ALTERNATING "STRANGE" FUNCTIONS

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ABSTRACT. In this note we consider infinite series similar to the "strange" function $F(q)$ of Kontsevich studied by Zagier, Bryson-Ono-Pitman-Rhoades, Bringmann-Folsom-Rhoades, Rolen-Schneider, and others in connection to quantum modular forms. We show that a class of "strange" alternating series that are well-defined almost nowhere in the complex plane can be added (using a modified definition of limits) to familiar infinite products to produce convergent q-hypergeometric series, of a shape that specializes to Ramanujan's mock theta function $f(q)$, Zagier's quantum modular form $\sigma(q)$, and other interesting number-theoretic objects. We also give Cesàro sums for these "strange" series.

1. Introduction and statement of results

In a 1997 lecture at the Max Planck Institute for Mathematics, Fields medalist Maxim Kontsevich discussed an almost nonsensical q-hypergeometric series [10]

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n, \tag{1}$$

where the *q-Pochhammer symbol* is defined by $(a;q)_0 := 1$, $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$, and $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$ for $a, q \in \mathbb{C}, |q| < 1$. This series $F(q)$ is often referred to in the literature as *Kontsevich's "strange" function*, and has since been studied deeply by Zagier [10] — it was one of his prototypes for quantum modular forms, which enjoy beautiful transformations similar to classical modular forms, and also resemble objects in quantum theory [9] — as well as by other authors [3, 4, 8] in connection to quantum modularity, unimodal sequences, and other topics.

There are many reasons to say the series (1) is "strange" (see [10]). For brevity, let us merely note that as $n \to \infty$, then $(q;q)_n$ converges on the unit disk, is essentially singular on the unit circle (except at roots of unity, where it vanishes), and diverges when $|q| > 1$. Thus $\sum_{n\geq 0}(q;q)_n$ converges almost nowhere in the complex plane. However, at $q = \zeta_m$ an mth order root of unity, F is suddenly very well-behaved: because $(\zeta_m;\zeta_m)_n = 0$ for $n \geq m$, then as $q \to \zeta_m$ radially, $F(\zeta_m) := \lim_{q\to\zeta_m} F(q)$ is just a polynomial in $\mathbb{Z}[\zeta_m]$.

Now let us turn our attention to the alternating case of this series, viz.

$$\widetilde{F}(q) := \sum_{n=0}^{\infty} (-1)^n (q;q)_n, \tag{2}$$

a summation that has been studied by Cohen [4], which is similarly "strange": it doesn't converge anywhere in $\mathbb{C}$ except at roots of unity, where it is a polynomial. In fact, computational examples suggest the odd and even partial sums of $\widetilde{F}(q)$ oscillate asymptotically between two convergent q-series.

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To capture this oscillatory behavior, let us adopt a notation we will use throughout. If S is an infinite series, we will write S_+ to denote the limit of the sequence of odd partial sums, and S_- for the limit of the even partial sums, if these limits exist (clearly if S converges, then $S_+ = S_- = S$).

Interestingly, like $F(q)$, the "strange" series $\widetilde{F}(q)$ is closely connected to a sum Zagier provided as another prototype for quantum modularity (when multiplied by $q^{1/24}$) [9], the function

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n} = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (q;q)_n \tag{3}$$

from Ramanujan's "lost" notebook, with the right-hand equality due to Andrews [2]. If we use the convention introduced above and write $\widetilde{F}_+(q)$ (resp. $\widetilde{F}_-(q)$) to denote the limit of the odd (resp. even) partial sums of $\widetilde{F}$, we can state this connection explicitly, depending on the choice of "+" or "−".

Theorem 1. *For $0 < |q| < 1$ we have*

$$\sigma(q) = 2\widetilde{F}_{\pm}(q) \pm (q;q)_\infty.$$

We can make further sense of alternating "strange" series such as this using Cesàro summation, a well-known alternative definition of the limits of infinite series (see [7]).

Definition 2. *The Cesàro sum of an infinite series is the limit of the arithmetic mean of successive partial sums, if the limit exists.*

In particular, it follows immediately that the Cesàro sum of the series S is the average $\frac{1}{2}(S_+ + S_-)$ if the limits S_+, S_- exist. Then Theorem 1 leads to the following fact.

Corollary 3. *We have that $\frac{1}{2}\sigma(q)$ is the Cesàro sum of the "strange" function $\widetilde{F}(q)$.*

A similar relation to Theorem 1 involves Ramanujan's prototype $f(q)$ for a mock theta function

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(-q;q)_n}, \tag{4}$$

the right-hand side of which is due to Fine (see (26.22) in [6], Ch. 3). Now, if we define

$$\widetilde{\phi}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(-q;q)_n}, \tag{5}$$

which is easily seen to be "strange" like the previous cases, and write $\widetilde{\phi}_\pm$ for limits of the odd/even partial sums as above, we can write $f(q)$ in terms of the "strange" series and an infinite product.

Theorem 4. *For $0 < |q| < 1$ we have*

$$f(q) = 2\widetilde{\phi}_\pm(q) \pm \frac{1}{(-q;q)_\infty}.$$

Again, the Cesàro sum results easily from this theorem.

Corollary 5. *We have that $\frac{1}{2}f(q)$ is the Cesàro sum of the "strange" function $\widetilde{\phi}(q)$.*

Theorems 1 and 4 typify a general phenomenon: the combination of an alternating Kontsevich-style "strange" function with a related infinite product is a convergent q-series when we fix the $\pm$ sign in this modified definition of limits. Let us fix a few more notations in order to discuss this succinctly. As usual, we write

$$(a_1, a_2, ..., a_r; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n,$$

along with the limiting case $(a_1, a_2, ..., a_r; q)_\infty$ as $n \to \infty$. Associated to the sequence $a_1, a_2, ..., a_r$ of complex coefficients, we will define a polynomial $\alpha_r(X)$ by the relation

$$(1 - a_1 X)(1 - a_2 X) \cdots (1 - a_r X) =: 1 - \alpha_r(X) X, \tag{6}$$

thus

$$(a_1 q, a_2 q, ..., a_r q; q)_n = \prod_{j=1}^{n} (1 - \alpha_r(q^j) q^j), \tag{7}$$

and we follow this convention in also writing $(1 - b_1 X)(1 - b_2 X) \cdots (1 - b_s X) =: 1 - \beta_s(X) X$ for complex coefficients $b_1, b_2, ..., b_s$. Moreover, we define a generalized alternating "strange" series:

$$\widetilde{\Phi}(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q) := \sum_{n=0}^{\infty} (-1)^n \frac{(a_1 q, a_2 q, ..., a_r q; q)_n}{(b_1 q, b_2 q, ..., b_s q; q)_n} \tag{8}$$

Thus $\widetilde{F}(q)$ is the case $\widetilde{\Phi}(1; 0; q)$, and $\widetilde{\phi}(q)$ is the case $\widetilde{\Phi}(0; -1; q)$. We note that if q is a kth root of $1/a_i$ for some i, then $\widetilde{\Phi}$ truncates after k terms like F and $\widetilde{F}$. As above, let $\widetilde{\Phi}_\pm$ denote the limit of the odd/even partial sums; then we can encapsulate the preceding theorems in the following statement.

Theorem 6. *For $0 < |q| < 1$ we have*

$$2\widetilde{\Phi}_\pm(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q) \pm \frac{(a_1 q, a_2 q, ..., a_r q; q)_\infty}{(b_1 q, b_2 q, ..., b_s q; q)_\infty}$$
$$= 1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^n \left(\alpha_r(q^n) - \beta_s(q^n)\right) (a_1 q, a_2 q, ..., a_r q; q)_{n-1}}{(b_1 q, b_2 q, ..., b_s q; q)_n}.$$

From this identity we can fully generalize the previous corollaries.

Corollary 7. *We have that $1/2$ times the right-hand side of Theorem 6 is the Cesàro sum of the "strange" function $\widetilde{\Phi}(a_1, ..., a_r; b_1, ..., b_s; q)$.*

The takeaway is that the Nth partial sum of an alternating "strange" series oscillates asymptotically as $N \to \infty$ between $\frac{1}{2}(S(q) + (-1)^N P(q))$, where S is an Eulerian infinite series and P is an infinite product as given in Theorem 6. We recover Theorem 1 from Theorem 6 as the case $a_1 = 1,\ a_i = b_j = 0$ for all $i > 1, j \geq 1$. Theorem 4 is the case $b_1 = -1,\ a_i = b_j = 0$ for all $i \geq 1, j > 1$.

Considering these connections together with diverse connections made by Kontsevich's $F(q)$ to important objects of study [3, 4, 10], it seems the ephemeral "strange" functions almost "enter into mathematics as beautifully"[1] as their convergent relatives, mock theta functions.

[1] To redirect Ramanujan's words

2. Proofs of results

In this section we quickly prove the preceding theorems, and justify the corollaries.

Proof of Theorem 1. Using telescoping series to find that

$$(q;q)_\infty = 1 - \sum_{n=0}^{\infty} (q;q)_n \left(1 - (1 - q^{n+1})\right) = 1 - \sum_{n=0}^{\infty} q^{n+1} (q;q)_n,$$

and combining this functional equation with the right side of (3) above, easily gives

$$\sigma(q) - (q;q)_\infty = 2 \sum_{n=0}^{\infty} q^{2n+1} (q;q)_{2n}.$$

On the other hand, manipulating symbols heuristically (for we are working with a divergent series $\widetilde{F}$) suggests we can rewrite

$$\widetilde{F}(q) = \sum_{n=0}^{\infty} \left((q;q)_{2n} - (q;q)_{2n+1}\right) = \sum_{n=0}^{\infty} (q;q)_{2n} \left(1 - (1 - q^{2n+1})\right) = \sum_{n=0}^{\infty} q^{2n+1} (q;q)_{2n},$$

which is a rigorous statement if by convergence on the left we mean the limit as $N \to \infty$ of partial sums $\sum_{n=0}^{2N-1} (-1)^n (q;q)_n$. We can also choose the alternate coupling of summands to similar effect, e.g. considering here the partial sums $1 + \sum_{n=1}^{N-1} [(q;q)_{2n} - (q;q)_{2n-1}] - (q;q)_{2N-1}$ as $N \to \infty$. Combining the above considerations proves the theorem for $|q| < 1$, which one finds to agree with computational examples. □

Proof of Theorem 4. Following the formal steps that prove Theorem 1 above, we can use

$$\frac{1}{(-q;q)_\infty} = 1 - \sum_{n=0}^{\infty} \frac{1}{(-q;q)_n} \left(1 - \frac{1}{1 + q^{n+1}}\right) = 1 - \sum_{n=1}^{\infty} \frac{q^n}{(-q;q)_n}$$

and rewrite the related "strange" series

$$\widetilde{\phi}(q) = \sum_{n=0}^{\infty} \frac{1}{(-q;q)_{2n}} \left(1 - \frac{1}{1 + q^{2n+1}}\right) = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(-q;q)_{2n+1}},$$

which of course fails to converge for $0 < |q| < 1$ on the left-hand side but makes sense if we use the modified definition of convergence in Section 1, to yield the identity in the theorem (which is, again, borne out by computational examples). □

Proof of Theorem 6. Using the definitions of the polynomials $\alpha_r(X), \beta_s(X)$, then following the exact steps that yield Theorems 1 and 4, i.e., manipulating and comparing telescoping-type series with the same modified definition of convergence, gives the theorem. □

Proof of corollaries. Clearly, for an alternating "strange" series in which the odd and even partial sums each approach a different limit, the average of these two limits will equal the Cesàro sum of the series. □

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