Example 1.5.6. *How many different foursomes can be formed from among a group of 12 golfers?*

Solution: From the wording of the question, a foursome is a set of four golfers. Thus, this is again a combination problem. That is, the number of foursomes is just

$$\binom{12}{4} = \frac{12!}{4! \times 8!} = \frac{11880}{24} = 495. \quad \square$$

Example 1.5.7. *How many possible first, second, and third place orders of finish are there in a 10-horse race?*

Solution: Clearly, in this problem, order matters. So we may apply the Multiplication Principle. Thus, there are

$$10 \times 9 \times 8 = 720$$

possible ways for the top three finishers to occur. $\square$

Permutations and combinations are major counting techniques. We will have occasion to use them many times, especially in the sections on casino games. The reader should become very familiar with each method and comfortable with when to apply them.

Exercises:

1.5.1 List all the permutations of $\{1, 2\}$.

1.5.2 List all the permutations of the elements in the set $\{8\spadesuit, 6\clubsuit, A\heartsuit\}$.

1.5.3 How many permutations of $\{1, 2, 3\}$ start with a 1?

1.5.4 How many permutations of $\{1, 2, 3\}$ end with a 2?

1.5.5 How many permutations are possible with a standard deck of 52 cards?

1.5.6 There are 5 marbles in a bowl, all distinct. In how many ways can you choose a set of two marbles? A set of four marbles?

1.5.7 In how many ways can a player arrange a 5-card hand?

1.5.8 In how many ways can a poker player arrange a three-card hand?

1.5.9 How many three-card poker hands are possible?

1.5.10 In the game of pinochle, each of 4 players are dealt 12 cards from a 48-card deck. How many possible pinochle hands are there?

1.5.11 How many ways are
right?

1.5.12 You and eight friend
one car available and i
you choose four people

1. there are no restri
2. you must go as it
3. you and your frier

1.5.13 How many ways are
in four consecutive sea
be done if you have tw

1.5.14 How many ways are
in five consecutive seat

1.5.15 Considering the last
two seats in the next r

1.5.16 If three couples go to
one row, how many wa

1. couples must be se
2. the men must be s
3. couples must not b

1.5.17 Your basketball team
two centers. If you are
center, how many starti

1.5.18 As in the previous pr
lineup of three guards,
starting lineups are pos
forwards?

1.5.19 If you hold three clu
and you keep cards of t
to arrange your hand so

1. the suits from left t
2. if the order of the s
3. if the suits are red,

1.5.20 You are preparing a b
exactly 10 players and e
the batting order if:

ꞌsomes can be formed from among

, a foursome is a set of four golfers. That is, the number of foursomes

30
— = 495. □

second, and third place orders of

ıtters. So we may apply the Mul-

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ccur. □

or counting techniques. We will ecially in the sections on casino liar with each method and com-

ts in the set {8♠, 6♣, A♡}.

ırt with a 1?

d with a 2?

th a standard deck of 52 cards?

ıct. In how many ways can you ır marbles?

a 5-card hand?

rrange a three-card hand?

ossible?

yers are dealt 12 cards from a e hands are there?

1.5.11 How many ways are there to arrange a pinochle hand from left to right?

1.5.12 You and eight friends wish to go to a hockey game. But there is only one car available and it seats just four people. In how many ways can you choose four people to go if:

1. there are no restrictions on who is choosen?
2. you must go as it is your car?
3. you and your friend must be in the group that goes?

1.5.13 How many ways are there for you and three firends to seat yourseves in four consecutive seats at a hockey game? In how may ways can this be done if you have two seats in two different rows?

1.5.14 How many ways are there for you and four friends to seat yourselves in five consecutive seats of one row at a basketball game?

1.5.15 Considering the last problem, what if three seats were in one row and two seats in the next row?

1.5.16 If three couples go to the Superbowl and have six consecutive seats in one row, how many ways are there to arrange the seating if:

1. couples must be seated together?
2. the men must be seated together?
3. couples must not be seated together?

1.5.17 Your basketball team has twelve players: five guards, five forwards and two centers. If you are going to start two guards, two forwards, and a center, how many starting lineups are possible?

1.5.18 As in the previous problem, suppose you are going to play a "small" lineup of three guards, one forward, and one center. Now how many starting lineups are possible? What if you start three guards and two forwards?

1.5.19 If you hold three clubs, four spades, two hearts, and four diamonds, and you keep cards of the same suit together, how many ways are there to arrange your hand so that:

1. the suits from left to right are clubs, diamonds, hearts, and spades?
2. if the order of the suits does not matter?
3. if the suits are red, black, red, black (left to right)?

1.5.20 You are preparing a batting order for a softball game. Your team has exactly 10 players and each will bat. In how many ways can you create the batting order if:

1. there are no restrictions?
2. you will bat first?
3. you will bat first and Bill will bat last?

1.5.21 Your team has 15 players and you must choose a team captain and a team representative to the league. In how many ways can this be done if:

1. there are no restrictions on either position?
2. the two players selected must be different?

1.6 Let's Play for Money!

Those words should always cause you to be suspicious. Anyone who wants to play a game for money also wants to make a profit at the game. In order not to be at a (potentially serious) disadvantage, we need to make some decisions as to how "fair" it is to play the game. Remember the warning from Cardano! In this section we develop one fundamental way of deciding this question.

Let us consider a simple example. How much would you be willing to pay to play the following game?

A single coin is tossed and we count how many tosses it takes before the first head appears. This might take one try and it might take many tries. Here we limit you to three tries. Based on the rules, the sample space for this game is

$$S = \{H, TH, TTH, TTT\}.$$

That is, the first three entries of S are the cases when a head is tossed and the game ends. The last entry corresponds to no head being tossed. We use the payments shown in Table 1.6.

To answer our question of how much you might be willing to pay in order to play this game, we want to consider the average amount you would win. If we know the average amount we will win upon playing this game, we should be willing to spend up to that amount to play.

To determine the average amount we win at this game we need to determine the probabilities of the various outcomes. These are given in Table 1.6 (note the probabilities sum to 1 as they should!).

We can see that based on these probabilities and the corresponding payments, our expected average winnings would be:

$$(1/2) \times \$1 + (1/4) \times \$2 + (1/8) \times \$4 + (1/8) \times \$0 = \$1.50.$$

That is, an expected value is a probability weighted average of the values taken by the random variable. If all $p_i = 1/k$, then we really have an average of the values of the random variable.

Thus, from our example above, the expected value of the coin flip game is \$1.50. Let X be the payoffs for each event. If we pay \$1.50 to play, our expected value computation for playing the game then becomes:

$$(1/2) \times \$1 + (1/4) \times \$2 + (1/8) \times \$4 + (1/8) \times \$0 - \$1.50 = \$0.00.$$

For experiments using payouts of money we use an expected value of zero as an indicator of a *fair game.* There is no clear advantage to either side.

Expected values can be computed for many different experiments and do not necessarily involve money. Remember, the expected value is a weighted average, and we may average many different sets of data.

Example 1.6.1. *What is the expected value of the roll of a single die?*

Solution: Here the expected value computation becomes straightforward since the (uniform) probability of each outcome is 1/6, our random variable X is the value rolled, and thus we have:

$$\begin{aligned} EV(X) &= (1/6)[1+2+3+4+5+6] \\ &= (1/6)(21) = 3.5. \quad \square \end{aligned}$$

Thus, the expected roll of a single die is 3.5. But we cannot actually roll 3.5. This example is important as it shows that the expected value is really an average and not necessarily even a possible outcome.

Exercises:

1.6.1 Given the following random variable $X(r) = r + 1$ if $r = 1, 2, 3$ and $X(r) = r - 1$ if $r = 4, 5, 6$; find the expected value of $X(r)$ if r is obtained by rolling one fair die.

1.6.2 Find the expected value for the game of Exercise 1.3.14.

1.6.3 We flip a fair coin and associated with the resulting flip f is the random variable $X(f) = 2$ if f is a head and $X(f) = -1$ if f is a tail. What is the expected value of X?

1.6.4 Find the expected value for the sum when we roll a pair of dice.

1.6.5 You have a deck composed of cards: two are 3s, one is a 4, two are 5s. You randomly select a card from your deck. What is the expected value (face value) of this selection?

1

1.6.6 There are seven marble
are blue. If we define t
color as $X(red) = 2$, X
expected value of a ran

1.6.7 Given the random varia
5, $X(2) = 7$ and $X(3) =$
randomly and each valu

1.6.8 For the random variabl
the expected value if P
$3) = 4/8$?

1.6.9 Below are the probabili
n games in the first half
USC will win in the firs

n	0
P(n)	.001 .0

1.6.10 You play a game whe
if you win the game and
of playing this game?

1.6.11 In the game from th
would win or lose \$1. N
game?

1.6.12 Odds makers try to p
much (the *spread*). If th
score would produce a t
bet if you can predict t
your expected value for

1.6.13 If the random variable
face value of that card, e
X(King) = 10. What is

1.6.14* You play a game wit
until you either win, or
expected number of time
sample space and the as

1.6.6 There are seven marbles in a jar. Two are red, two are white, and three are blue. If we define the random variable $X(m)$ based on the marble color as $X(red) = 2$, $X(white) = 4$, and $X(blue) = 3$, then what is the expected value of a randomly selected marble?

1.6.7 Given the random variable X defined for values $m = 1, 2, 3$ with $X(1) = 5$, $X(2) = 7$ and $X(3) = 2$, what is the expected value of X if m is chosen randomly and each value of m is equally likely?

1.6.8 For the random variable $X(m)$ shown in the previous problem, what is the expected value if $P(m = 1) = 1/8$, $P(m = 2) = 3/8$, and $P(m = 3) = 4/8$?

1.6.9 Below are the probabilities for the USC football team winning a total of n games in the first half season. What is the expected number of games USC will win in the first half season?

n	0	1	2	3	4	5	6
P(n)	.001	.010	.060	.185	.304	.332	.108

1.6.10 You play a game where the probability of winning is .45. You win \$3 if you win the game and lose \$2 otherwise. What is the expected value of playing this game?

1.6.11 In the game from the previous problem, suppose instead that you would win or lose \$1. Now what is the expected value of playing this game?

1.6.12 Odds makers try to predict which football team will win and by how much (the *spread*). If they are correct, adding the spread to the loser's score would produce a tie. Suppose you can win \$6 for every dollar you bet if you can predict the winner of three consecutive games. What is your expected value for this bet?

1.6.13 If the random variable X assigns to each card of a standard deck the face value of that card, except $X(\text{Ace}) = 1$ and $X(\text{Jack}) = X(\text{Queen}) = X(\text{King}) = 10$. What is the expected value of X?

1.6.14* You play a game with probability p that you will win. You will play until you either win, or have lost three consecutive times. What is the expected number of times you will play this game? (Hint: Think of the sample space and the associated probabilities.)

1.6.15* There are five marbles in a jar. Three of the marbles are red and two are blue. What is the expected number of times you must randomly select a marble in order to select a blue marble, assuming that you do not replace selected marbles? What if you do replace the selected marbles?

1.7 Is That Fair?

In this section we wish to consider some other examples of the use of probabilities, expected values and in general, ideas about fairness in games. Since we have not given fairness a mathematical definition, we have many possible ways to interpret what we might mean. Our examples are intended to show there are many ways to determine fairness and many subtle ways to tip the scales in your own favor.

Our first application is in the idea of fair division of a prize. We consider the problem mentioned earlier that intrigued Pascal and Fermat, called **The Problem of Points.** We demonstrate the idea behind this problem with an example.

Example 1.7.1. *Suppose Jane and Tom are playing a game that requires the winner to reach a total of 5 points. Jane is leading* 4 *points to* 2 *when the game is forced to halt. How shall we fairly divide the prize money for this unfinished game?*

Solution: Our proposed solution is to divide the prize money based on the probability of winning for each player. Since the players need to reach 5 points to win, a total of 9 points are possible (5 for one and 4 for the other). We shall project the possible play forward to this total of 9 points, assuming each outcome is equally likely, to see the chances for each player. Can you determine how many possible sequences we must consider?

Now, of the eight possible ending scenarios to the game (see Table 1.7), Jane wins in seven of them. It would seem fair to give Jane 7/8 of the prize money and Tom just 1/8. □

The general Problem of Points asked for a solution to this type of problem, no matter what the partial score or the final winning score. The process above lends itself to a solution of this general problem. However, we shall not go through the details for the solution to the general problem. That we should just accept this process, which checks all the possible sequences of play until we reach the maximum number of games possible, is a reasonable solution.

Next let's take a look at another expected value problem often called the *St. Petersburg Paradox.* As you might expect from the name, this question raises a big issue.

Example 1.7.2. The St. P
peatedly until a head comes up
\$2. If it first comes up heads o
that if the first head appears
What is the expected value of

Solution: Note that this is a
considered in the previous sec
1/2. In general, the probability
the expected value of this gam

$$EV(X) =$$

Therefore, arguing as we did b
an INFINITE number of dolla
amount. To recover your inves
did not appear until 2^n was a
the head not to appear for an

Thus, the moral of the sto
fairness indicator, but only wh
often enough to allow the play
is only going to be played once
guide. A game requiring an i
impossible. The point we nee
(and any probability argume
play where there is a real pos
must be repeated over and o
significance, and the number

Now let's take a different

TABLE 1.5: The

Point 7
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