

NIM GAMES

We will now consider a class of games that offers a number of mathematical possibilities. The first and most basic of this class of games is NIM.

GAME: NIM

Players: Two players

Board: Any number of piles of chips.

Moves: During your turn you may remove some number (> 0) of chips from exactly one pile.

Winner: Last person able to make a move wins.

Strategy: There is a definite strategy to playing NIM. It depends upon the board itself. The losing position has zero chips remaining on the board. You wish to place your opponent in this position. We do this by considering the chip piles in base 2.

Example: Suppose the piles are 3, 5 and 8 chips. Then

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8: 01000
5: 00101
3: 00011
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01110 (NIM sum equals 0 when even number of 1's are counted and 1 otherwise.)

This is a nonzero position. Our move is to place the board in a zero position. This can always be done. In this case if we change the pile of 8 chips to 6 we will place the board in zero position.

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6: 00110
5: 00101
3: 00011
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00000

The opponent must now move and any move will place the board in a nonzero position. We then repeat our strategy and return the board to a zero position on our move. By keeping this strategy our opponent will be the only one ever facing zero position and eventually that will be the empty board, and we will win.

VARIATION 1: LASKAR NIM

In this variation the game is played as NIM with one other move allowed. On any turn a player may split one pile of chips into two piles.

Now to handle the strategy for this game we wish to fall back on the strategy for NIM if possible. This will work, but we have to adjust for the new move.

To do this we use the GRUNDY FUNCTION. The Grundy function places a value on the chip stack. The value placed is the first unused value of in the set of possible subconfigurations of the stack.

Examples: a stack with one chip has Grundy value 1 as the only subconfiguration is a stack with no chips and it clearly is valued 0. This the first available value is 1. So $g(1) = 1$.

For a stack with 2 chips the Grundy value is 2, as 0 and 1 stacks are the subconfigurations. So $g(2) = 2$.

For a stack with 3 chips the subconfigurations are 0,1,2 and 1+2 (stacks with 1 and 2 chips). These give values of 0,1,2,3. Thus, the first available value is 4. thus $g(3) = 4$.

For a stack with 4 chips, the subconfigurations are 0,1,2,3, 1+3, and 2+2. These in turn have values 0,1,2,4,5,4. Thus $g(4) = 3$.

From this point on the values follow a recursive pattern:

$$g(n) = g(n - 4) + 4, \text{ for } n \geq 5.$$

VARIATION 2: Red and Blue NIM

In this variation there are two colors of chips, red and blue. One player (Red) can only remove red chips (and all chips above them in any one pile) while the other player (Blue) can only remove blue chips (and all chips above them in any one pile). We can also allow white chips that either player may remove.

It is easy to find example of Red - Blue NIM where red always wins, or blue always wins, or the first player always wins or the second player always wins. These 4 options will be given values.

If the second player always wins: value 0

If the blue player always wins: value > 0

If the red player always wins: value < 0

If the first player always wins: "fuzzy" value

We now wish to be able to find these values, see if all numbers are the value of some game, and learn an arithmetic of games that allows us to find the value of a game from its parts.

Clearly, a pile of x blue chips has value x and y red chips has value $-y$.

Example 1: What is the value of a game with one pile of chips, a red chip on top a blue chip. It is easy to see Blue always wins this game so by our definitions its value should be positive. Adding a pile with one red chip to this game produces a game where red always wins. So in terms of values, adding -1 to this game produces a negative value. Thus, the original game has value between 0 and 1. It is a good test to see the value is actually $1/2$. This is done by considering the game with two piles (red on blue) and a third pile with one red chip. It is easy to see this game has second player winning, and hence value 0. But this shows the original game has value $1/2$.

Practice problems:

1. Show a pile of blue (bottom) and two red chips on top has value $1/4$.
2. Then show one blue and three reds has value $1/8$.
3. Can you conjecture as to the value of one blue chip with x red chips on top?