"The Book proof" of Vizing's Generalized Theorem and Shannon's Theorem (proof obtained from B. Toft)

Let G be a multigraph and let $k \geq \Delta(G)$. Let ϕ be a k-edge coloring of $G-e$ for some $e \in E(G)$. Assume that G is not k-edge colorable.

For a vertex v, let $\phi(v)$ be the set of colors of ϕ present at vertex v. Similarly, let $\overline{\phi}(v)$ be the set of colors of ϕ not present at v.

A *fan* F_x is an ordered sequence of edges $(e_1, e_2, \ldots, e_n)$ at vertex x such that for every j, $2 \leq j \leq n$, there exists an i, $1 \leq i \leq j-1$ such that $\phi(e_j) \in \overline{\phi}(y_i)$.

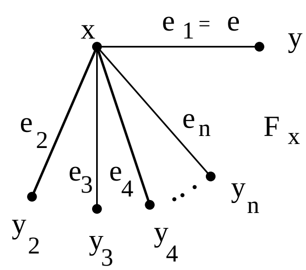


Figure 1: The Fan F_x.

Claim 1: In a fan F_x, $\overline{\phi}(y_j) \cap \overline{\phi}(x) = \emptyset$ for all j, $1 \leq j \leq n$.

Proof: Assume this is not the case. Choose the fan F_x and coloring ϕ such that $\overline{\phi}(y_j) \cap \overline{\phi}(x) \neq \emptyset$ with j as small as possible. Let $\alpha \in \overline{\phi}(y_j) \cap \overline{\phi}(x)$.

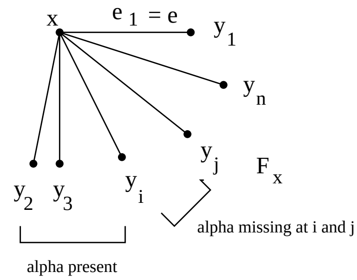
If $y_j = y_1$, then color α is missing at both x and y_1. Then e can be colored α ($e_1 = e$ and $G-e$ is k-edge colorable) and G is k-edge colorable. Since this is not the case, $y_j \neq y_1$.

Let $\beta = \phi(e_j)$ Then there is an i, $1 \leq i \leq j-1$ such that $\beta \in \overline{\phi}(y_i)$. Recolor e_j with the color α. The result is a new k-edge coloring ϕ' of $G-e$. Then, $(e_1, e_2, \ldots, e_i)$ is a fan with respect to ϕ' and $\overline{\phi'}(y_i) \cap \overline{\phi'}(x) \neq \emptyset$ since β is in this intersection. This contradicts the minimality of j and completes the proof of the claim.
□

Claim 2: In a fan $\overline{\phi}(y_i) \cap \overline{\phi}(y_j) = \emptyset$ for all i and j where $y_i \neq y_j$.

Proof: Assume this is not the case. Choose the fan F_x and coloring ϕ such that $\overline{\phi}(y_i) \cap \overline{\phi}(y_j) \neq \emptyset$ with $y_i \neq y_j$, and with i as small as possible and subject to this $j-i$ as small as possible.

Let $\alpha \in \overline{\phi}(y_i) \cap \overline{\phi}(y_j)$. Let $\beta \in \overline{\phi}(x)$. Such a β exists since $k \geq \Delta(G)$ and there is an uncolored edge at x. By Claim 1, $\beta \in \phi(y_h)$ for all h and $\alpha \in \phi(x)$.



For $1 \leq h \leq n$ let P_h denote the alternating $\alpha - \beta$ chain containing y_h.

Case 1: Suppose $x \notin P_i$.

Change α and β on P_i and obtain ϕ'. The color β is then missing at y_i and at x. Then $(e_1, e_2, \ldots, e_i)$ is a fan with respect to ϕ', contradicting Claim 1. □

Case 2: Suppose $x \in P_i$ and $x \notin P_j$.

Change color α and β on P_j and obtain ϕ'. The color β is then missing at y_j and x. Then $(e_1, e_2, \ldots, e_j)$ is a fan with respect to ϕ', contradicting Claim 1. □

Case 3: Suppose $x \in P_i$ and $x \in P_j$.

Then $P_i = P_j$ and x, y_i, y_j all have degree 1 in P_i. This is impossible. □

Let F_x be maximal. Let $\phi(\overline{F}_x)$ be the colors of ϕ at x not in the fan F_x.

Claim 3: $\phi(\overline{F}_x) \cap \overline{\phi}(y_i) = \emptyset$ for all i, $1 \leq i \leq n$.

Proof: This follows directly from F_x being maximal and the definition of a fan. □

Now let $z_1 (= y_1), z_2, \ldots, z_m$ be the different y_i (recall we are in a multigraph so y's may be repeated) $(2 \leq m \leq n)$. Claims 1, 2, 3 imply that $\overline{\phi}(z_1), \overline{\phi}(z_2), \ldots, \overline{\phi}(z_m), \overline{\phi}(x)$ and $\phi(\overline{F}_x)$ are disjoint subsets of the set of k colors of ϕ. Hence,

$$|\overline{\phi}(z_1)| + |\overline{\phi}(z_2)| + \ldots + |\overline{\phi}(z_m)| + |\overline{\phi}(x)| + |\phi(\overline{F}_x)| \leq k.$$

Hence,

$$k - (deg(z_1) - 1) + (k - (deg(z_2)) + \ldots + (k - deg(z_m)) + (k - (deg(x) - 1)) + (deg(x) - 1 - (n - 1)) \leq k.$$

Thus,

$$k(m+1) + 2 - n - (\sum_{i=1}^{m} deg(z_i)) \leq k$$

or

$$2 \leq (\sum_{i=1}^{m} deg(z_i)) + n - mk.$$

If $\mu(x, z_i)$ denotes the number of edges between x and z_i, then $n \leq \sum_{i=1}^{m} \mu(x, z_i)$. With this the following inequality holds:

$$(*) \;\; 2 \leq \sum_{i=1}^{m} (deg(z_i) + \mu(x, z_i) - k)$$

with $m \geq 2$.

From (*) we get the following:

A. There exists a z_i such that $deg(z_i) + \mu(x, z_i) - k \geq 1$.

B. There exists $z_i, z_j (z_i \neq z_j)$ such that

$$deg(z_i) + deg(z_j) + \mu(x, z_i) + \mu(x, z_j) - 2k \geq 2.$$

Further, since $deg(x) \geq \mu(x, z_i) + \mu(x, z_j)$, B implies:

C. There exists z_i, z_j $(z_i \neq z_j)$ such that

$$deg(z_i) + deg(z_j) + deg(x) - 2k \geq 2.$$

If $k \geq \Delta(G) + \mu(G)$, where $\mu(G)$ is the max. multiplicity of G, then A gives a contradiction. Hence the assumption that G is not k-edge colorable must be wrong and Vizing's theorem holds.

Further note: If $k \geq \lfloor \frac{3}{2}\Delta(G) \rfloor$, then C (B) gives a contradiction. Hence, again the assumption that G is not k-edge colorable must be wrong. From this we conclude:

Thm: G is $\Delta(G) + \mu(G)$ edge colorable. (generalized Vizing, 1964)

Thm: G is $\frac{3}{2}\Delta(G)$- edge colorable. (Shannon, 1949).