Connectivity and Forbidden Families for Hamiltonian Properties

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Definition

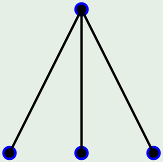
We say the collection of graphs $\mathcal{H} = \{H_1, \ldots, H_k\}$ is **forbidden** in G if no H_i is an induced subgraph of G.

We also say G is **$\mathcal{H}$-free.**

Each H_i is called a **forbidden subgraph**.

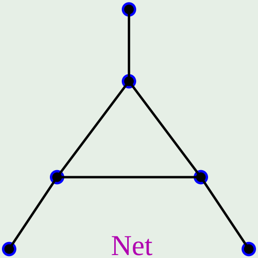
Graphs of interest

Example



Claw

$K_{1,3}$

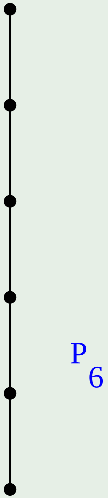


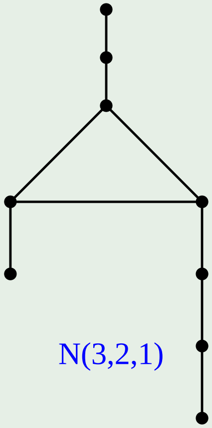
Net

N(1,1,1)

Graphs of interest

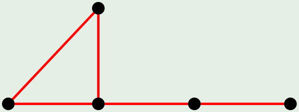
Example



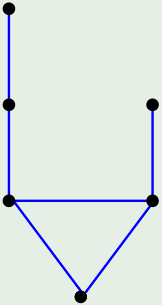


Graphs of interest

Example



$Z_2 = N(2,0,0)$



$W = N(2,1,0)$

Ancient History -

Since 1980, a great many results have been shown concerning families of forbidden subgraphs implying various cycle (especially hamiltonian) properties. The first such result was the following:

Theorem

D. Duffus, RG, M. Jacobson, 1980.

*If G is a $\{K_{1,3}, N(1,1,1)\}$-**free** graph, then*

1. *if G is 2-connected, then G is hamiltonian;*
2. *if G is connected, then G is traceable.*

This Theorem spurred interest in the area and a number of other results soon followed. Of special interest are the following pairs, each of which implies a 2-connected G is hamiltonian:

1. **(RG, Jacobson, 1982)**
   $K_{1,3}$, $N(2,0,0)$

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   $K_{1,3}$, $N(2,0,0)$
2. **(Broersma and Veldman, 1990)**
   $K_{1,3}$, P_6

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3. **(Bedrosian, 1991)**
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3. **(Bedrosian, 1991)**
   $K_{1,3}$, $N(2,1,0)$
4. **(Faudree, RG, Ryjáček, Schiermeyer, 1995)**
   $K_{1,3}$, $N(3,0,0)$ for $n \geq 10$.

Theorem

Bedrosian 91 - R. Faudree and RG 97.

Let R *and* S *be connected graphs* $(R, S \neq P_3)$ *and* G *a 2-connected graph of order* n*. Then* G *is* $\{R, S\}$*-free implies* G *is hamiltonian*

if, and only if,

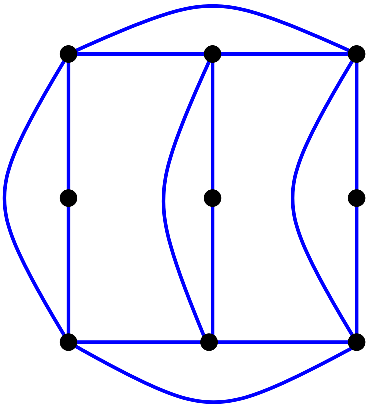
$\mathbf{R} = \mathbf{K_{1,3}}$ *and* S *is one of the graphs*

$$\mathbf{P_6}, \mathbf{N(1,1,1)}, \mathbf{N(2,1,0)}, \mathbf{N(2,0,0)},$$

$($*or* $\mathbf{N(3,0,0)}$ *when* $n \geq 10)$*, or a connected induced subgraph of one of these graphs.*

Note: $N(i, j, k)$ where $i + j + k = 3$.

No claw and no Z(3,0,0), but not hamiltonian



$P_{T,T,T}$

Sir William Rowan Hamilton

We now turn our attention to pancyclic graphs.

Theorem

Faudree

If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph of order $n \geq 3$, then G is pancyclic.

Theorem

RG and Jacobson

If $G \neq C_n$ is a 2-connected $\{K_{1,3}, N(2,0,0)\}$-free graph of order $n \geq 3$, then G is pancyclic.

Theorem

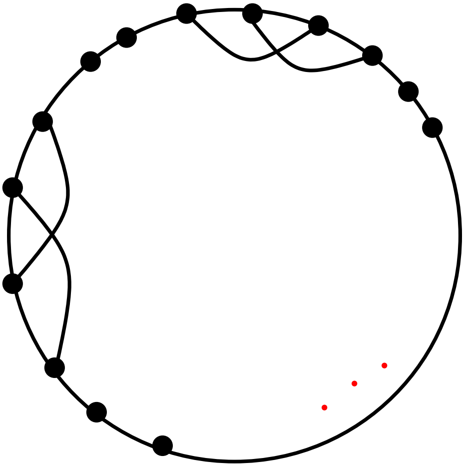
Faudree and RG

Let R and S be connected graphs $(R, S \neq P_3)$ and let $G \neq C_n$ be a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is pancyclic

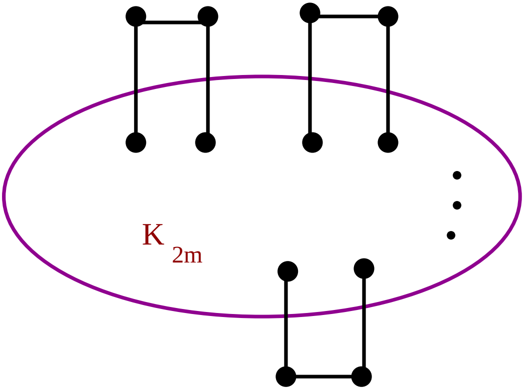
if, and only if,

$R = K_{1,3}$ *and* S *either* P_6 *or* $N(2, 0, 0)$
or a connected induced subgraph of one of these graphs.

Claw free, hamiltonian but not pancyclic



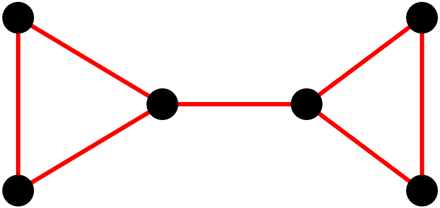
No N(2,1,0), N(1,1,0) or N(1,1,1)



2−connected, non−pancyclic, claw−free and N(3,0,0)−free

Question: What changes if we consider **3-connected** graphs?

A new graph is now needed.



L

Bigger properties call for bigger nets!

Theorem

RG, T. Luczak, F. Pfender, 2004.
Let X and Y be connected graphs on at least three vertices such that $X, Y \neq P_3$ and $Y \neq K_{1,3}$. Then the following statements are equivalent:

1. *Every 3-connected $\{X, Y\}$-free graph G is* **pancyclic.**
2. *$X = K_{1,3}$ and Y is a subgraph of one of the graphs from the family $\mathcal{F} = \{P_7, L, N(4,0,0), N(3,1,0), N(2,2,0), N(2,1,1)\}$*
.

Question

What about 4-connected?

Of course, there is the famed conjecture of Matthews and Sumner (1985):

Conjecture

Every 4-connected $K_{1,3}$-free graph is hamiltonian.

with M. Ferrara, S. Gehrke, C. Magnant, J. Powell.

Now we consider the generalized nets $N(i, j, k)$ where $i + j + k = 5$.

Theorem

Let G be a 4-connected $\{K_{1,3}, N(i, j, k)\}$-free graph with $i + j + k = 5$. Then G is pancyclic.

Clearly, we had to show that such graphs are hamiltonian. This builds on the known set of forbidden pairs that imply a 4-connected graph is hamiltonian.

Question

1. Can we characterize the 4-connected forbidden pairs that imply a graph is pancyclic?

2. Are there pairs not involving the claw?

Recently, with Chen, Egawa, and Saito we showed the following:

Theorem

For $k \geq 2$, every k-connected $\{K_{1,k+1}, P_4\}$-free graph is either pancyclic or isomorphic to $K_{k,k}$.

Theorem

Each forbidden pair for a k-connected graph contains $K_{1,s}$ for some $s \leq k+1$.

Hamiltonian Connected

Definition

A graph G is called hamiltonian connected if any pair of distinct vertices are joined by a spanning path.

Theorem

(F.B. Shepard, 1991)
If G is a 3-connected claw and net free graph, then G is hamiltonian connected.

Theorem

Chen and RG, 2000

If G is a 3-connected claw-free graph, then G is hamiltonian connected if any of the following holds.

1. *G is $N(3,0,0)$-free,*
2. *G is P_6-free*
3. *G is $N(2,1,0)$-free.*

Theorem

Faudree and RG

Let X and Y be connected graphs ($\neq P_3$) and let G be a 3-connected graph. If G being $\{X, Y\}$-free implies G is hamiltonian connected, then $X = K_{1,3}$ and Y satisfies each of the following:

1. $\Delta(Y) \leq 3$,
2. *A longest induced path in Y has order at most 12,*
3. *Y contains no cycles of length at least 4,*
4. *All triangles in Y are vertex disjoint,*
5. *Y is claw-free.*

Theorem

Broersma, Faudree, Huck, Trommel and Veldman, 2002

Let X and Y be connected graphs ($\neq P_3$) and let G be a 3-connected graph. If G being $\{X, Y\}$-free implies G is hamiltonian connected, then $X = K_{1,3}$ and Y satisfies each of the following:

1. $\Delta(Y) \leq 3$,
2. *A longest induced path in Y has order at most 9,*
3. *Y contains no cycles of length at least 4,*
4. *The distance between triangles in Y is either 1 or at least 3,*
5. *There are at most 2 triangles in Y,*
6. *Y is claw-free.*

They also showed

Theorem

If G is a 3-connected claw-free and L-free graph, then G is hamiltonian connected.

The previous results imply the remaining possible forbidden graphs:

1. P_k with $k \leq 9$,
2. $N(i, j, k)$ with some restrictions on how large i, j, k can be,
3. L_k $(k \neq 2)$, (like L only the triangles are joined by a path of length k so $L = L_1$),
4. L_k $(k \neq 2)$ with tree components attached to either of the triangles.

with Bian, Horn, Janiszewski, La Fleur and Wrayno

Theorem

If G being 3-connected, claw and L_k-free implies G is hamiltonian connected, then $k \leq 6$ and k is odd, and there are no tree components attached to either triangle.

Theorem

If G being 3-connected claw and $N(i,j,k)$-free implies G is hamiltonian connected, then $i + j + k \leq 7$.

Theorem

If G is 3-connected and claw and P_9-free, then G is hamiltonian connected.