Online and First-Fit Chain Partitioning

An *online poset* $P^{\prec}$ is a poset $P = (V, \leq)$ together with a total ordering $\prec$ of V, called the *presentation* of P. The online poset induced by the first i vertices of $P^{\prec}$ is denoted by $P_i^{\prec}$. An *online partitioning algorithm* $\mathcal{A}$ is a deterministic algorithm that partitions the vertices of $P^{\prec}$ so that the part V_j of the i-th vertex v_i depends only on $G_i^{\prec}$. First-Fit is the online partitioning algorithm that assigns the i-th vertex v_i of $P^{\prec}$ to part V_j, where j is the least positive integer such that v_i is *allowed* in V_j (by some requirement).

By Dilworth's Theorem every poset with width $w \in \mathbb{N}$ can be partitioned into w chains. However this cannot always be done online. For example, for every online algorithm $\mathcal{A}$ there is an online poset $P^{\prec}$ with width 2 and four vertices such that $\mathcal{A}$ uses at least 3 chains for $P^{\prec}$. In 1981 I gave an online algorithm to partition any online poset with width w into at most $\frac{1}{4}(5^w - 1)$ chains. Since then it has been an open problem to determine whether this can be done with polynomially many chains. There was essentially no progress on this question until 2009, when Bosek and Krawczyk proved the sub-exponential bound $w^{14 \lg w}$.

The Bosek-Krawczyk algorithm makes use of First-Fit on the class of posets that do not induce $\mathbf{w} + \mathbf{w}$, the poset consisting of two disjoint chains of length w with no comparabilities between them. They needed a result of Bosek, Krawczyk and Szczypka showing that First-Fit requires at most $3kw^2$ chains on this class. This was improved to $8(k-1)^2 w$ by Joret and Milans, and then to $16kw$ by Dujmović, Joret and Woods. The *ladder* L_k is the poset formed from two disjoint chains $x_1 \dots x_k$ and $y_1 \dots y_k$ with the additional comparabilities $x_i \leq y_j$ in L_k iff $i \leq j$. Bosek and Krawczyk observed that a polynomial bound for online chain partitioning general posets would follow from a polynomial bound (in terms of k and w) on the performance of First-Fit on L_k-free posets.

Matthew Smith and I have proved that the performance of First-Fit on L_2-free online posets is at most w^2 and that this is tight. We have also proved a performance bound of $(kw)^{O(\lg w)}$ for First-Fit on L_k-free posets. This allows us to give a simpler algorithm and proof for the Bosek-Krawczyk Theorem.