FAÀ DI BRUNO'S FORMULA WITH PRODUCT REPRESENTATION

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Abstract. In this expository note we give a quick proof of Faà di Bruno's formula—framed in terms of integer partitions—using the multinomial theorem, and adjoin an infinite product representation to the classical power series identity. As examples, we give an identity related to the Riemann zeta function, as well as formulas for Ramanujan's tau function and the counting function for k-color partitions.

Francesco Faà di Bruno was an Italian priest and mathematician active in the mid-nineteenth century. For the convenience of fellow students, we record an easy proof of the useful formula that bears his name [2], and also adjoin an infinite product representation to the usual statement of the identity, based on elementary ideas. We follow up with a few examples related to objects from number theory.

First we must fix some notations. Let $\mathcal{P}$ denote the set of all integer partitions, including the empty partition $\emptyset$, and let $\lambda = (1^{m_1}\, 2^{m_2}\, 3^{m_3} \ldots\, k^{m_k} \ldots)$ denote a given partition with m_k representing the *multiplicity* of k as a part of $\lambda \in \mathcal{P}$, noting that λ has only finitely many nonzero m_k. Then we take $|\lambda|$ to be the *size* of λ (i.e., the number being partitioned) with the convention $|\emptyset| := 0$, take $\ell(\lambda) = m_1 + m_2 + m_3 + \ldots$ to be its *length* (i.e., the number of parts) with $\ell(\emptyset) := 0$, and we write "$\lambda \vdash n$" to mean λ is a partition of n. We let $q \in \mathbb{C}$, $|q| < 1$ unless noted otherwise, and define the usual exponential function $\exp(x) := e^x$.

Then we can write Faà di Bruno's identity (in a slightly simplified, equivalent form) as a sum over all partitions λ, and add in a product identity too.

Proposition 1 (Faà di Bruno's formula with product representation)**.** *For $a(n)$ an arithmetic function and $a_n \in \mathbb{C}$, we have*

$$\exp\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} \ldots}{m_1!\; m_2!\; m_3!\; \ldots} = \prod_{n=1}^{\infty} (1-q^n)^{a(n)},$$

where a_n and $a(n)$ are related by

$$a_n = -\frac{1}{n}\sum_{d|n} a(d)d, \qquad a(n) \;=\; -\frac{1}{n}\sum_{d|n}\mu(n/d)a_d d$$

with the sums taken over divisors of n, and μ being the classical Möbius function.

Proof. To prove the first equality, we begin with the well-known *multinomial theorem*, re-written as a sum over partitions λ in the set $\mathcal{P}_{[k]} \subset \mathcal{P}$ whose parts are all $\leq k$, having length $\ell(\lambda) = n$:

$$(a_1 + a_2 + a_3 + \ldots + a_k)^n = n! \sum_{\substack{\lambda \in \mathcal{P}_{[k]} \\ \ell(\lambda)=n}} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} \ldots a_k^{m_k}}{m_1!\; m_2!\; m_3!\; \ldots\; m_k!} \tag{1}$$

If we let k tend to infinity, assuming the infinite sum $a_1 + a_2 + a_3 + \ldots$ converges, the series on the right becomes a sum over all partitions of length n. Then dividing both sides of (1) by $n!$ and summing over $n \geq 0$, the left-hand side yields the Maclaurin series expansion for $\exp(a_1 + a_2 + a_3 + ...)$, and the right side can be rewritten as a sum over all partitions:

$$\exp(a_1 + a_2 + a_3 + ...) = \sum_{\lambda \in \mathcal{P}} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} ...}{m_1!\ m_2!\ m_3!\ ...} \tag{2}$$

Now, taking $a_k \mapsto a_k q^k$ in (2), we can write

$$\begin{aligned} a_1^{m_1} a_2^{m_2} a_3^{m_3} ... \mapsto\ & (a_1 q)^{m_1} (a_2 q^2)^{m_2} (a_3 q^3)^{m_3} ... \\ =& q^{m_1 + 2m_2 + 3m_3 + ...} a_1^{m_1} a_2^{m_2} a_3^{m_3} ... = q^{|\lambda|} a_1^{m_1} a_2^{m_2} a_3^{m_3} ... \end{aligned}$$

in the summands on the right-hand side, which completes the series aspect of the proof.

The product representation follows from Bruinier, Kohnen and Ono [1], and also is immediate from the proofs in [4] if we replace $-f(n)/n$ with an arithmetic function $a(n)$ in the final equation of that paper. For a given $a(n)$, if we set

$$a_n \ = \ -\frac{1}{n} \sum_{d|n} d \cdot a(d), \tag{3}$$

we have

$$\prod_{n=1}^{\infty} (1 - q^n)^{a(n)} = \exp\left(\sum_{n=1}^{\infty} a_n q^n \right).$$

Applying Möbius inversion to (3) gives the converse divisor sum identity for $a(n)$.

□

So we can view Faà di Bruno's formula as a generating function for coefficients of certain partition-theoretic sums involving the form $(a_1^{m_1} a_2^{m_2} a_3^{m_3} ...)/(m_1!\ m_2!\ m_3!\ ...)$. As examples, we give a few simple substitutions that lead to interesting partition sum identities. In what follows, let $(a;q)_\infty := \prod_{n \geq 0} (1 - aq^n)$ denote the infinite *q-Pochhammer symbol*, and $n_\lambda := \prod_{k=1}^{\infty} k^{m_k}$ denote the so-called *integer* of λ (the product of the parts) in the notation of [5, 6], with $n_\emptyset := 1$.

Example 2. *Setting $a_i = i^{-s}, \mathrm{Re}(s) > 1$, and $q = 1$ in Proposition 1, gives*

$$\exp\left(\zeta(s)\right) = \sum_{\lambda \in \mathcal{P}} \frac{1}{n_\lambda^s\ m_1!\ m_2!\ m_3!\ ...}$$

with $\zeta(s) := \sum_{n \geq 1} n^{-s}$ the Riemann zeta function, the right-hand side of which resembles the "partition zeta functions" in [3, 6].

Example 3. *Setting $a \equiv 24$ in Proposition 1 yields $a_i = -24\sigma(i)$, where $\sigma(i) = \sum_{d|i} d$ as usual. Then Ramanujan's tau function $\tau(n)$, which is the nth coefficient of $q(q;q)_\infty^{24}$, can be written*

$$\tau(n) = \sum_{\lambda \vdash (n-1)} (-24)^{\ell(\lambda)} \frac{\sigma(1)^{m_1} \sigma(2)^{m_2} \sigma(3)^{m_3} ...}{n_\lambda\ m_1!\ m_2!\ m_3!\ ...}.$$

Example 4. *Setting $f \equiv -k$ with $k \geq 1$ in Proposition 1 yields $\delta_i = k\sigma(i)$. Then the number $P_k(n)$ of k-color partitions of n, which is the nth coefficient of $(q;q)_\infty^{-k}$, can be written*

$$P_k(n) = \sum_{\lambda \vdash n} k^{\ell(\lambda)} \frac{\sigma(1)^{m_1} \sigma(2)^{m_2} \sigma(3)^{m_3} ...}{n_\lambda\ m_1!\ m_2!\ m_3!\ ...}.$$

References

[1] J. H. Bruinier, W. Kohnen, and K. Ono. *The arithmetic of the values of modular functions and the divisors of modular forms*. Composition Math. **140** (2004), 552-566.

[2] F. Faà di Bruno, *Sullo sviluppo delle funzioni*, Annali di Scienze Matematiche e Fisiche **6** (1855): 479-480.

[3] K. Ono, L. Rolen and R. Schneider, *Explorations in the theory of partition zeta functions*, Exploring the Riemann Zeta Function 190 years from Riemann's Birth, Springer, 2016.

[4] R. Schneider, *A Golden Product Identity for e*, Mathematics Magazine **87.2** (2014): 132-134.

[5] R. Schneider, *Arithmetic of partitions and the q-bracket operator*, Proceedings of the American Mathematical Society (2016).

[6] R. Schneider, *Partition zeta functions*, Research in Number Theory **2:9** (2016).

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