ON MINIMUM DEGREE IMPLYING THAT A GRAPH IS $H$-LINKED

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ABSTRACT. Given a fixed multigraph $H$, possibly containing loops, with $V(H) = \{h_1, \ldots, h_m\}$, we say that a graph $G$ is $H$-linked if for every choice of $m$ vertices $v_1, \ldots, v_m$ in $G$, there exists a subdivision of $H$ in $G$ such that $v_i$ is the branch vertex representing $h_i$ (for all $i$). This generalizes the concept of $k$-linked graphs (as well as a number of other well-known path or cycle properties). In this paper we determine a sharp lower bound on $\delta(G)$ (which depends upon $H$) such that each graph $G$ on at least $10(|V(H)| + |E(H)|)$ vertices satisfying this bound is $H$-linked.

1. Introduction

For terms not defined here, see [8]. A graph is $k$-linked if for every sequence of $2k$ vertices, $v_1, \ldots, v_k, w_1, \ldots, w_k$, there are internally disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins $v_i$ and $w_i$. The literature contains numerous results and important open problems dealing with $k$-linked graphs. In this paper we are concerned with the following generalization of $k$-linked graphs.

Let $H$ be a multigraph. An $H$-subdivision in a graph $G$ is a pair of mappings $f : V(H) \rightarrow V(G)$ and $g : E(H)$ into the set of paths in $G$ such that:

(a) $f(u) \neq f(v)$ for all distinct $u, v \in V(H)$;
(b) for every $uv \in E(H)$, $g(uv)$ is an $f(u), f(v)$-path in $G$, and distinct edges map to internally disjoint paths in $G$.

A graph $G$ is $H$-linked if every injective mapping $f : V(H) \rightarrow V(G)$ can be extended to an $H$-subdivision in $G$.

This idea originated with Jung [6], but had not been significantly developed until recently, when the concept was considered independently in [3] and [7], then in [4], [5], [1], and [2]. In [4] and [5], for loopless multigraphs $H$ with $k$ edges and minimum degree at least two, sharp lower bounds on the minimum degree sufficient to imply that any graph $G$ on at least $7.5|E(H)|$ vertices is $H$-linked were provided. In [1], a similar bound on the minimum degree was given for more general multigraphs $H$ but only for graphs $G$ with exponentially (in $|V(H)|$) many vertices. Also, in [2] and [5], degree conditions for extending $H$-linkages to span the vertex set of $G$ were explored, showing ties of $H$-linked graphs to a number of well-known path and cycle questions. The purpose of this paper is to provide, for all multigraphs $H$ possibly containing...
loops, a lower bound on $\delta(G)$, sufficient to ensure that any such sufficiently large $G$ will be $H$-linked. The bound on $\delta(G)$ will be sharp for every $H$ and the bound on the order of $G$ will be linear in $|E(H)| + |V(H)|$.

We begin with some definitions. With a slight abuse of notation, we will say that a multigraph $H$ is acyclic if it does not contain cycles other than loops. Denote by $c(H)$ the number of acyclic components of $H$. Let

$$b(H) = \begin{cases} |V(H)| - 1, & \text{if } H \text{ is a forest,} \\ \max_{X \subseteq V(H)} e(X, V(H) - X) + c(H), & \text{otherwise.} \end{cases}$$

Our main result is the following:

**Theorem 1.** Let $H$ be a multigraph with $e(H)$ edges (loops or non-loops) and let $k_1 = k_1(H) = e(H) + c(H)$. Let $G$ be a simple graph of order at least $9.5(k_1 + 1)$. If

$$\delta(G) \geq \left\lceil \frac{n + b(H)}{2} \right\rceil - 1,$$

then $G$ is $H$-linked. Moreover, every injective mapping $f : V(H) \to V(G)$ can be extended to an $H$-subdivision in $G$ containing at most $5k_1 + 2$ vertices.

Observe that the restriction (1) cannot be weakened. To see this, suppose first that the multigraph $H$ has no acyclic components and hence $b(H) = \max_{X \subseteq V(H)} e(X, V(H) - X)$. Suppose that this cut determines a partition of $V(H)$ into sets $X$ and $Y$. Let $G$ be formed from two complete graphs $G_1$ and $G_2$ of order $l$ that intersect on $b(H) - 1$ vertices. If the set $S$ chosen as the image of $V(H)$ under $f$ is such that the vertices of $X$ lie in $G_1 - G_2$ and the vertices of $Y$ lie in $G_2 - G_1$, then $G_1 \cap G_2$ is not large enough to allow an embedding of $H$. Further, $\delta(G) = l - 1$. Since $|V(G)| = 2l - b(H) + 1$, we see that $\delta(G) = \frac{n + b(H) - 3}{2}$. Thus, (1) is necessary in this case. To see the reason for the definition of $b(H)$ when $H$ has both acyclic components and components containing cycles, consider the same graph $G$ as above. Now, if the set $S$ which is the image of $V(H)$ under $f$ has the vertices of each acyclic component in $G_1 \cap G_2$, then the need to define $b(H)$ as above becomes clear. If every component of $H$ is acyclic, we will choose $S$ to have the vertices of all but one component from $H$ in $G_1 \cap G_2$ and then place the vertices of the remaining component into $G_1 - G_2$ and $G_2 - G_1$ according to its natural bipartition.

In the next three sections we prove Theorem 1 for the case of loopless $H$, and in the final section we prove the theorem in full generality.

### 2. Preliminaries

In this and the next two sections we consider only loopless $H$. If $H'$ is obtained from $H$ by adding an edge $e'$ and if $k_1(H') \leq k_1(H)$ and $b(H') \leq b(H)$, then, since $H' \supset H$, the fact that a graph $G$ is $H'$-linked implies that $G$ is $H$-linked. If $H$ has at least two components and a component $H_1$ of $H$ is acyclic, then by adding an edge connecting $H_1$ with another component, we decrease $c(H)$. This means that $b(H)$ and $k_1(H)$ do not change. Thus, it is enough to consider only the cases when $H$ is a tree or has no acyclic components. Further, if $H$ is a
tree on at least 3 vertices, then adding to $H$ an edge connecting two vertices at distance two decreases $c(H)$ and keeps $b(H) = |V(H)| - 1$. If $H$ is a tree on 2 vertices, then $H = K_2$ and hence $b(H) = 1$. Thus, it suffices to prove the case when $H$ has no acyclic components or $H = K_2$.

Suppose that $c(H) = k$. Let $f : V(H) \to V(G)$ be an injective mapping and $W = f(V(H))$. Let $E(H) = \{e_j = u_j^0v_j^0 : 1 \leq j \leq k\}$. Let $u_j = f(u_j^0)$ and $v_j = f(v_j^0)$.

If $H = K_2$, then $k = 1$ and $b(H) = 1$. In this case, if an $n$-vertex graph $G$ satisfies the conditions of the theorem, then $\delta(G) \geq (n - 1)/2$. Therefore $u_1$ and $v_1$ either are adjacent or have a common neighbor. This settles the case of $H = K_2$, and from now on we assume that $H$ has no acyclic components. In this case, $k_1 = k$ by the definition of $k_1$ and $|W| = |V(H)| \leq k$.

For each edge $e_j = u_j^0v_j^0 \in E(H)$, we define functions $\beta(e_j, u_j^0), \beta(e_j, v_j^0)$ inductively as follows:

If $H$ has no vertices of degree one, then for every $j$, let $\beta(e_j, u_j^0) = 1/\deg_H(u_j^0)$ and $\beta(e_j, v_j^0) = 1/\deg_H(v_j^0)$.

If $H$ has a pendant vertex $u_s^0$ (which is incident with the edge $e_s = u_s^0v_s^0$), let $H' = H - u_s^0$. Since $H'$ is a smaller graph without acyclic components, we can define $\beta(e_j, u_j^0), \beta(e_j, v_j^0)$ for every $j \neq s$ and then let $\beta(e_s, u_s^0) = 1$ and $\beta(e_s, v_s^0) = 0$.

For simplicity, we denote $\beta(e_j, u_j^0)$ by $\beta_j$, and $\beta(e_j, v_j^0)$ by $\gamma_j$. By construction, for every $j = 1, \ldots, k$,

\begin{equation}
0 \leq \beta_j, \gamma_j \leq 1 \text{ and } \beta_j + \gamma_j \leq 1.
\end{equation}

Also, for every $u^0 \in V(H)$,

\begin{equation}
\sum_{\{e \in E(H) : u^0 \in e\}} \beta(e, u^0) = 1, \quad \text{and hence} \quad \sum_{j=1}^{k} (\beta_j + \gamma_j) = |V(H)| = |W|.
\end{equation}

Say that a family $C$ of the form $\{P_1, \ldots, P_k\}$ is a partial $H$-linkage if each $P_j$ is either the set $\{u_j, v_j\}$ or a $u_j, v_j$-path and the following properties hold:

(I) $|X| \leq |W| + 3k - 2b(H) + 2\alpha + 3$, where $X = \bigcup_{j=1}^{k} V(P_j)$ and $\alpha$ is the number of $P_j$-s that are paths;

(II) the internal vertices of the paths $P_j$’s are pairwise disjoint and disjoint from $W$.

Consider $C_0 = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$. This family satisfies the properties (I) and (II) above with $X = \bigcup_{j=1}^{k} \{u_j, v_j\} = W$ and $\alpha = 0$. Therefore, $C_0$ is a partial $H$-linkage.

A partial $H$-linkage $C = \{P_1, \ldots, P_k\}$ is optimal, if as many as possible of the $P_j$-s are paths and subject to this the set $X = \bigcup_{j=1}^{k} V(P_j)$ is as small as possible. We will prove that an optimal partial $H$-linkage is an $H$-subdivision. This will imply our theorem (for loopless $H$).

Suppose, to the contrary, that $C = \{P_1, \ldots, P_k\}$ is an optimal partial $H$-linkage but is not an $H$-subdivision. Let, for definiteness, $P_k = \{u_k, v_k\}$ and $u_kv_k \notin E(G)$. Denote $X = \bigcup_{j=1}^{k} V(P_j)$, $x = u_k$, and $y = v_k$. Let $A = N(x) - X$, $B = N(y) - X$, and $R = V(G) - (X \cup A \cup B)$.

It is well known (see e.g. [8, p. 51]) that

\begin{equation}
b(H) \geq \frac{(k + 1)}{2}
\end{equation}
for every $H$ with $k > 0$ edges.

Therefore, each of $A$ and $B$ has size at least

$$\delta(G) - (|X| - 2) \geq \frac{n + b(H) - 2}{2} - (|W| + 3k - 2b(H) + 2(k - 1) + 3 - 2)$$

$$\geq \frac{9.5k + b(H) - 2}{2} - 6k + 1 + 2b(H) = 2.5b(H) - 1.25k \geq 1.25.$$  

It follows that we may choose distinct $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

For $v \in V(G)$, let $d_j(v)$ denote the number of neighbors of $v$ in the interior of $P_j$ plus $\beta_j$ if $u_j \in N_G(v)$ and plus $\gamma_j$ if $v_j \in N_G(v)$ ($\beta_j$ and $\gamma_j$ are defined above (2)). By (3), we have

$$(5) \quad \sum_{j=1}^k d_j(v) = |N_G(v) \cap X| \quad \forall v \in V(G).$$

Let $l_p$ be the number of $P_j$'s of length $p$ for $p \geq 1$, and $l_0$ be the number of $P_j$'s that are not paths. Then

$$(6) \quad |X| = |W| + \sum_{p \geq 1} (p - 1)l_p = \sum_{j=1}^k (\beta_j + \gamma_j) + \sum_{p \geq 1} (p - 1)l_p$$

and

$$(7) \quad k = \sum_{p \geq 0} l_p = \alpha + l_0.$$

We will assume that every path $P_j$ is of the form $P_j = u_j, w_{1,j}, \ldots, w_{p_j-1,j}, v_j$. Sometimes, for simplicity we will write $p$ instead of $p_j$ and $w_i$ instead of $w_{i,j}$ if $j$ is clear from the context. In the rest of the paper, for every $j = 1, \ldots, k$ and fixed $a_1, a_2 \in A$, $b_1, b_2 \in B$, we denote $M_j = d_j(x) + d_j(y)$ and $L_j = d_j(a_1) + d_j(a_2) + d_j(b_1) + d_j(b_2)$.

3. Main Lemma

Lemma 3 in this section is important for the proof. It provides that an optimal partial linkage (if it is not an $H$-subdivision) has fewer vertices than condition (I) in the definition of partial linkage requires. Thus, more vertices are available to improve the linkage. We begin with a lemma needed in the proof of Lemma 3.

Lemma 2. Let $a_1, a_2 \in A$, $b_1, b_2 \in B$. For a $P_j = u_j, w_1, \ldots, w_{p-1}, v_j$, let $s_j = M_j + 0.5L_j$, $\beta = \beta_j$, and $\gamma = \gamma_j$. Define

$$D_1(p, \beta, \gamma) = \begin{cases} p + 2 + 2\beta + 2\gamma, & \text{for } p \leq 1, \\ p + 4 + 2\beta + 2\gamma, & \text{for } p \geq 2. \end{cases}$$

Then

(a) $s_j \leq D_1(p, \beta, \gamma)$;

(b) $s_k \leq 2(\beta_k + \gamma_k)$. Furthermore, if $xy = u_kv_k \not\in E(G)$, then $s_k = \beta_k + \gamma_k$.  

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Proof. Let \( \lambda = \max\{\beta, \gamma\} \). Since \( H \) has no acyclic components, we have \( \lambda \leq 1 \) and \( \min\{\beta, \gamma\} \leq 0.5 \).

By definition, \( L_k = 2\beta_k + 2\gamma_k \). If \( xy \in E(G) \), then \( M_k = \beta_k + \gamma_k \); otherwise, \( M_k = 0 \). This proves (b).

Claim 1. Let \( Z = \{a_1, a_2, b_1, b_2\} \).

(i) For each \( z \in Z \), the distance in \( P_j \) between any two neighbors of \( z \) is at most two. In particular, each \( z \in Z \) has at most 3 neighbors in \( P_j \).

(ii) If \( p \geq 3 \), then no \( z \in Z \) is a common neighbor of \( u_j \) and \( v_j \).

(iii) If \( p \geq 3 \), then \( x \) and \( y \) have no interior neighbors of distance at most \( p - 3 \) in \( P_j \).

(iv) If \( p \geq 3 \), then \( x \) (respectively, \( y \)) has no interior neighbors at distance at most \( p - 4 \) in \( P_j \) from interior neighbors of \( b_1 \) and \( b_2 \) (respectively, of \( a_1 \) and \( a_2 \)).

Proof. If some \( z \in Z \) is adjacent to \( w_i \) and \( w_{i+m} \) for some \( m \geq 3 \) (we treat \( u_j \) as \( w_0 \) and \( v_j \) as \( w_p \)), then we can replace \( P_j \) by a shorter \( u_j, v_j \)-path, a contradiction to the optimality of \( C \). This proves (i), and (ii) is a partial case of (i).

If \( x \) and \( y \) have interior neighbors at distance at most \( p - 3 \) in \( P_j \), then we can delete \( P_j \) from \( C \) and add a shorter \( x, y \)-path. This proves (iii). The same trick proves (iv), completing the proof of the claim.

In order to prove (a), we consider several cases (depending on \( p \)).

CASE 1. \( p = 0 \). By (2), \( L_j \leq 4(\beta + \gamma) \leq 4 \). Therefore \( s_j = M_j + 0.5L_j \leq 2(\beta + \gamma) + 2 = D_1(0, \beta, \gamma) \).

CASE 2. \( p = 1 \). Trivially,

\[
s_j \leq 2(\beta + \gamma) + 0.5(4(\beta + \gamma)) \leq 2(\beta + \gamma) + 2 < D_1(1, \beta, \gamma).
\]

CASE 3. \( p = 2 \). If each of \( x \) and \( y \) is adjacent to \( w_1 \) and some \( z \in Z \) is adjacent to both \( u_j \) and \( v_j \), then \( C \) is not optimal: we can replace \( P_j \) by the path \( u_j, z, v_j \) and add the path \( xw_1y \). Otherwise, either \( M_j \leq 2(\beta + \gamma) + 1 \) and hence

\[
s_j \leq 2(\beta + \gamma) + 1 + 0.5(4(\beta + \gamma + 1)) \leq 2(\beta + \gamma) + 6 = D_1(2, \beta, \gamma),
\]

or \( L_j \leq 4(\lambda + 1) \) and hence

\[
s_j \leq 2(\beta + \gamma + 1) + 0.5(4(\lambda + 1)) \leq 2(\beta + \gamma) + 4 < D_1(2, \beta, \gamma).
\]

CASE 4. \( p = 3 \). By (iii), \( M_j \leq 2(\beta + \gamma) + 2 \). If \( L_j \leq 10 \), then \( s_j \leq D_1(3, \beta, \gamma) \). Otherwise, because of the symmetry between \( A \) and \( B \), we may assume that \( d_j(a_1) + d_j(a_2) > 5 \) and that \( d_j(a_1) > 2.5 \). Then by (ii), we may assume that \( a_1 \) is adjacent to \( w_1, w_2 \) and \( v_j \) and that \( a_2 \) is adjacent to \( w_1 \) and \( w_2 \) (and maybe to one more vertex). If \( yw_2 \in E(G) \), then we can replace \( P_j \) with \( u_j, w_1, a_1, v_j \) and add the path \( x, a_2, w_2, y \), a contradiction to the optimality of \( C \). If neither of \( x \) and \( y \) is adjacent to \( w_2 \), then by (iii), \( M_j \leq 2(\beta + \gamma) + 1 \), by (ii), \( L_j \leq 4(2 + \lambda) \leq 12 \), and therefore \( s_j \leq 2(\beta + \gamma) + 7 = D_1(3, \beta, \gamma) \). If \( xw_2 \in E(G) \) and some \( b \in \{b_1, b_2\} \) is adjacent to \( w_2 \), then we can replace \( P_j \) with \( u_j, w_1, a_1, v_j \) and add the path \( x, w_2, b, y \). Finally, if neither
of \( b_1w_2 \) and \( b_2w_2 \) is in \( E(G) \), then by (i), \( d_j(b_1) + d_j(b_2) \leq 2(1 + \lambda) \leq 4 \), and hence by (ii) \( L_j \leq 6 + 4 = 10 \).

**CASE 5.** \( p \geq 4 \). If \( x \) has \( r \) interior neighbors and \( r \geq 2 \), then by (iii), \( d_j(y) \leq \beta + \gamma \) and by (iv), \( d_j(b_i) \leq \max\{0, 3 - r\} + \lambda \). In this case,

\[
s_j \leq 2\beta + 2\gamma + r + 3 + \max\{0, 3 - r\} + \lambda.
\]

If \( r \geq 3 \), then \( s_j \leq 2\beta + 2\gamma + p - 1 + 3 + \lambda \leq p + 3 + 2\beta + 2\gamma \leq D_1(p, \beta, \gamma) \). If \( r = 2 \), then \( s_j \leq 2\beta + 2\gamma + r + 4 + \lambda \leq 2\beta + 2\gamma + p + 3 \leq D_1(p, \beta, \gamma) \), again.

Thus, we can assume that each of \( x \) and \( y \) has at most one interior neighbor in \( P_j \). By (iv) \( d_j(a_i) + d_j(y) \leq \beta + \gamma + \lambda + 3 \) and \( d_j(b_i) + d_j(x) \leq \beta + \gamma + \lambda + 3 \) for \( i = 1, 2 \). Therefore,

\[
s_j \leq 2\lambda + 6 + 2\beta + 2\gamma \leq 2\beta + 2\gamma + p + 2 + 2 = D_1(p, \beta, \gamma).
\]

This completes the proof of (a) and hence, of Lemma 2.

**Lemma 3.** Let \( a_1, a_2 \in A, \ b_1, b_2 \in B, \ Z = \{a_1, a_2, b_1, b_2\} \), and \( V_0 = (A \cup B) - Z - N_G(Z) \). Then \( |X| \leq |W| + 3k - 2b(H) + 2\alpha - |R| - |V_0| \).

**Proof.** Let

\[
(8) \quad \Sigma' = \deg_G(x) + \deg_G(y) + \frac{1}{2}(\deg_G(a_1) + \deg_G(a_2) + \deg_G(b_1) + \deg_G(b_2)).
\]

Observe that every vertex \( w \notin X \) contributes to \( \Sigma' \) at most 2: if \( w \in R \), then it is not adjacent to \( x \) and \( y \), and if \( w \in A \) (respectively, \( w \in B \)), then it is not adjacent to \( y, b_1, \) and \( b_2 \) (respectively, to \( x, a_1, \) and \( a_2 \)). By the definition, every vertex in \( V_0 \) is not adjacent to any vertex in \( Z \), and therefore contributes at most 1 to \( \Sigma' \). Furthermore, every \( z \in Z \) contributes at most 1.5 to \( \Sigma' \), since it is not adjacent to itself. Therefore,

\[
(9) \quad \Sigma' \leq 4 \cdot 1.5 + 2(|A \cup B| - 4) + 2|R| + \sum_{j=1}^{k} s_j - |V_0|.
\]

By Lemma 2 and (7),

\[
\sum_{j=1}^{k} s_j \leq k + l_0 + 2l_1 + \sum_{p \geq 2} (p + 3)l_p + 2 \sum_{j=1}^{k} (\beta_j + \gamma_j) - 1
\]

\[
= k + l_0 + 2l_1 + \sum_{p \geq 2} (p + 3)l_p + 2|W| - 1.
\]

Therefore,

\[
\Sigma' \leq 2(|A \cup B| + |R|) - 2 - |V_0| + 2(|W| + l_0 + \sum_{p \geq 1} p l_p) - 1 - l_0 + \sum_{p \geq 2} (3 - p)l_p + k
\]

\[
= 2n + 3k - |V_0| - 3 - l_0 - \sum_{p \geq 2} (p - 3)l_p.
\]

Combining with (6) and (7), we get

\[
|X| + \Sigma' \leq 2n + |W| + 3k + 2\alpha - 3 - l_0 - 2l_1 - |V_0|.
\]
By (4), \( \delta(G) \geq \frac{n+b(H)}{2} - 1 \) and hence \( \Sigma' \geq 2n + 2b(H) - 4 \). Thus,

\[(12) \quad |X| \leq |W| + 3k - 2b(H) + 2\alpha - l_0 - 2l_1 - |V_0| + 1 \leq |W| + 3k - 2b(H) + 2\alpha - |V_0|.
\]

If an \( r \in R \) has a neighbor \( a_0 \in A \) and a neighbor \( b_0 \in B \), then one can add to \( C \) the path \( P_k = x, a_0, r, b_0, y \). The new set of paths will be a better partial linkage, since the new \( X \) would have size at most \( |W| + 3k - 2b(H) + 2(\alpha + 1) + 1 \). Since this contradicts the choice of \( C \), no \( r \in R \) has both a neighbor in \( A \) and a neighbor in \( B \). Thus every \( r \in R \) contributes at most 1 to \( \Sigma' \), and (9) becomes

\[
\Sigma' \leq 4 \cdot 1.5 + 2(|A \cup B| - 4) + |R| + \sum_{j=1}^{k} s_j - |V_0|.
\]

Correspondingly, (12) transforms into

\[(13) \quad |X| \leq |W| + 3k - 2b(H) + 2\alpha - |V_0| - |R|.
\]

\[\square\]

4. Completion of the case of loopless \( H \)

Lemma 3 has the following two immediate consequences.

**Lemma 4.** \(|A| + |B| > 2k|.

*Proof.*** By Lemma 3, \(|A| + |B| = n - (|X| + |R|) \geq n - (|W| + 3k - 2b(H) + 2\alpha) \geq 9.5k - (k + 3k - 2\frac{k+1}{2} + 2(k - 1)) = 4.5k + 3 > 2k|.

\[\square\]

**Lemma 5.** Each \( v \in V(G) \) is adjacent to at least 3 vertices in \( A \cup B - V_0 \). In particular, either \( v \) has 2 neighbors in \( A \) that belong or are adjacent to the set \( \{a_1, a_2\} \), or 2 neighbors in \( B \) that belong or are adjacent to the set \( \{b_1, b_2\} \).

*Proof.*** By Lemma 3, \( \delta(G) - (|X| + |R| + |V_0|) \geq 0.5(9.5k + b(H) - 2) - |W| - 3k + 2b(H) - 2\alpha \geq 4.75k + 0.5b(H) - 1 - k - 3k + 2b(H) - 2(k - 1) = 2.5b(H) - 1.25k + 1 \geq 2.25 > 2 \). Thus each vertex has at least 3 neighbors in \( V(G) - X - R - V_0 \).

For given \( a_1, a_2 \in A \), \( b_1, b_2 \in B \), let \( A'' = A''(a_1, a_2) \) (respectively, \( B'' = B''(b_1, b_2) \)) denote the set of vertices in \( X \) having at least 2 neighbors in \( A \) (respectively, in \( B \)) that belong or are adjacent to the set \( \{a_1, a_2\} \) (respectively, \( \{b_1, b_2\} \)). The above lemma yields that for every choice of \( a_1, a_2, b_1, \) and \( b_2 \),

\[(14) \quad A'' \cup B'' = X.
\]

**Lemma 6.** For every non-adjacent \( s, t \in A \) (or \( B \)), \(|N(s) \cap N(t) - X| \geq 3|.

*Proof.*** Suppose to the contrary that \( a_1, a_2 \in A \), \( a_1a_2 \notin E(G) \) and the cardinality of the set \( T \) of common neighbors of \( a_1 \) and \( a_2 \) is at most two. Consider arbitrary \( b_1, b_2 \in B \) and let \( Z = \{a_1, a_2, b_1, b_2\} \). Then the contribution of every \( a \in A - Z - T \) to the sum \( \Sigma' \)
defined in (8) is at most 1.5. Thus, repeating the proof of Lemma 3, instead of (13), we will get
$$|X| \leq |W| - |R| + 3k - 2b(H) + 2\alpha - |V_0| - 0.5(|A - V_0| - 4).$$
In other words,
$$|X| + 0.5|A| + |R| \leq |W| + 3k - 2b(H) + 2\alpha + 2 \leq 6k - 2b(H).$$

On the other hand, $\deg_{G - X}(a_1) + \deg_{G - X}(a_2) \leq |A| + |T| + |R| - 2$ (the $-2$ arises because
neither of $a_1$ and $a_2$ is adjacent to $a_1$ or $a_2$).
It follows that
$$2 \left(\frac{n + b(H)}{2}\right) - 2 \leq \delta(G) \leq 2|X| + |A| + |R|,$$
which together with (15) yields $n + b(H) - 2 \leq 2(6k - 2b(H))$. Thus, $n \leq 12k - 5b(H) + 2 \leq
12k - 5\frac{k+1}{2} + 2 = 9.5k - 0.5$, a contradiction. \hfill \Box

For the rest of the section, we fix some distinct $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and let $A'' = A''(a_1, a_2)$ and $B'' = B''(b_1, b_2)$.

**Lemma 7.** Let $X$ be optimal, $1 \leq j \leq k - 1$, and either $\{u_j, v_j\} \subset A''$ or $\{u_j, v_j\} \subset B''$. Then
for each $a \in A$ and $b \in B$,
$$(N(a) \cap N(b) \cap P_j) \setminus \{u_j, v_j\} = \emptyset.$$

*Proof.* Assume to the contrary that $r \in N(a) \cap N(b) \cap P_j \setminus \{u_j, v_j\}$. Let $P'_k = (x, a, r, b, y)$. Without loss of generality, assume that $\{u_j, v_j\} \subset A''$. Then there exist $s \in N(u_j) \cap A \setminus \{a\}$ and $t \in N(v_j) \cap A \setminus \{a\}$. If $s = t$ or $s$ is adjacent to $t$, then let $P'_j = (u_j, s, t, v_j)$.

If $s$ and $t$ are non-adjacent, then by Lemma 6, we have $|(N(s) \cap N(t)) \setminus X| \geq 3$, and therefore there exists $q \in N(s) \cap N(t) \setminus \{a, b\}$. In this case, let $P'_j = (u_j, s, q, t, v_j)$. In both cases, $P'_j$ is a path disjoint from $P'_k$. Thus, in both cases we increase the number of $P_j$s that are paths by one and, by (13), maintain $|X| \leq |W| + 3k - 2b(H) + 2(\alpha + 1) + 3$. This is a contradiction which completes the proof. \hfill \Box

**Lemma 8.** Let $X$ be optimal, $1 \leq j \leq k - 1$, $P_j = (w_0, w_1, \ldots, w_p)$, where $w_0 = u_j \in A''$ and $w_p = v_j \in B''$. If some $w_i$, $1 \leq i \leq p - 1$ has a neighbor $a_0 \in A \cup \{x\}$ and a neighbor $b_0 \in B \cup \{y\}$, then each $w_{i'}$ for $i < i' \leq p$ has no neighbors in $A - a_0$ and each $w_{i''}$ for $0 \leq i'' < i$ has no neighbors in $B - b_0$.

*Proof.* Suppose some $w_{i'}$ for $i < i' \leq p$ has a neighbor $a' \in A - a_0$. By the definition of $A''$, $u_j$ has a neighbor $a'' \in A - a_0$. By Lemma 6, the length of a shortest path $P'$ from $a''$ to $a'$ in $G[A - a_0]$ is at most two. Thus, we can replace $P_j$ by the path $(u_j, a'', P', a', w_{i'}, P'_j, v_j)$ (where $P'_j$ is the part of $P_j$ connecting $w_{i'}$ with $v_j$) and add the path $P_k = (x, a_0, w_i, b_0, y)$. The new set of $\alpha + 1$ paths has at most $|X| + 5$ vertices, which by (13) is at most $|W| + 3k - 2b(H) + 2(\alpha + 1) + 3$, a contradiction to the choice of $C$. \hfill \Box

Similarly to $d_j(v)$, let $d_j(u, v)$ denote the number of common neighbors of $u$ and $v$ ‘inside’ $P_j$ plus $\beta_j \cdot |N(u) \cap N(v) \cap \{u_j\}|$ plus $\gamma_j \cdot |N(u) \cap N(v) \cap \{v_j\}|$. Let $X$ be optimal, $a \in A$, $b \in B$. Since $N(a) \cap N(b) \cap (V(G) - X + x + y) = \emptyset$, we have $\sum_{j=1}^{k-1} d_j(a, b) \geq 2\delta(G) - (n - 2) \geq b(H)$.

By Lemma 7, if $d_j(a, b) > 1$, then either $u_j \in A'' - B''$ and $v_j \in B'' - A''$ or $v_j \in A'' - B''$ and $u_j \in B'' - A''$. Recall that $e_k$ also connects $A'' - B''$ with $B'' - A''$. It follows that the set
\( E' = \{ e_k \} \cup \{ e_j : d_j(a, b) > 1 \} \) spans a bipartite subgraph in \( H \) and hence \( |\{ e_j : d_j(a, b) > 1 \}| \leq b(H) - 1 \). Thus, there exists some \( j = j(a, b) \) such that \( d_j(a, b) > 1 \).

**Lemma 9.** Let \( X \) be optimal, \( 1 \leq j \leq k - 1 \). Then there is at most one \( a \in A \), such that there is more than one \( b \in B \) with \( j = j(a, b) \).

**Proof.** Let \( P_j = (w_0, w_1, \ldots, w_p) \), where \( w_0 = u_j \) and \( w_p = v_j \). Assume to the contrary that there are \( a_1, a_2 \in A \) and \( b_1, b_2, b_3, b_4 \in B \) such that \( j(a_1, b_1) = j(a_1, b_2) = j(a_2, b_3) = j(a_2, b_4) = j \), where \( a_1 \neq a_2, b_1 \neq b_2, b_3 \neq b_4 \). By Lemma 7, we may assume that \( u_j \in A'' \setminus B'' \) and \( v_j \in B'' \setminus A'' \).

Since \( \beta_j + \gamma_j \leq 1 \), there exists \( i, 1 \leq i \leq p - 1 \), such that \( w_i \in N(a_1) \cap N(b_1) \). Since \( b_3 \neq b_4 \), we may assume that \( b_3 \neq b_1 \). By Lemma 8, no vertex in \( V(P_j) - w_i \) can belong to \( N(a_2) \cap N(b_3) \). However, this contradicts the fact that \( d_j(a_2, b_3) > 1 \).

By Lemma 4, \( |A| + |B| > 2k \). We may assume that \( |A| \leq |B| \). Thus \( |B| \geq k \). If \( |A| \geq k \), then since \( |B| \geq k \), for each \( a \in A \) there is some \( j(a) \) and \( b_1(a) \) and \( b_2(a) \) such that \( j(a) = j(a, b_1(a)) = j(a, b_2(a)) \). Furthermore, since \( |A| \geq k \), for some \( a_1, a_2 \in A \), the indices \( j(a_1) \) and \( j(a_2) \) are the same. This contradicts Lemma 9.

Thus we may assume that \( |A| < k \). Since \( |B| \geq k \), for each \( a \in A \) there is some \( j(a) \) and \( b_1(a) \) and \( b_2(a) \) such that \( j(a) = j(a, b_1(a)) = j(a, b_2(a)) \). Let \( J = \{ j(a) : a \in A \} \). By Lemma 9, the indices \( j(a) \) are distinct for distinct \( a \in A \) and hence \( |J| = |A| \).

**Lemma 10.** Suppose that \( j \in J \). Then \( x \) is not adjacent to some interior vertex of \( P_j \).

**Proof.** Let \( P_j = (w_0, w_1, \ldots, w_p) \), where \( w_0 = u_j \) and \( w_p = v_j \). By the definition of \( J \), there exists \( a \in A \) and \( b_1, b_2 \in B \) such that \( d_j(a, b_1), d_j(a, b_2) > 1 \). Since \( \beta_j + \gamma_j \leq 1 \), this implies that \( p \geq 2 \). Assume that \( u_j \in A'' \setminus B'' \) and \( v_j \in B'' \setminus A'' \).

Since \( u_j \notin B'' \), we may assume that \( u_j b_1 \notin E(G) \). Let \( w_{i'}, w_{i''} \in N(a) \cap N(b_1) \) and \( i' < i'' \). By the choice, \( 1 \leq i' \leq p - 1 \). If \( x w_{i'} \in E(G) \), then we get a contradiction to Lemma 8 with \( a_0 = x \), since \( w_{i'} a \in E(G) \). Thus, \( x w_{i'} \notin E(G) \).

By Lemma 10, \( x \) is not adjacent to at least \( |J| \) vertices in \( X - W \). It also is not adjacent to itself. Thus, \( |N(x) \cap X| \leq |X| - |J| - 1 \leq |W| + 3k - 2b(H) + 2(k - 1) - |J| - 1 \leq 6k - 2b(H) - 3 - |J| \). Since \( |J| = |A| = |N(x) \cap X| \), we get

\[
\frac{n + b(H)}{2} - 1 \leq \deg(x) \leq 6k - 2b(H) - 3,
\]

which yields \( n \leq 12k - 5b(H) - 4 \leq 9.5k - 6.5 \), a contradiction. This contradiction proves that an optimal partial \( H \)-linkage is an \( H \)-linkage in the case of loopless \( H \).

By condition (I) in the definition of a partial \( H \)-linkage, \( |X| \leq |W| - 2b(H) + 5k + 3 \leq 5k + 2 \).

5. **Proof of the general case**

As in Section 2, it is enough to consider \( H \) that either has no acyclic components or is connected and has at most two vertices. Let \( H \) have \( k' \) non-loop edges and \( k'' \) loops, in total \( k = k' + k'' \) edges. Recall that \( n \geq 9.5(k_1 + 1) \), where \( k_1 = k + c(H) \). Note that \( b(H) \) does not depend on \( k'' \), thus \( b(H) \geq 0.5k' \).
Let $f: V(H) \to V(G)$ be an injective mapping and $W = f(V(H))$. Let $E(H) = \{e_j = u_j^0v_j^0 : 1 \leq j \leq k\}$. We may assume that the first $k'$ edges are not loops. Let $u_j = f(u_j^0)$ and $v_j = f(v_j^0)$.

Let $H'$ be the multigraph obtained from $H$ by deleting all loops and let $k' = k' + c(H')$. Since $H'$ is loopless, our theorem is proved for it, and thus $f$ can be extended to an $H'$-subdivision in $G$ on at most $5k' + 2$ vertices. Recall that if $H'$ has no acyclic components, then $k' = k$.

If $H'$ has an acyclic component, then so does $H$, and hence by the above, $|V(H')| \leq 2$. It was observed in Section 2 that in this case $G$ has a subdivision of $H$ on at most $3$ vertices. Thus, in either case, $f$ can be extended to an $H'$-subdivision in $G$ on at most $5k' + 2$ vertices. Among such $H'$-subdivisions choose one, say, $F_1$, with the fewest vertices and let $X_1 = V(F_1)$. We will extend $F_1$ to a partial $H$-subdivision $F$ such that

(I') as many loops as possible are mapped to internally disjoint cycles of length at most 4 and

(II') among partial $H$-subdivisions satisfying (I'), the set $X = V(F)$ has the smallest size.

We claim that such a partial $H$-subdivision is actually an $H$-subdivision. Suppose not. Then we may assume that $F$ represents the images $g(e_j)$ for $1 \leq j \leq q$, where $k' \leq q \leq k - 1$.

First we observe that by the minimality of $F_1$ and $F$, every vertex outside $X$ has at most 3 neighbors in $g(e_j)$ for each $1 \leq j \leq q$.

Let $e_{q+1}$ be a loop at vertex $u_{q+1}^0$ and $u_{q+1} = f(u_{q+1}^0)$. Consider graph $G' = G - (X - u_{q+1})$.

If $H$ is not an isolated vertex, then every $v \in W$ is in $X_1$ (in fact, $v$ belongs to $g(e_j)$ for some $1 \leq j \leq k'$), therefore, $u_{q+1}$ has at most $3(q - k')$ neighbors in $X - X_1$ by (I'). If $H$ is an isolated vertex, then $k' = 0$, $V(H) = \{u_{q+1}\}$ and $u_{q+1}$ has at most $2q$ neighbors in $X$. It follows that

$$\deg_{G'}(u_{q+1}) \geq \deg_{G}(u_{q+1}) - 5k' - 2 - 3(q - k') \geq \frac{n + k'/2}{2} - 1 - 5k' - 2 - 3(q - k')$$

$$\geq \frac{n}{2} - 4.75q - 3 \geq \frac{9.5(k + 1)}{2} - 4.75(k - 1) - 3 \geq 6.5.$$

Let $S = N_{G'}(u_{q+1})$. If some vertices of $S$ are adjacent or have a common neighbor in $G'$ other than $u_{q+1}$, then we extend our partial $H$-linkage. If this is not the case, then all neighbors in $G'$ of vertices in $S$, apart from $u_{q+1}$, are distinct. Thus,

$$\sum_{s \in S} (\deg_{G'}(s) - 1) + |S| + 1 \leq n - (|X| - 1).$$

Since $S \cap X = \emptyset$, by the above, $\deg_{G'}(s) \geq \deg_{G}(s) - \min\{|X|, 3q\}$ for every $s \in S$. Thus, (16) yields $|S| (\delta(G) - \min\{|X|, 3q\}) + 1 \leq n - |X| + 1$. Since $|S| > 6$, we have

$$\frac{6n}{2} \leq 6 \min\{|X|, 3q\} + n - |X| \leq 15q + n \leq 15(k - 1) + n.$$

It follows that $2n < 15k$, a contradiction.

**References**


[7] T. Whalen, Degree conditions and relations to distance, extendability, and levels of connectivity in graphs.