Neighborhood Unions and Independent Cycles

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Abstract
We prove that if $G$ is a simple graph of order $n \geq 3k$ such that $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x$ and $y$, then $G$ contains $k$ vertex independent cycles.

1 Introduction

(Notation will go here. FYI I use $N(x_1, x_2, ..., x_n)$ to mean $N(x_1) \cup N(x_2) \cup ...N(x_n)$.)

In 1963 Corradi and Hajnal in [1] produced the following result which proved a conjecture of Erdos:

**Theorem 1** If $G$ is a graph of order $n \geq 3k$, $k \geq 1$, with $\delta(G) \geq 2k$, then $G$ contains $k$ independent cycles.

In 1989, Justesen in [2] generalized this result to degree sums of nonadjacent pairs and in 1999 Justesen’s result was improved by Wang in [4] with the following sharp result:

**Theorem 2** If $G$ is a graph of order $n \geq 3k$ such that $\text{deg}(u) + \text{deg}(v) \geq 4k - 1$ for all pairs $u, v$ of nonadjacent vertices, then $G$ contains $k$ independent cycles.

A summary of results on independent cycles in graphs can be found in [3].

In this paper, we look at neighborhood unions that imply the existence of $k$ independent cycles. Specifically we prove the following result:
Theorem 3 If $G$ is a graph of order $n \geq 3k$ such that $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x$ and $y$, then $G$ contains $k$ vertex independent cycles.

(I don’t know if this is useful or not but...) The result is sharp in the sense that for any $k$ the graph $G = K_{3k-1} \cup K_2$ has $|N(x, y)| = 3k - 1$ for all nonadjacent vertices $x$ and $y$ and does not have $k$ independent cycles. Also, for $k = 1$ and for any $n$, we need $|N(x, y)| \geq 3k$ in order to be guaranteed the existence of a cycle.

2 Proof of Theorem 3

The proof will proceed by double induction on $n$ and $k$.

The theorem is clearly true for small values of $n$. Thus, we assume the statement of the theorem is true for graphs of order less than $n$.

Let $G$ be a graph of order $n$ satisfying the hypothesis of the theorem. Let $k = 1$. Then $|N(x, y)| \geq 3$ for all nonadjacent pairs of vertices. Thus $G$ must contain a cycle.

Assume $G$ does not contain $k$ independent cycles for $k \leq n/3$. If $G$ contains a triangle, $T$, then $G - T$ contains $k - 1$ independent cycles by the inductive hypothesis. Thus, $G$ contains $k$ independent cycles. So we assume $g(G) \geq 4$.

Let $\mathcal{C} = \{C_1, C_2, C_3, ..., C_{k-1}\}$ be a collection of $k - 1$ vertex disjoint cycles which exist by the inductive hypothesis. Choose $\mathcal{C}$ so that $|V(\mathcal{C})|$ is minimized. Let $L = G - V(\mathcal{C})$. Note that our choice of $\mathcal{C}$ implies that $|V(L)| \geq 3$ since $G - \{v_1, v_2, v_3\}$ contains $k - 1$ independent cycles for any choice of $v_1, v_2, v_3$.

Of all collections $\mathcal{C}$ such that $|V(\mathcal{C})|$ is minimized, choose one such that $L$ has a minimum number of connected components. Finally, of all collections $\mathcal{C}$ with a minimum number of connected components, pick one such that the order of a maximum component of $L$ is maximized.

Claim 1: $L$ has at most one connected component.

Assume $L$ has two or more components. Let $v$ and $w$ be end vertices of distinct trees in $L$ such that $w$ is in a component of maximum order. Then $|N_C(v, w)| \geq 3k - 2$. So there exists $C_i \in \mathcal{C}$ such that $|N_{C_i}(v, w)| \geq 4$. By the minimality of $|V(\mathcal{C})|$, we know that $C_i$ must be a 4-cycle with vertices (in order), $u_1u_2u_3u_4$, such that $vu_1, vu_3, wu_2, wu_4 \in E(G)$.

Let $C_i'$ be the cycle $u_1vu_3u_4$. Let $C' = \mathcal{C} - C_i \cup \{C_i'\}$. Now $L' = G - V(C')$ has a larger maximum connected component than $L$. This contradicts our
choice of $C$. Thus, $L$ has at most one component.

Claim 2: We can assume $L$ is a path.

If $L$ is not a path, pick a path $P$ of maximum length in $L$. Let $w$ be an end vertex of this path. Let $v$ be an end vertex of $L$ not on this path. As in the proof of claim 1, we can simultaneously insert $v$ into $C$ and append $u_2$ to $P$. Continue this process until $L$ is a path.

Claim 3: We can assume that at least one penultimate vertex on the path $L$ has degree at least $3k/2$.

Pick $v, w$ to be end vertices of $L$. Without loss of generality, we assume $d(w) \geq 3k/2$. If neither (or possibly the) penultimate vertex has degree at least $3k/2$, then, as in the proof of claim 1, we can simultaneously insert $v$ into $C$ and append $u_2$ to $L$. Now $w$ is a penultimate vertex with degree at least $3k/2$.

Label the vertices of the path $L : x_1x_2...x_m$. Now, $|N_C(x_1, x_2, x_3)| = |N_C(x_1, x_3)| + |N_C(x_2)| \geq 3k - 2 + \frac{3k}{2} - 2 = \frac{9k}{2} - 4 > 4(k - 1)$ for $k \geq 1$. But this means there exists $C_i \in C$ such that $|N_{C_i}(x_1, x_2, x_3)| \geq 5$ which contradicts the minimality of $|V(C)|$. Thus, $G$ has $k$ independent cycles.

References


