Jacobi's triple product, mock theta functions and the q-bracket

Robert P. Schneider, Emory University

January 4, 2017

Sea change

Ramanujan set sail from Madras for England in 1914.

Sea change

In Europe, a revolution in physics was underway:

- Quantum theory (Bohr, 1913)
- General relativity (Einstein, 1915)
- Black holes (Schwarzschild, 1916)

Ramanujan's creative flights were expressions of the *Zeitgeist* of his era.

Ramanujan's enigma

In his final letter, Ramanujan told of an exciting discovery:

mock theta functions

- List of curious interrelated q-series
- "Enter into mathematics as beautifully" as classical modular forms
- No proofs (or even a hint)

New horizons

Take $q, z \in \mathbb{C}$.

Universal mock theta function

Gordon-McIntosh (2011): For $|q| < 1, z \neq 0$, we define

$$g_3(z, q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(z;q)_n (z^{-1}q;q)_n}.$$

- $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$ (q-Pochhammer symbol)
- Specializes to all of Ramanujan's MTFs

Space-time storm

Mock theta functions and mock modular forms connected to physics of *black holes*

Jacobi triple product formula

For $|q| < 1,\ z \neq 0$:

$$j(z;q) := (z;q)_\infty (z^{-1}q;q)_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\frac{n(n-1)}{2}}$$

- Foundational in number theory (q-series, mod. forms), combinatorics, physics

Quantum sea

JTP proved via *bosons* and *fermions* (Borcherds)

Big questions

- Mock theta functions and black holes?
- Jacobi's triple product and quantum physics??
- What is the connection between *q*-series and phenomena at the extremes of nature???

Big answer

??

Big questions

- Mock theta functions and black holes?
- Jacobi's triple product and quantum physics??
- Is there a connection between q-series and phenomena at the extremes of nature???

Smaller answer

There is a connection between MTFs and the JTP via an operator from statistical physics...

The q-bracket of Bloch–Okounkov

Let $\mathcal{P}$ denote the integer partitions ($3+2$, $7+1+1$, etc.).

q-bracket operator

For $|q| < 1, f : \mathcal{P} \to \mathbb{C}$, Bloch–Okounkov (2000) define:

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} = (q;q)_\infty \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}$$

Statistical interpretation

- Average (i.e. expected) value of f over all partitions
- Appropriate choice of q, e.g. $q = e^{-1/k_B T}$

The q-bracket of Bloch–Okounkov

Let $\mathcal{P}$ denote the integer partitions ($3+2$, $7+1+1$, etc.).

q-bracket operator

For $|q| < 1, f : \mathcal{P} \to \mathbb{C}$, Bloch–Okounkov (2000) define:

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} = (q;q)_\infty \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}$$

Deep roots in number theory

- Connected to quasi-MFs, p-adic MFs
- Natural role in partition theory

Connecting the UMTF to the JTP

For $|q| < 1, z \neq 0$, recall the UMTF

$$g_3(z, q) = \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(z; q)_n (z^{-1}q; q)_n}$$

and the Jacobi triple product

$$j(z; q) = (z; q)_\infty (z^{-1}q; q)_\infty (q; q)_\infty.$$

- Note the resemblance in the q-Pochhammer symbols.

Connecting the UMTF to the JTP

We write the reciprocal of $j(z;q)$ as a sum over partitions:

$$j(z;q)^{-1} = \frac{1}{(z;q)_\infty (z^{-1}q;q)_\infty (q;q)_\infty} =: \sum_{\lambda\in\mathcal{P}} j_z(\lambda) q^{|\lambda|}$$

Then the q-bracket of $j_z(\lambda)$ is

$$\langle j_z \rangle_q = (q;q)_\infty \sum_{\lambda\in\mathcal{P}} j_z(\lambda) q^{|\lambda|} = \frac{1}{(z;q)_\infty (z^{-1}q;q)_\infty}.$$

- Average value of j_z over all partitions (dep. on q)
- Modular function (up to mult. by power of q) for $z \neq 1$ a root of unity, $q := e^{2\pi i\tau}, \tau \in \mathbb{H}$

Connecting g_3 to the JTP

Theorem (S.)

For $0 < |q| < 1 \ll |z|$ we have:

$$\langle j_z \rangle_q \sim z(1-q) g_3(z^{-1}, q^{-1}) \text{ as } |z| \to \infty$$

For $0 < |q| < 1, 0 < |z| \ll 1$ we have:

$$\langle j_z \rangle_q \sim z^{-1} q \, g_3(z^{-1}, q^{-1}) \text{ as } |z| \to 0$$

- For z *away* from the unit circle, $g_3(z^{-1}, q^{-1})$ (times linear factors) gives avg. value of $j_z(\lambda)$.

Connecting g_3 to the JTP

Theorem (continued)

In general, for $0 < |q| < 1$, $z \neq 0$ or 1:

$$\langle j_z \rangle_q = \left[z(1-q) + z^{-1}q \right] \; g_3(z^{-1}, q^{-1}) + \frac{zq^2}{1-z} \widetilde{U}(z,q)$$

- $\widetilde{U}$ is the rank gen. function for weakly unimodal seq. (and first item printed from "lost" notebook)
- For z *near* the unit circle, $\widetilde{U}$ more prominent
- For $z \neq 1$ a root of unity, g_3 and $\widetilde{U}$ work together to produce modularity of the q-bracket

Connecting g_3 to the JTP

Theorem (continued)

In general, for $0 < |q| < 1, z \neq 0$ or 1:

$$\langle j_z \rangle_q = \left[z(1-q) + z^{-1}q \right] \; g_3(z^{-1}, q^{-1}) + \frac{zq^2}{1-z} \widetilde{U}(z,q)$$

What up with the “inverted” UMTF?

- $g_3(z^{-1}, q)$ converges when $|q| > 1, z \neq 0$
- “Entangled” with a dual q-series conv. when $|q^{-1}| < 1$

Jacobi triple product and mock theta functions

- Associated to opposite physical extremes
- Subatomic (JTP) and supermassive (MMFs)
- Intertwined via *q*-bracket from statistical physics
- Stat. phys. applies to phenomena at every scale

Big question (again)

What is the connection between *q*-series and phenomena at the extremes of nature?

Gratitude

Thank you to the session organizers for having me to speak, and to **you** for listening :)