. Let $v_i \in N_C(x)$ with $1 \le i \le 3$, and let v_1, v_2, v_3 be in that order on C. Let $X = \{v_1, v_2, v_3\}$. These neighbors define three intervals $C[v_i, v_{i+1}), 1 \le i \le 3$, where $v_4 = v_1$. If $|C| \ge 7$, then some interval contains at least two vertices of $C - X$. Without loss of generality, we may assume $C[v_1, v_2)$ contains them. Then $x, C[v_2, v_1], x$ is a shorter cycle with chord xv_3, contradicting the minimality of $\mathscr{C}$. Thus $|C| \le 6$. □

Lemma 3.4. *Suppose there exist at least five edges connecting two vertex-disjoint paths P_1 and P_2. Then there exist at least three mutually parallel edges or at least three mutually crossing edges.*

Proof. Let $x_iy_i \in E(P_1, P_2)$ for each $1 \le i \le 5$. Without loss of generality, let $x_1, x_2, \ldots, x_5$ appear in that order on P_1. Also we may assume that y_1, y_5 are in that order on P_2, otherwise, we consider the reverse orientation of P_2. Let $P_2 = u_1, u_2, \ldots, u_s$ $(s \ge 1)$. If $s = 1$, then all the edges connecting P_1 and P_2 are mutually parallel. Thus we may assume that $s \ge 2$. Now we claim that $y_1 \ne u_1$. Suppose not. Then there exist at least two parallel edges in $\{x_iy_i \,|\, 2 \le i \le 5\}$, otherwise, the lemma holds. Let $x_{i_1}y_{i_1}, x_{i_2}y_{i_2}$ for $2 \le i_1 < i_2 \le 5$ be the parallel edges. Then $x_1y_1, x_{i_1}y_{i_1}, x_{i_2}y_{i_2}$ are three mutually parallel edges. Thus the claim holds. By symmetry, $y_5 \ne u_s$. If $y_i \in P_2[y_1, y_5]$ for some $2 \le i \le 4$, then x_1y_1, x_iy_i, x_5y_5 are three mutually parallel edges. Thus $y_i \notin P_2[y_1, y_5]$ for each $2 \le i \le 4$. Then $|P_2[u_1, y_1) \cap \{y_2, y_3, y_4\}| \ge 2$ or $|P_2(y_5, u_s] \cap \{y_2, y_3, y_4\}| \ge 2$. By symmetry, we may assume that $|P_2[u_1, y_1) \cap \{y_2, y_3, y_4\}| \ge 2$. Let i_1, i_2 be integers such that $2 \le i_1 < i_2 \le 4$ and $y_{i_1}, y_{i_2} \in P_2[u_1, y_1)$. If y_{i_1}, y_{i_2} are in that order on P_2, then $x_{i_1}y_{i_1}, x_{i_2}y_{i_2}$ are parallel edges, and $x_{i_1}y_{i_1}, x_{i_2}y_{i_2}, x_5y_5$ are three mutually parallel edges. On the other hand, if y_{i_2}, y_{i_1} are in that order on P_2, then $x_{i_1}y_{i_1}, x_{i_2}y_{i_2}$ are crossing edges, and $x_1y_1, x_{i_1}y_{i_1}, x_{i_2}y_{i_2}$ are three mutually crossing edges. Thus the lemma holds. □

Lemma 3.5. *Suppose there exist at least three mutually parallel edges or at least three mutually crossing edges connecting two vertex-disjoint paths P_1 and P_2. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$.*

Proof. If there exist at least three mutually crossing edges connecting the paths P_1 and P_2, then we consider the reverse orientation of P_2. Then the edges are all mutually parallel. Thus we have only to consider the case where all the edges are mutually parallel. Now let x_1y_1, x_2y_2, x_3y_3 be the edges. Without loss of generality, let x_1, x_2, x_3 appear in that order on P_1. Note that the endpoints y_1, y_2, y_3 appear in that order on P_2. Then $P_1[x_1, x_3], y_3, P_2^-[y_3, y_1], x_1$ is a cycle with chord x_2y_2. □

Lemma 3.6. *Suppose there exist at least five edges connecting two vertex-disjoint paths P_1 and P_2 with $|P_1 \cup P_2| \ge 7$. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$ not containing at least one vertex of $\langle P_1 \cup P_2 \rangle$.*

Proof. By Lemma 3.4, there must be at least three mutually parallel edges or at least three mutually crossing edges. Then by Lemma 3.5, there exists a chorded cycle C in $\langle P_1 \cup P_2 \rangle$. If $V(C) \ne V(P_1 \cup P_2)$, then the lemma holds. Thus suppose $V(C) = V(P_1 \cup P_2)$. Let C' be a cycle obtained from C by removing all chords. Since $|E(\langle P_1 \cup P_2 \rangle) - E(C')| \geq 3$, C has at least three chords. By $|C| = |P_1 \cup P_2| \geq 7$, a shorter chorded cycle exists in $\langle P_1 \cup P_2 \rangle$ as in the proof of Lemma 3.2. Thus the lemma holds. □

Lemma 3.7. *Let P_1, P_2 be two vertex-disjoint paths, and let u_1, u_2 ($u_1 \neq u_2$) be in that order on P_1. Suppose $d_{P_2}(u_i) \geq 2$ for each $i \in \{1, 2\}$. Then there exists a chorded cycle in $\langle P_1[u_1, u_2] \cup P_2 \rangle$.*

Proof. Let $P_2 = v_1, \dots, v_t$, and let $v_i, v_j \in N_{P_2}(u_1)$ with $i < j$. If u_2 has a neighbor that lies in $P_2[v_1, v_i]$ or $P_2[v_j, v_t]$, then we can easily form a chorded cycle in $\langle P_1[u_1, u_2] \cup P_2 \rangle$. Thus both of u_2's neighbors in P_2 must lie in $P_2(v_i, v_j)$, call them $v_\ell, v_{\ell'}$ with $\ell < \ell'$. Then $P_1[u_1, u_2], v_{\ell'}, P_2^-[v_{\ell'}, v_i], u_1$ is a cycle with chord $u_2 v_\ell$. □

Lemma 3.8. *Let H be a connected graph of order at least 4. Suppose H contains neither a chorded cycle nor a Hamiltonian path. Let $P_1 = u_1, \dots, u_s$ ($s \geq 3$) be a longest path in H, and let $P_2 = v_1, \dots, v_t$ ($t \geq 1$) be a longest path in $H - P_1$. Then the following statements hold.*
(i) $N_{H-P_1}(u_i) = \emptyset$ *for each* $i \in \{1, s\}$.
(ii) $d_H(u_i) = d_{P_1}(u_i) \leq 2$ *for each* $i \in \{1, s\}$.
(iii) $N_{H-(P_1 \cup P_2)}(v_j) = \emptyset$ *for each* $j \in \{1, t\}$.
(iv) $d_{P_2}(v_j) \leq 2$ *for each* $j \in \{1, t\}$.
(v) $u_1 u_s \notin E(H)$.
(vi) *If* $d_H(v_1) \leq d_H(v_t)$, *then* $d_H(\{u_1, u_s, v_1\}) \leq 6$.

Proof. Since P_1 is a longest path, clearly, (i) holds. By (i), $d_H(u_i) = d_{P_1}(u_i)$ for each $i \in \{1, s\}$. Since H does not contain a chorded cycle, $d_{P_1}(u_i) \leq 2$ for each $i \in \{1, s\}$. Thus (ii) holds. Since P_2 is a longest path in $H - P_1$, clearly, (iii) holds. Also, since H does not contain a chorded cycle, (iv) holds. Furthermore, since H is connected and P_1 is a longest path in H, $u_1 u_s \notin E(H)$. Thus (v) holds.

Finally, we prove (vi). Let $X = \{u_1, u_s, v_1\}$. By (ii), $d_H(u_i) \leq 2$ for each $i \in \{1, s\}$. If $d_H(v_1) \leq 2$, then $d_H(X) \leq 6$, and (vi) holds. Thus we may assume $d_H(v_1) \geq 3$. Then $d_H(v_t) \geq 3$ by the assumption. If $t = 1$, then $d_{P_1}(v_1) \geq 3$. Thus there exists a chorded cycle in $\langle v_1 \cup P_1 \rangle$, a contradiction. If $t = 2$, then $d_{P_1}(v_1) \geq 2$ and $d_{P_1}(v_2) \geq 2$ by (iii), and so by Lemma 3.7, there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$, a contradiction. Thus we may assume $t \geq 3$. By Lemma 3.7, $d_{P_1}(v_j) \leq 1$ for some $j \in \{1, t\}$. Suppose $j = 1$, that is, $d_{P_1}(v_1) \leq 1$. By (iii) and (iv), $d_{P_2}(v_1) = 2$. Since $N_{P_1}(v_\ell) \neq \emptyset$ for each $\ell \in \{1, t\}$ by (iii) and (iv), there exists a cycle with chord adjacent to v_1 in $\langle P_1 \cup P_2 \rangle$, a contradiction. If $j = t$, that is, $d_{P_1}(v_t) \leq 1$, then we get a contradiction as in the case where $j = 1$. Thus (vi) holds. □

Lemma 3.9. *Let H be a graph containing a path P. If there exist nested edges between vertices of P, then H contains a chorded cycle.*

Proof. Let v_1, v_2, v_3, v_4 be in that order on P. Suppose $v_1 v_4$ and $v_2 v_3$ are nested edges. Then $P[v_1, v_4], v_1$ is a cycle with chord $v_2 v_3$. □

Lemma 3.10. *Let H be a graph containing a path $P = v_1, v_2, \dots, v_t$ ($t \geq 4$). For any $2 \leq i \leq t-2$, if v_i has a right edge and v_{i+1} has a left edge, then H contains a chorded cycle.*

Proof. Let $v_i v_j \in E(H)$ with $i + 2 \leq j \leq t$ and $v_{i+1} v_\ell \in E(H)$ with $1 \leq \ell \leq i - 1$. Then $P[v_\ell, v_i], v_j, P^-[v_j, v_{i+1}], v_\ell$ is a cycle with chord $v_i v_{i+1}$. □

Lemma 3.11. *Let H be a graph containing a path $P = v_1, \ldots, v_t$ $(t \geq 3)$, and not containing a chorded cycle. If $v_1v_i \in E(H)$ for some $i \geq 3$, then $d_P(v_j) \leq 3$ for any $j \leq i-1$ and in particular, $d_P(v_{i-1}) = 2$. And if $v_tv_i \in E(H)$ for some $i \leq t-2$, then $d_P(v_j) \leq 3$ for any $j \geq i+1$ and in particular, $d_P(v_{i+1}) = 2$.*

Proof. Suppose $v_1v_i \in E(H)$ for some $i \geq 3$. No vertex v_j with $j \leq i-1$ has a left edge, otherwise the edge nests with v_1v_i, and by Lemma 3.9, H contains a chorded cycle, a contradiction. Also, no vertex v_j with $j \leq i-1$ has two or more right edges, otherwise the edges nest, and again H contains a chorded cycle, a contradiction. Thus $d_P(v_j) \leq 3$ for any $j \leq i-1$. Furthermore, v_{i-1} cannot have a right edge by Lemma 3.10. Thus $d_P(v_{i-1}) = 2$. By symmetry, the same proof shows that if $v_tv_i \in E(H)$ for some $i \leq t-2$, then $d_P(v_j) \leq 3$ for any $j \geq i+1$ and $d_P(v_{i+1}) = 2$. □

Lemma 3.12. *Let H be a graph containing a path $P = v_1, \ldots, v_t$ $(t \geq 6)$, and not containing a chorded cycle. If $d_P(v_1) = 1$, then $d_P(v_i) = 2$ for some $3 \leq i \leq 5$, or if $v_1v_3 \in E(H)$, then $d_P(v_i) = 2$ for some $4 \leq i \leq 6$.*

Proof. Suppose either $d_P(v_1) = 1$ or $v_1v_3 \in E(H)$. If $d_P(v_1) = 1$, then we let $i = 3$, and if $v_1v_3 \in E(H)$, then we let $i = 4$. Vertex v_i cannot have a left edge, otherwise in the first case, we have $d_P(v_1) = 2$, and in the second case, we get a chorded cycle by Lemmas 3.9 and 3.10. Thus we have a contradiction in either case. If $d_P(v_i) = 2$, then the lemma holds. Thus suppose $d_P(v_i) \geq 3$. Then v_i must have a right edge, say v_iv_j with $j \geq i+2$. If $j = i+2$, then $d_P(v_{i+1}) = 2$, otherwise we get a contradiction by Lemma 3.10. Thus $j > i+2$. By Lemma 3.10, v_{i+1} cannot have a left edge. If $d_P(v_{i+1}) = 2$, then the lemma holds. Thus $d_P(v_{i+1}) \geq 3$, and v_{i+1} has a right edge, say $v_{i+1}v_\ell$ for some $\ell \geq i+3$. If $\ell \leq j$, then we have nested edges and a chorded cycle by Lemma 3.9, a contradiction. Thus $\ell > j$. By the same arguments as for v_{i+1}, either $d_P(v_{i+2}) = 2$, or v_{i+2} has a right edge $v_{i+2}v_{\ell'}$ for some $\ell' > \ell$. In the later case, $P[v_i, v_{i+2}], v_{\ell'}, P^-[v_{\ell'}, v_j], v_i$ is a cycle with chord $v_{i+1}v_\ell$, a contradiction. Thus $d_P(v_{i+2}) = 2$, and the lemma holds. □

Lemma 3.13. *Let H be a graph containing a path $P = v_1, \ldots, v_t$ $(t \geq 6)$, and not containing a chorded cycle. If $d_P(v_t) = 1$, then $d_P(v_i) = 2$ for some $t-4 \leq i \leq t-2$, or if $v_tv_{t-2} \in E(H)$, then $d_P(v_i) = 2$ for some $t-5 \leq i \leq t-3$.*

Proof. The lemma follows from the proof of Lemma 3.12 by symmetry. □

Lemma 3.14. *Let H be a graph of order at least 13. Suppose H does not contain a chorded cycle. If H contains a Hamiltonian path, then there exists an independent set X of four vertices in H such that $d_H(X) \leq 8$.*

Remark 3. We consider the following graph H of order 12. (See Fig. 1.) Then H satisfies all the conditions except for the order in Lemma 3.14. However, H does not contain an independent set X of four vertices such that $d_H(X) \leq 8$. Thus $|H| \geq 13$ is necessary.

Proof. Let $P = v_1, \ldots, v_t$ $(t \geq 13)$ be a Hamiltonian path in H. If $v_1v_t \in E(H)$, then $d_H(v) = 2$ for any $v \in V(H)$, otherwise, a chorded cycle exists in H, a contradiction. Then $X = \{v_1, v_3, v_5, v_7\}$ is an independent set of four vertices such that $d_H(X) = 8$. Thus we may now assume $v_1v_t \notin E(H)$. Since P is a Hamiltonian path in H, note $d_P(v) = d_H(v)$ for any $v \in V(P)$. Also, $d_H(v_1) \leq 2$ and $d_H(v_t) \leq 2$ by Lemma 3.9.

Case 1. Suppose $d_H(v_1) = 1$ and $d_H(v_t) = 1$.

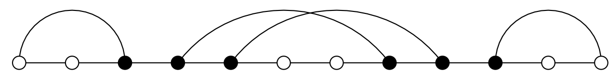

Fig. 1. The graph H of order 12. The white vertex ($\circ$) shows degree 2, and the black vertex ($\bullet$) shows degree 3.

By Lemmas 3.12 and 3.13, $d_H(v_i) = 2$ for some $3 \le i \le 5$ and $d_H(v_j) = 2$ for some $t-4 \le j \le t-2$. Since $t \ge 13$, $v_iv_j \notin E(H)$. Thus $X = \{v_1, v_i, v_j, v_t\}$ is the desired set.

Case 2. Suppose $d_H(v_1) = 1$ and $d_H(v_t) = 2$, or $d_H(v_1) = 2$ and $d_H(v_t) = 1$.

In this case, we may assume $d_H(v_1) = 1$ and $d_H(v_t) = 2$, otherwise, we consider the reverse orientation of P. Let $v_tv_j \in E(H)$ for some $2 \le j \le t-2$. Suppose $2 \le j \le t-5$. Since $d_H(v_t) = 2$, $v_{j+1}v_t \notin E(H)$ and $v_{j+3}v_t \notin E(H)$. By Lemma 3.11, $d_H(v_{j+1}) = 2$ and $d_H(v_{j+3}) \le 3$. Then $X = \{v_1, v_{j+1}, v_{j+3}, v_t\}$ is the desired set. Thus $t-4 \le j \le t-2$. By Lemma 3.12, $d_H(v_i) = 2$ for some $3 \le i \le 5$. If $j \in \{t-4, t-3\}$, then v_{j+1} is still non-adjacent to v_t and $d_H(v_{j+1}) = 2$ by Lemma 3.11. Since $t \ge 13$, $v_iv_{j+1} \notin E(H)$. Then $X = \{v_1, v_i, v_{j+1}, v_t\}$ is the desired set. Thus $j = t-2$. By Lemma 3.13, $d_H(v_\ell) = 2$ for some $t-5 \le \ell \le t-3$. Since $t \ge 13$, $v_iv_\ell \notin E(H)$. Then $X = \{v_1, v_i, v_\ell, v_t\}$ is the desired set.

Case 3. Suppose $d_H(v_1) = 2$ and $d_H(v_t) = 2$.

Suppose $v_1v_3 \in E(H)$ or $v_tv_{t-2} \in E(H)$. Then we may assume $v_1v_3 \in E(H)$, otherwise, we consider the reverse orientation of P. By Lemma 3.12, $d_H(v_i) = 2$ for some $4 \le i \le 6$. If $v_tv_{t-2} \in E(H)$, then $d_H(v_j) = 2$ for some $t-5 \le j \le t-3$ by Lemma 3.13. As before, since $t \ge 13$, $v_iv_j \notin E(H)$. Then $X = \{v_1, v_i, v_j, v_t\}$ is the desired set. Thus $v_tv_{t-2} \notin E(H)$. Then $v_tv_s \in E(H)$ for some $s \le t-3$. By Lemma 3.11, $d_H(v_{s+1}) = 2$. Note $s \ge 3$ since $v_1v_3 \in E(H)$. If $v_{s+1} \notin \{v_{i-1}, v_i, v_{i+1}\}$, then $X = \{v_1, v_i, v_{s+1}, v_t\}$ is the desired set. Thus $v_{s+1} \in \{v_{i-1}, v_i, v_{i+1}\}$. This implies that $v_s \in \{v_{i-2}, v_{i-1}, v_i\}$. Note $v_s \ne v_i$ since $v_tv_s \in E(H)$ and $d_H(v_i) = 2$. Thus $v_s \in \{v_{i-2}, v_{i-1}\}$. Since $v_i \in \{v_4, v_5, v_6\}$ and $s \ge 3$, $v_s \in \{v_3, v_4, v_5\}$. If $d_H(v) = 2$ for some $v \in \{v_{s+4}, v_{s+5}\}$, then $X = \{v_1, v_i, v, v_t\}$ is the desired set. Thus $d_H(v) \ge 3$ for each $v \in \{v_{s+4}, v_{s+5}\}$. Furthermore, neither v_{s+4} nor v_{s+5} has a right edge, otherwise, this edge nests with v_sv_t, and H contains a chorded cycle by Lemma 3.9, a contradiction. Thus both v_{s+4} and v_{s+5} have left edges. It follows that $v_{s+4}v_\ell, v_{s+5}v_{\ell'} \in E(H)$, and then $\ell < \ell' < s$, otherwise, we have nested edges and a chorded cycle by Lemma 3.9, a contradiction. Then $P[v_\ell, v_s], v_t, P^-[v_t, v_{s+4}], v_\ell$ is a cycle with chord $v_{\ell'}v_{s+5}$, a contradiction.

Suppose $v_1v_3 \notin E(H)$ and $v_tv_{t-2} \notin E(H)$. Then $v_1v_i \in E(H)$ for some $4 \le i \le t-1$ and $v_tv_j \in E(H)$ for some $2 \le j \le t-3$. Note $i \ne j+1$, otherwise, H contains a cycle with chord v_jv_{j+1}, a contradiction. By Lemma 3.11, $d_H(v_{i-1}) = 2$ and $d_H(v_{j+1}) = 2$. If $i \notin \{j+2, j+3\}$, then $X = \{v_1, v_{i-1}, v_{j+1}, v_t\}$ is the desired set. Thus $i \in \{j+2, j+3\}$. Now we claim that $d_H(v_{\ell_1}) = 2$ for some $\ell_1 \in \{3, 4\}$. If $j \in \{2, 3\}$, then $d_H(v_{j+1}) = 2$ by Lemma 3.11. Suppose $4 \le j \le t-3$. If $d_H(v_3) \ge 3$, then $v_3v_{i'} \in E(H)$ for some $i' > i$ by Lemma 3.9. Then $P[v_1, v_j], v_t, P^-[v_t, v_i], v_1$ is a cycle with chord $v_3v_{i'}$, a contradiction. Thus $d_H(v_3) = 2$. In all cases, the claim holds. By symmetry, $d_H(v_{\ell_2}) = 2$ for some $\ell_2 \in \{t-3, t-2\}$. Then $X = \{v_1, v_{\ell_1}, v_{\ell_2}, v_t\}$ is the desired set. Thus Lemma 3.14 holds. $\square$

Lemma 3.15. *Let $k \geq 2$ be an integer, and let G be a graph. Suppose G does not contain k vertex-disjoint chorded cycles. Let $\{C_1, \ldots, C_{k-1}\}$ be a minimal set of $k-1$ vertex-disjoint chorded cycles in G, $H = G - \mathscr{C}$, where $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, and $X \subseteq V(H)$ with $|X| = 4$. Suppose H contains a Hamiltonian path. Then $d_{C_i}(X) \leq 12$ for each $1 \leq i \leq k-1$.*

Proof. Suppose not, then $d_{C_i}(X) \geq 13$ for some $1 \leq i \leq k-1$. Let $X = \{x_1, x_2, x_3, x_4\}$. By Lemma 3.3, $d_{C_i}(x_j) \leq 4$ for each $1 \leq j \leq 4$. Now we consider degree sequences defined in Section 1 (Introduction) from four vertices of X to C_i. Recall that when we write (d_1, d_2, d_3, d_4), we assume $d_{C_i}(x_j) = d_j$ for each $1 \leq j \leq 4$, since it is sufficient to consider the case of equality. Without loss of generality, we may assume $d_{C_i}(x_1) \geq d_{C_i}(x_2) \geq d_{C_i}(x_3) \geq d_{C_i}(x_4)$. Then the possible degree sequences from X to C_i are $(4,4,4,1)$, $(4,4,3,2)$ or $(4,3,3,3)$. Since $d_{C_i}(x_1) = 4$, $|C_i| = 4$ by Lemma 3.3. Let $C_i = v_1, v_2, v_3, v_4, v_1$. We show the existence of two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, and then G contains k vertex-disjoint chorded cycles, a contradiction. Now we consider the following three cases based on the degree sequences.

Case 1. The sequence is $(4,4,4,1)$.

Then $d_{C_i}(x_j) = 4$ for each $1 \leq j \leq 3$ and $d_{C_i}(x_4) = 1$. Without loss of generality, we may assume $x_4 v_1 \in E(G)$. Since H is connected, there exists a path from x_4 to some other $x \in X$ not containing $X - \{x_4, x\}$. Without loss of generality, we may assume there exists a path P in H connecting x_4 and x_3. Since $d_{C_i}(x_3) = 4$, $v_1, v_2 \in N_{C_i}(x_3)$. Then $x_4, v_1, v_2, x_3, P[x_3, x_4]$ is a cycle with chord $x_3 v_1$. For each $j \in \{1,2\}$, since $d_{C_i}(x_j) = 4$, $v_3, v_4 \in N_{C_i}(x_j)$. Then x_1, v_3, x_2, v_4, x_1 is the other cycle with chord $v_3 v_4$. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, a contradiction.

Case 2. The sequence is $(4,4,3,2)$.

Then $d_{C_i}(x_1) = d_{C_i}(x_2) = 4$, $d_{C_i}(x_3) = 3$, and $d_{C_i}(x_4) = 2$. Since H is connected, there exists a path P from x_4 to some other $x \in X$ not containing $X - \{x_4, x\}$.

First suppose $x = x_3$, that is, the path P connects x_4 and x_3. Since $d_{C_i}(x_3) = 3$, without loss of generality, we may assume $v_j \in N_{C_i}(x_3)$ for each $1 \leq j \leq 3$. Assume $v_1 \in N_{C_i}(x_4)$. Then $P[x_3, x_4], v_1, v_2, x_3$ is a cycle with chord $x_3 v_1$. For each $j \in \{1,2\}$, since $d_{C_i}(x_j) = 4$, $v_3, v_4 \in N_{C_i}(x_j)$. Then x_1, v_3, x_2, v_4, x_1 is the other cycle with chord $v_3 v_4$. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, a contradiction. Hence $v_1 \notin N_{C_i}(x_4)$. Similarly, $v_3 \notin N_{C_i}(x_4)$ by symmetry. Since $d_{C_i}(x_4) = 2$, $v_2 \in N_{C_i}(x_4)$. Then $P[x_3, x_4], v_2, v_1, x_3$ is a cycle with chord $x_3 v_2$. Since $v_3, v_4 \in N_{C_i}(x_j)$ for each $j \in \{1,2\}$, x_1, v_3, x_2, v_4, x_1 is the other cycle with chord $v_3 v_4$. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, a contradiction.

Next suppose $x = x_1$ (or x_2), that is, the path P connects x_4 and x_1 (or x_2). Without loss of generality, we may assume P connects x_4 and x_1. Since $d_{C_i}(x_3) = 3$, without loss of generality, we may assume $v_j \in N_{C_i}(x_3)$ for each $1 \leq j \leq 3$. Assume $v_1 \in N_{C_i}(x_4)$. Since $d_{C_i}(x_1) = 4$, $v_1, v_4 \in N_{C_i}(x_1)$. Then $P[x_1, x_4], v_1, v_4, x_1$ is a cycle with chord $x_1 v_1$. Since $d_{C_i}(x_2) = 4$, $v_2, v_3 \in N_{C_i}(x_2)$. Then x_2, v_2, x_3, v_3, x_2 is the other cycle with chord $v_2 v_3$. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, a contradiction. Hence $v_1 \notin N_{C_i}(x_4)$. Similarly, $v_3 \notin N_{C_i}(x_4)$ by symmetry. Since $d_{C_i}(x_4) = 2$, $v_4 \in N_{C_i}(x_4)$, and since $d_{C_i}(x_1) = 4$, $v_3, v_4 \in N_{C_i}(x_1)$. Then $P[x_1, x_4], v_4, v_3, x_1$ is a cycle with chord $x_1 v_4$. Since $d_{C_i}(x_2) = 4$, $v_1, v_2 \in N_{C_i}(x_2)$. Then x_2, v_1, x_3, v_2, x_2 is the other cycle with chord $v_1 v_2$. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, a contradiction.

Case 3. The sequence is $(4,3,3,3)$.

Then $d_{C_i}(x_1) = 4$ and $d_{C_i}(x_j) = 3$ for each $2 \leq j \leq 4$. Since H contains a Hamiltonian path by the assumption, we let P be the Hamiltonian path. We may assume the order of x_1, x_2, x_3, x_4 on P is either x_1, x_2, x_3, x_4 or x_2, x_1, x_3, x_4, otherwise we consider the reverse orientation of P. Since $d_{C_i}(x_4) = 3$, the vertex x_4 is adjacent to at least two consecutive vertices on C_i. Without loss of generality, we may assume $v_1, v_2 \in N_{C_i}(x_4)$. Since $d_{C_i}(x_3) = 3$, without loss of generality, we may assume $v_1 \in N_{C_i}(x_3)$. Then $P[x_3, x_4], v_2, v_1, x_3$ is a cycle with chord x_4v_1.

Next we prove that if x_1, x_2 (resp. x_2, x_1) are in that order on P, then there exists the other chorded cycle in $\langle P[x_1, x_2] \cup \{v_3, v_4\}\rangle$ (resp. $\langle P[x_2, x_1] \cup \{v_3, v_4\}\rangle$). Suppose that x_1, x_2 are in that order on P. (If x_2, x_1 are in that order on P, then we consider the reverse orientation of $P[x_2, x_1]$.) Since $d_{C_i}(x_1) = 4$, $v_3, v_4 \in N_{C_i}(x_1)$, and since $d_{C_i}(x_2) = 3$, $v_\ell \in N_{C_i}(x_2)$ for some $\ell \in \{3, 4\}$. If $v_3 \in N_{C_i}(x_2)$, then $P[x_1, x_2], v_3, v_4, x_1$ is the other cycle with chord x_1v_3. If $v_4 \in N_{C_i}(x_2)$, then $P[x_1, x_2], v_4, v_3, x_1$ is the other cycle with chord x_1v_4. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i\rangle$, a contradiction. □

4 Proof of Theorem 1.3

Suppose G does not contain a chorded cycle.

Claim 4.1. *G is connected.*

Proof. Suppose not, then $\mathrm{comp}(G) \geq 2$. Let $G_1, G_2, \ldots, G_{\mathrm{comp}(G)}$ be the components of G. First suppose $\mathrm{comp}(G) \geq 3$. By Theorem 1.1, there exists $x_i \in V(G_i)$ for each $1 \leq i \leq 3$ such that $d_{G_i}(x_i) \leq 2$. Then $X = \{x_1, x_2, x_3\}$ is an independent set and $d_G(X) \leq 6$. This contradicts the $\sigma_3(G)$ condition. Next suppose $\mathrm{comp}(G) = 2$. Without loss of generality, we may assume $|G_1| \geq |G_2|$. Since $|G| \geq 7$, $|G_1| \geq 4$. If G_1 is complete, then G_1 contains a chorded cycle. Thus G_1 is not complete. By Theorem 1.2, there exist non-adjacent $x_0, x_1 \in V(G_1)$ such that $d_{G_1}(\{x_0, x_1\}) \leq 4$. On the other hand, by Theorem 1.1, there exists $x_2 \in V(G_2)$ such that $d_{G_2}(x_2) \leq 2$. Then $X = \{x_0, x_1, x_2\}$ is an independent set and $d_G(X) \leq 6$. This contradicts the $\sigma_3(G)$ condition. Thus Claim 4.1 holds. □

Let $P_1 = u_1, \ldots, u_s$ be a longest path in G. Note $s \geq 3$ since $|G| \geq 7$ and G is connected by Claim 4.1.

Claim 4.2. *G contains a Hamiltonian path.*

Proof. Suppose not, then P_1 is not a Hamiltonian path in G. Thus $V(G - P_1) \neq \emptyset$. Let $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) be a longest path in $G - P_1$. Without loss of generality, we may assume $d_G(v_1) \leq d_G(v_t)$. Let $X = \{u_1, u_s, v_1\}$. By Lemma 3.8 (i), (v), and (vi), X is an independent set and $d_G(X) \leq 6$. This contradicts the $\sigma_3(G)$ condition. Thus Claim 4.2 holds. □

By Claim 4.2, P_1 is a Hamiltonian path in G. Note $s = |G| \geq 7$. If $u_1u_s \in E(G)$, then $d_G(u) = 2$ for any $u \in V(G)$, otherwise a chorded cycle exists in G, a contradiction. Then $X = \{u_1, u_3, u_5\}$ is an independent set and $d_G(X) = 6$. This contradicts the $\sigma_3(G)$ condition. Thus $u_1u_s \notin E(G)$. Since P_1 is a Hamiltonian path in G, note $d_{P_1}(u) = d_G(u)$ for any $u \in V(P_1)$. We also note $d_{P_1}(u_i) \leq 2$ for each $i \in \{1, s\}$. Suppose $d_{P_1}(u_1) = 1$. By Lemma 3.12, $d_G(u_i) = 2$ for some $3 \leq i \leq 5$. Since $s \geq 7$, $X = \{u_1, u_i, u_s\}$ is an independent set and $d_G(X) \leq 6$, a contradiction. Thus $d_{P_1}(u_1) = 2$. Now suppose $u_1u_3 \in E(G)$. By Lemma 3.12, $d_G(u_i) = 2$ for some $4 \leq i \leq 6$. If $s \geq 8$, then $X = \{u_1, u_i, u_s\}$ is an independent set and $d_G(X) \leq 6$, a contradiction. Thus $s = 7$. Then $d_G(u_j) \geq 3$ for each $j \in \{4, 5\}$, otherwise we get a contradiction, since $X = \{u_1, u_j, u_7\}$ for some $j \in \{4,5\}$ would be an independent set with $d_G(X) \leq 6$. Thus $d_G(u_6) = 2$ by Lemma 3.12. Since u_4 does not have a left edge by Lemmas 3.9 and 3.10, u_4 must have a right edge. Since $d_G(u_6) = 2$, $u_4u_7 \in E(G)$. By Lemma 3.11, $d_G(u_5) = 2$, a contradiction. Thus $u_1u_3 \notin E(G)$, that is, $u_1u_i \in E(G)$ for some $4 \leq i \leq s-1$. By Lemma 3.11, $d_G(u_{i-1}) = 2$. Then $X = \{u_1, u_{i-1}, u_s\}$ is an independent set and $d_G(X) \leq 6$, a contradiction. This completes the proof of Theorem 1.3. $\square$

5 Proof of Theorem 1.4

By Theorem 1.3, we may assume $k \geq 2$. Suppose Theorem 1.4 does not hold. Let G be an edge-maximal counter-example. If G is complete, then G contains k vertex-disjoint chorded cycles. Thus we may assume G is not complete. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Since G' is not a counter-example by the edge-maximality of G, G' contains k vertex-disjoint chorded cycles $C_1, \dots, C_k$. Without loss of generality, we may assume $xy \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ vertex-disjoint chorded cycles. Over all sets of $k-1$ vertex-disjoint chorded cycles in G, choose $C_1, \dots, C_{k-1}$ with $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, $H = G - \mathscr{C}$, and with P_1 be a longest path in H, such that

(A1) $|\mathscr{C}|$ is as small as possible,

(A2) subject to (A1), $\mathrm{comp}(H)$ is as small as possible, and,

(A3) subject to (A1) and (A2), $|P_1|$ is as large as possible.

We may assume H does not contain a chorded cycle, otherwise G contains k vertex-disjoint chorded cycles, a contradiction.

Claim 5.1. *H has order at least* 13.

Proof. Suppose $|H| \leq 12$. First suppose $|C_i| \leq 8$ for each $1 \leq i \leq k-1$. Since by assumption, $|G| \geq 8k+5$, it follows that $|H| \geq (8k+5) - 8(k-1) = 13$, a contradiction. Thus $|C_i| \geq 9$ for some $1 \leq i \leq k-1$. Without loss of generality, we may assume C_1 is a longest cycle in $\mathscr{C}$. Then $|C_1| \geq 9$. By Lemma 3.2, C_1 has at most two chords, and if C_1 has two chords, then these chords must be crossing. For integers t and r, let $|C_1| = 3t + r$, where $t \geq 3$ and $0 \leq r \leq 2$.

Subclaim 5.1.1. *The cycle C_1 contains t (≥ 3) vertex-disjoint sets $X_1, \dots, X_t$ of three independent vertices each in G such that $d_{C_1}(\cup_{i=1}^{t} X_i) \leq 6t + 4$.*

Proof. For any $3t$ vertices of C_1, their degree sum in C_1 is at most $3t \times 2 + 4 = 6t + 4$, since C_1 has at most two chords. Thus it only remains to show that C_1 contains t vertex-disjoint sets of three independent vertices each. Start anywhere on C_1 and label the first $3t$ vertices of C_1 with labels 1 through t in order, starting over again with 1 after using label t. If $r \geq 1$, label the remaining r vertices of C_1 with the labels $t+1, \dots, t+r$. (See Fig. 2.) The labeling above yields t vertex-disjoint sets of three vertices each, where all the vertices labeled with 1 are one set, all the vertices labeled with 2 are another set, and so on. Given this labeling, since $t \geq 3$, any vertex x in C_1 has a different label than x^- and x^+. Let C_0 be the cycle obtained from C_1 by removing all chords. Then the vertices in each of the t sets are independent in C_0. Thus the only way vertices in the same set are not independent in C_1 is if the endpoints of a chord of C_1 were given the same label. Note any vertex labeled i is distance at least 3 in C_0 from any other vertex labeled i. Thus even if we exchange the label of x in C_0 for the one of x^- (or x^+), the vertices in each of the resulting t sets are still independent in C_0.

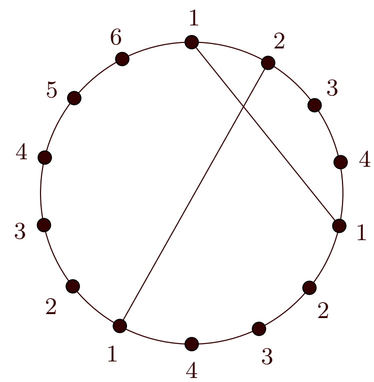

Fig. 2. An example when $t = 4$ and $r = 2$.

Case 1. No chord of C_1 has both endpoints with the same label.

Then there exist t vertex-disjoint sets of three independent vertices each in C_1.

Case 2. Exactly one chord of C_1 has both endpoints with the same label.

Recall that C_1 has at most two chords, and if C_1 has two chords, then these chords must be crossing. Since $|C_1| \geq 9$, even if C_1 has two chords, each chord has an endpoint x such that there exists some vertex $x' \in \{x^-, x^+\}$ which is equal to no endpoint of the other chord. Choose such an endpoint x of the chord whose endpoints were assigned the same label, and exchange the label of x for the one of x'. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of three independent vertices each in C_1.

Case 3. Two chords of C_1 each have both endpoints with the same label.

Then the two chords are crossing. Since endpoints of a chord have the same label in this case, recall that these endpoints have distance at least 3. Suppose there is an endpoint x of one chord of C_1 which is adjacent to an endpoint $y\,(= x^+)$ of the other chord on C_1. (See Fig. 3 (a).) Now we exchange the label of x for the one of y. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of three independent vertices each in C_1.

Suppose no endpoint of one chord of C_1 is adjacent to an endpoint of the other chord on C_1. (See Fig. 3 (b).) Let x_1x_2, y_1y_2 be the two distinct chords of C_1. Since the two chords are crossing, without loss of generality, we may assume x_1, y_1, x_2, y_2 are in that order on C_1. Now we exchange the labels of x_1 and x_1^+, and next the ones of y_2 and y_2^-. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of three independent vertices each in C_1. □

Since $|C_1| \geq 9$, $d_{C_1}(v) \leq 2$ for any $v \in V(H)$ by (A1) and Lemma 3.3. Thus, since $|H| \leq 12$ by our assumption, it follows that $|E(H, C_1)| \leq 24$. Let $X_1, \ldots, X_t$ be as in Subclaim 5.1.1, and let $\mathscr{X} = X_1 \cup \cdots \cup X_t$. By the $\sigma_3(G)$ condition, $d_G(\mathscr{X}) \geq t(9k-2)$. Suppose $k = 2$. Then $\mathscr{C}$ has only one cycle C_1. Since $k = 2$ and $t \geq 3$, $|E(C_1, H)| \geq d_H(\mathscr{X}) \geq t(9k-2) - (6t+4) = 10t - 4 \geq 26$, a contradiction.

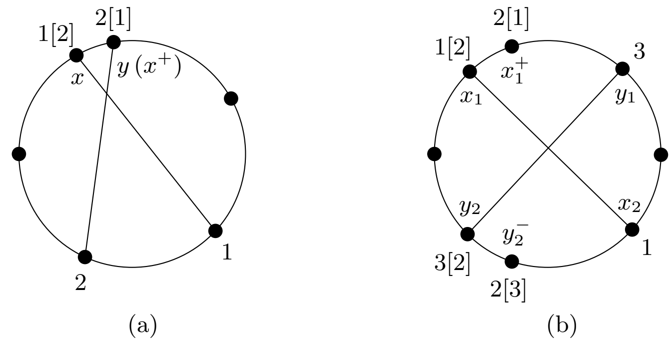

Fig. 3. Examples: (a) – the labels of x and y are 1 and 2, (b) – the labels of x_1 and y_2 are 1 and 3. ($[i]$ means i is a new label for a vertex after the exchange.)

Now suppose $k \geq 3$. Then we have

$$\begin{aligned} |E(\mathscr{X}, \mathscr{C} - C_1)| &= d_G(\mathscr{X}) - d_{C_1}(\mathscr{X}) - d_H(\mathscr{X}) \\ &\geq t(9k-2) - (6t+4) - 24 \\ &= 9kt - 8t - 28, \end{aligned}$$

and since $t \geq 3$,

$$\begin{aligned} 9kt - 8t - 28 &= 9t(k-1) + t - 28 \geq 9t(k-1) - 25 \\ &> 9t(k-1) - 9t \\ &= 9t(k-2). \end{aligned}$$

Thus $|E(\mathscr{X}, C')| > 9t$ for some C' in $\mathscr{C} - C_1$, since $\mathscr{C} - C_1$ contains $k-2$ vertex-disjoint chorded cycles. Let $h = \max\{d_{C'}(v) \,|\, v \in \mathscr{X}\}$. Let v^* be a vertex of $\mathscr{X}$ such that $d_{C'}(v^*) = h$. If $h \leq 3$, then $|E(\mathscr{X}, C')| \leq 3 \times 3t = 9t$, a contradiction. Thus $h \geq 4$. By the maximality of C_1, $|C'| \leq |C_1| = 3t + r$. It follows that $h = d_{C'}(v^*) \leq |C'| \leq 3t + r$. Recall $t \geq 3$ and $0 \leq r \leq 2$. Then

$$\begin{aligned} |E(\mathscr{X} - \{v^*\}, C')| &\geq (9t+1) - d_{C'}(v^*) \geq (9t+1) - (3t+r) \\ &= 6t - r + 1 \geq 17. \end{aligned} \tag{1}$$

Since $h = d_{C'}(v^*) \geq 4$, let v_1, v_2, v_3, v_4 be neighbors of v^* in that order on C'. Note v_1, v_2, v_3, v_4 partition C' into four intervals $C'[v_i, v_{i+1})$ for all $1 \leq i \leq 4$, where $v_5 = v_1$. By (1), there exist at least 17 edges from $C_1 - v^*$ to C'. Thus $C'[v_i, v_{i+1})$ for some $1 \leq i \leq 4$ contains at least five of these edges. Without loss of generality, we may assume $i = 4$, that is, $C'[v_4, v_1)$. Then by Lemma 3.6, $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$ contains a chorded cycle not containing at least one vertex of $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$. Note $v^*, C'[v_1, v_3], v^*$ is a cycle with chord v^*v_2, and it uses no vertices from $C'[v_4, v_1)$. Thus we have two shorter vertex-disjoint chorded cycles in $\langle C_1 \cup C' \rangle$, contradicting (A1). Hence Claim 5.1 holds. □

Claim 5.2. *H is connected.*

Proof. Suppose not. First we prove the following subclaim.

Subclaim 5.2.1. *Let X be an independent set of three vertices in H such that $d_H(X) \leq 6$. Then there exists some C in $\mathscr{C}$ such that the degree sequences from the vertices of X to C are $(4, 4, 2)$ or $(4, 3, 3)$. Furthermore, then $|C| = 4$.*

Proof. By the $\sigma_3(G)$ condition, $d_{\mathscr{C}}(X) \geq (9k-2)-6 = 9k-8 > 9(k-1)$. Thus there exists some C in $\mathscr{C}$ such that $d_C(X) \geq 10$. By Lemma 3.3, $d_C(x) \leq 4$ for any $x \in X$. It follows that the degree sequences from three vertices of X to C are $(4,4,2)$ or $(4,3,3)$. Then by Lemma 3.3, $|C| = 4$. □

Now we consider the following two cases based on comp(H).

Case 1. Suppose comp$(H) \geq 3$.

Let H_1, H_2, H_3 be three distinct components of H. For each $1 \leq i \leq 3$, let x_i be an endpoint of a longest path in H_i. Since H does not contain a chorded cycle, $d_{H_i}(x_i) \leq 2$ for each $1 \leq i \leq 3$. Note x_i for each $1 \leq i \leq 3$ is not a cutvertex of H_i, since x_i is an endpoint of a longest path. Then $X = \{x_1, x_2, x_3\}$ is an independent set and $d_H(X) \leq 6$. By Subclaim 5.2.1, the degree sequences from three vertices of X to some C in $\mathscr{C}$ are $(4,4,2)$ or $(4,3,3)$, and $|C| = 4$. Without loss of generality, we may assume $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3)$. Let $C = v_1, v_2, v_3, v_4, v_1$. By the degree sequences, x_2 and x_3 have a common neighbor in C. Without loss of generality, we may assume $v_4 \in N_C(x_2) \cap N_C(x_3)$. Then $\langle H_2 \cup H_3 \cup v_4 \rangle$ is connected. Since $d_C(x_1) = 4$, $v_i \in N_C(x_1)$ for each $1 \leq i \leq 3$. Then $C' = x_1, v_1, v_2, v_3, x_1$ is a 4-cycle with chord x_1v_2. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Since $H_1 - x_1$ is connected, comp$(H') \leq$ comp$(H) - 1$. This contradicts (A2).

Case 2. Suppose comp$(H) = 2$.

Let H_1, H_2 be two distinct components of H. Recall P_1 is a longest path in H. Without loss of generality, we may assume H_1 contains P_1. Let $P_1 = u_1, \ldots, u_s$. Then $|H_1| \geq |P_1| = s$. By Claim 5.1, $|H| \geq 13$. Thus $|H_i| \geq 7$ for some $i \in \{1,2\}$. Since H_i is connected, there exists a path of order at least 3 in H_i. Thus $s \geq 3$, since P_1 is a longest path in H. Also, we let $P_2 = v_1, \ldots, v_t$ $(t \geq 1)$ be a longest path in H_2. Since P_i for each $i \in \{1,2\}$ is a longest path in H_i, $d_{H_1}(u_j) = d_{P_1}(u_j) \leq 2$ for each $j \in \{1, s\}$ and $d_{H_2}(v_\ell) = d_{P_2}(v_\ell) \leq 2$ for each $\ell \in \{1, t\}$. Let $X = \{u_1, u_s, v_1\}$. Then $d_H(X) \leq 6$.

First suppose $u_1u_s \notin E(H_1)$. Then X is an independent set. By Subclaim 5.2.1, the degree sequences from three vertices of X to some C in $\mathscr{C}$ are $(4,4,2)$ or $(4,3,3)$, and $|C| = 4$. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_s)$. Let $C = x_1, x_2, x_3, x_4, x_1$.

Suppose the degree sequence is $(4,4,2)$. By the degree sequence, since u_s and v_1 have a common neighbor in C, without loss of generality, we may assume $x_4 \in N_C(u_s) \cap N_C(v_1)$. Note u_1 is not a cutvertex of H_1, since u_1 is an endpoint of a longest path. Thus $H_1 - u_1$ is connected, and $\langle (H_1 - u_1) \cup H_2 \cup x_4 \rangle$ is also connected. Since $d_C(u_1) = 4$, $x_j \in N_C(u_1)$ for each $1 \leq j \leq 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord u_1x_2. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then comp$(H') \leq$ comp$(H) - 1 = 2 - 1 = 1$. This contradicts (A2).

Suppose the degree sequence is $(4,3,3)$. If $d_C(u_1) = 4$ and $d_C(u_s) = d_C(v_1) = 3$, then we get a contradiction similar to the case where $(4,4,2)$. Thus $d_C(u_1) = d_C(u_s) = 3$ and $d_C(v_1) = 4$. Without loss of generality, we may assume $x_1 \in N_C(u_1)$. Since $d_C(v_1) = 4$, $x_i \in N_C(v_1)$ for each $2 \leq i \leq 4$. Then $C' = v_1, x_2, x_3, x_4, v_1$ is a 4-cycle with chord v_1x_3. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Assume $|H_2| = 1$. Then comp$(H') = 1$, a contradiction. Thus $|H_2| \geq 2$. Note $H_2 - v_1$ is connected. By (A2), comp$(H') =$ comp(H). Then $x_1, P_1[u_1, u_s]$ is a longer path than P_1 in H'. This contradicts (A3).

Next suppose $u_1u_s \in E(H_1)$. Since H_1 is connected and P_1 is a longest path, $C_1 = P_1[u_1, u_s], u_1$ is a Hamiltonian cycle. Assume $s \geq 4$. Let $X = \{u_1, u_3, v_1\}$. Since H_1 does not contain a chorded cycle, $u_1u_3 \notin E(H_1)$ and $d_{H_1}(u_i) = 2$ for each $i \in \{1,3\}$. Thus X is an independent set and $d_H(X) \leq 6$. Now, letting u_3 play the role of u_s in the case where $u_1u_s \notin E(H_1)$, we get a contradiction, similarly. Hence, $s = 3$. Since C_1 is a Hamiltonian cycle in H_1, $|H_1| = 3$. Note $|H_2| \geq 10$ by Claim 5.1, and H_2 does not contain a longer path than P_1. Thus $H_2 = K_{1,p}$, where $p \geq 9$. Let $V(K_{1,p}) = \{a_1\} \cup \{b_1, b_2, \ldots, b_p\}$, and let $X = \{b_1, b_2, b_3\}$. Since $d_{H_2}(b_i) = 1$ for each $1 \leq i \leq 3$, $d_{H_2}(X) = 3$. Also, X is an independent set. By Subclaim 5.2.1, the degree sequences from three vertices of X to some C in $\mathscr{C}$ are $(4,4,2)$ or $(4,3,3)$, and $|C| = 4$. Let $C = x_1, x_2, x_3, x_4, x_1$. Without loss of generality, we may assume $d_C(b_1) \geq d_C(b_2) \geq d_C(b_3)$. Since $d_C(b_2) \geq 3$ by the degree sequences, without loss of generality, we may assume $x_i \in N_C(b_2)$ for each $2 \leq i \leq 4$. Then $C' = b_2, x_2, x_3, x_4, b_2$ is a 4-cycle with chord b_2x_3. Since $d_C(b_1) = 4$, $x_1 \in N_C(b_1)$. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note $H_2 - b_2$ is connected. By (A2), $\mathrm{comp}(H') = \mathrm{comp}(H)$. Then x_1, b_1, a_1, b_3 is a longer path than P_1. This contradicts (A3). □

Claim 5.3. *H contains a Hamiltonian path.*

Proof. Suppose not, then by Claims 5.1 and 5.2, $|H| \geq 13$ and H is connected. Recall P_1 is a longest path in H. Then $V(H - P_1) \neq \emptyset$. Let $P_1 = u_1, \ldots, u_s$ $(s \geq 3)$, and let $P_2 = v_1, \ldots, v_t$ $(t \geq 1)$ be a longest path in $H - P_1$. Without loss of generality, we may assume $d_H(v_1) \leq d_H(v_t)$. Let $X = \{u_1, u_s, v_1\}$. Then by Lemma 3.8 (i), (v), and (vi), X is an independent set and $d_H(X) \leq 6$. Noting $\sigma_3(G) \geq 9k - 2$ and Lemma 3.3, as in Subclaim 5.2.1 in the proof of Theorem 1.4, there exists some C in $\mathscr{C}$ such that the degree sequences from three vertices of X to C are $(4,4,2)$ or $(4,3,3)$, and $|C| = 4$. Let $C = x_1, x_2, x_3, x_4, x_1$ be a 4-cycle with chord x_1x_3. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_s)$.

Suppose $d_C(u_1) = 4$. By the degree sequence, u_s and v_1 have a common neighbor in C, say x_ℓ for some $1 \leq \ell \leq 4$. Note u_1 is not a cutvertex of H, since u_1 is an endpoint of a longest path. Thus $H - u_1$ is connected. Since $d_C(u_1) = 4$, $\langle u_1 \cup (C - x_\ell) \rangle$ contains a chorded 4-cycle, say C'. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note H' is connected. Then $P_1[u_2, u_s], x_\ell, P_2[v_1, v_t]$ is a longer path than P_1 in H'. This contradicts (A3). Thus $d_C(u_1) \leq 3$, that is, $d_C(u_1) = d_C(u_s) = 3$ and $d_C(v_1) = 4$. Since $d_C(u_1) = 3$, $x_1, x_3 \in N_C(u_1)$ or $x_2, x_4 \in N_C(u_1)$.

First suppose $x_1, x_3 \in N_C(u_1)$. Recall x_1x_3 is a chord of C. Since $d_C(u_s) = 3$, without loss of generality, we may assume $x_4 \in N_C(u_s)$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord x_1x_3. Since $d_C(v_1) = 4$, $x_4 \in N_C(v_1)$. Note $H - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $P_1[u_2, u_s], x_4, P_2[v_1, v_t]$ is a longer path than P_1 in H'. This contradicts (A3).

Next suppose $x_2, x_4 \in N_C(u_1)$. Since $d_C(u_1) = 3$, without loss of generality, we may assume $x_3 \in N_C(u_1)$. Since $d_C(u_s) = 3$, without loss of generality, we may assume $x_4 \in N_C(u_s)$. Then $C' = u_1, x_2, x_1, x_3, u_1$ is a 4-cycle with chord x_2x_3. Since $d_C(v_1) = 4$, $x_4 \in N_C(v_1)$. Note $H - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $P_1[u_2, u_s], x_4, P_2[v_1, v_t]$ is a longer path than P_1 in H'. This contradicts (A3). □

By Claims 5.1, 5.3, and Lemma 3.14, there exists an independent set X of four vertices in H such that $d_H(X) \leq 8$. Let $X = \{x_1, x_2, x_3, x_4\}$, and let $X_1 = \{x_1, x_2, x_3\}, X_2 = \{x_1, x_2, x_4\}, X_3 = \{x_1, x_3, x_4\}$, and $X_4 = \{x_2, x_3, x_4\}$. Then $3|X| = \sum_{i=1}^{4} |X_i|$. Note X_i for each $1 \leq i \leq 4$ is an independent set. By the $\sigma_3(G)$ condition,

$$3 \cdot d_G(X) = \sum_{i=1}^{4} d_G(X_i) \geq 4\sigma_3(G) \geq 4(9k-2) = 36k - 8.$$

On the other hand, by Claim 5.3 and Lemma 3.15,

$$3 \cdot d_G(X) = 3(d_{\mathscr{C}}(X) + d_H(X)) \leq 3(12(k-1) + 8) = 36k - 12,$$

a contradiction. This completes the proof of Theorem 1.4. □

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References

[1] S. Chiba, S. Fujita, Y. Gao, G. Li, On a sharp degree sum condition for disjoint chorded cycles in graphs, *Graphs Combin.* 26 (2010), 173–186.

[2] S. Chiba, T. Yamashita, Degree conditions for the existence of vertex-disjoint cycles and paths: A survey, *Graphs Combin.* 34 (2018), 1–83.

[3] K. Corrádi, A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* 14 (1963), 423–439.

[4] H. Enomoto, On the existence of disjoint cycles in a graph, *Combinatorica* 18, No. 4 (1998), 487–492.

[5] D. Finkel, On the number of independent chorded cycles in a graph, *Discrete Math.* 308, Issue 22 (2008), 5265–5268.

[6] S. Fujita, H. Matsumura, M. Tsugaki, T. Yamashita, Degree sum conditions and vertex-disjoint cycles in a graph, *Australas. J. Combin.* 35 (2006), 237–251.

[7] Y. Gao, X. Lin, H. Wang, Vertex-disjoint double chorded cycles in bipartite graphs, *Discrete Math.* 342, Issue 9 (2019), 2482–2492.

[8] R.J. Gould, Graph Theory, Dover Pub. Inc. Mineola, N.Y. 2012.

[9] R.J. Gould, K. Hirohata, A. Keller, On vertex-disjoint cycles and degree sum conditions, *Discrete Math.* 341, Issue 1 (2018), 203–212.

[10] T. Molla, M. Santana, E. Yeager, Disjoint cycles and chorded cycles in a graph with given minimum degree, *Discrete Math.* 343, Issue 6 (2020), 111837.

[11] H. Wang, On the maximum number of independent cycles in a graph, *Discrete Math.* 205, Issues 1–3 (1999), 183–190.