Math 190

Counting Poker Hands

**Poker:** Poker is played with a 52 card deck. Each card has two attributes, a rank and a suit. The rank of a card can be any of 13 possibilities:

\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\},

while the suit can be any of 4 possibilities:

\{♠, ♦, ♥, ♣\}.

We exclude jokers and wildcards.

**Purpose:** The purpose of this handout is to review the counting techniques we applied in class to count the number of each of the various types of poker hands.

**Example 1.** The number of different 5 card poker hands was our first example. This is a standard combinations question and the solution is \(\binom{52}{5} = 2,598,560\).

**Straight flush:** A straight flush is a hand with 5 consecutive ranks, all of which are of the same suit. Straights may begin with an ace, two, ..., 10. Thus there are \(\binom{10}{1}\) ways to begin the straight and \(\binom{4}{1}\) ways to select the suit, for a total of \(\binom{10}{1}\binom{4}{1} = 40\) possible straight flushes. Note that 4 of these are the royal flushes (10, J, Q, K, A), which we counted separately.

**Four of a kind:** This is 4 cards of the same rank. The 5th card is necessarily of a different rank. Our computation is: \(\binom{13}{1}\) ways to select the rank, and \(\binom{4}{1}\) ways to select the 4 cards of that rank. The last card can be selected in \(\binom{48}{1}\) ways (any card of another rank). Thus, by the multiplication principle:

\[
\binom{13}{1}\binom{4}{1}\binom{48}{1} = 624.
\]

**Full house:** This consists of three cards of one rank and two cards of another rank. Thus, the number of such hands is:

\[
\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744.
\]

**Flush:** This consists of 5 cards in one suit (but not a straight). Thus, we choose the suit, then the 5 cards, then subtract the 40 straight flush hands computed above:

\[
\binom{4}{1}\binom{13}{5} - 40 = 5108.
\]

**Straight:** This consists of 5 consecutive ranks (not all of one suit). As noted above, there are \(\binom{10}{1}\) ways to select the straight (equivalent to selecting the starting card). Then, there are \(\binom{4}{1}\) ways to select the card in each of the 5 positions of the straight. But this over counts as it allows a straight flush, so we need to subtract that value. Our computation is then:

\[
\binom{10}{1}\binom{4}{1}^5 - 40 = 10,200.
\]

**Three of a kind:** Consists of three cards of one rank and two more cards, of two further ranks. Thus, our computation is:

\[
\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1} = 54,912.
\]

**Two Pair:** Consists of one pair of one rank and a second pair of another rank and a 5th card of yet a third rank. Thus, by the multiplication principle our count becomes:

\[
\binom{13}{2}\binom{4}{2}^2\binom{44}{1} = 123,552.
\]
**One pair:** Consists of 2 cards of one rank and three other cards that are of three other ranks. By the mult. principle the count is:

\[
\binom{13}{1} \binom{4}{1} \binom{12}{3} \binom{4}{3}^3 = 1,098,240.
\]

**High card:** Consists of all other hands. So by subtracting we have:

\[
\]

<table>
<thead>
<tr>
<th>HAND</th>
<th>HOW TO COUNT</th>
<th>NUMBER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight Flush</td>
<td>(\binom{10}{1}\binom{4}{1})</td>
<td>40</td>
</tr>
<tr>
<td>Four of a kind</td>
<td>(\binom{13}{1}\binom{48}{1})</td>
<td>624</td>
</tr>
<tr>
<td>Full house</td>
<td>(\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2})</td>
<td>3744</td>
</tr>
<tr>
<td>Flush</td>
<td>(\binom{12}{1}\binom{4}{2}) - 40</td>
<td>5108</td>
</tr>
<tr>
<td>Straight</td>
<td>(\binom{10}{1}\binom{4}{5}) - 40</td>
<td>10,200</td>
</tr>
<tr>
<td>Three of a kind</td>
<td>(\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2})</td>
<td>54,912</td>
</tr>
<tr>
<td>Two Pair</td>
<td>(\binom{13}{2}\binom{4}{2}\binom{48}{1})</td>
<td>123,552</td>
</tr>
<tr>
<td>One pair</td>
<td>(\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{3})</td>
<td>1,098,240</td>
</tr>
<tr>
<td>High card</td>
<td>(\binom{52}{5}) - all other counts</td>
<td>1,302,540</td>
</tr>
</tbody>
</table>

**3 Card Poker**

This is poker played with only 3 cards. Using the rules you have learned and the experience from computing the number of 5 card poker hands to complete the table below. Try to justify your computations as we did before.

<table>
<thead>
<tr>
<th>HAND</th>
<th>HOW TO COUNT THE NO. of HANDS</th>
<th>NUMBER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight flush</td>
<td></td>
<td>48</td>
</tr>
<tr>
<td>Three of a kind</td>
<td></td>
<td>52</td>
</tr>
<tr>
<td>Straight</td>
<td></td>
<td>720</td>
</tr>
<tr>
<td>Flush</td>
<td></td>
<td>1096</td>
</tr>
<tr>
<td>One pair</td>
<td></td>
<td>3744</td>
</tr>
</tbody>
</table>