Partition Zeta Functions

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My muse

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- q-series

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- partitions

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- zeta function

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- zeta function
- product-sum formulas

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Euler product formula

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}, \ \mathrm{Re}(s) > 1$$

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Explicit zeta values

$$\zeta(2N) := \pi^{2N} \times \textit{rational number}$$

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Another Eulerian product-sum identity

$$\prod_{n=0}^{\infty}(1-zq^n)^{-1} = 1 + zq^{-1}\sum_{n=1}^{\infty}\frac{q^n}{\prod_{k=0}^{n-1}(1-zq^k)}$$

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Partition generating function

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty}p(n)q^n$$

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- These two identities are specializations of a single product-sum identity.

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- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, denote a generic partition, $l(\lambda) := r$ denote its *length*, $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ denote its *size*.

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- For the empty partition $\emptyset$, let $l(\emptyset) = |\emptyset| = 0, n_\emptyset = 1$.

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We consider sums of the shape

$$\sum_{\lambda \in \mathcal{P}'} \phi(\lambda), \quad \mathcal{P}' \subseteq \mathcal{P}, \quad \phi : \mathcal{P}' \to \mathbb{C}.$$

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Example

$$\sum_{\lambda \in \mathcal{P}_n} \mathrm{rk}(\lambda) = 0$$

where $\mathcal{P}_n$ denotes partitions of n, $\mathrm{rk}(\lambda)$ is the rank of λ.

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- $\varphi(z;q)$ reduces to $(zq;q)_\infty$ if $f = z \in \mathbb{C}$.

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We intend "$\lambda_i \in \lambda$" to mean λ_i is a part of $\lambda \in \mathcal{P}$.

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Similar to Euler's proof of the partition generating function identity.

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For convenience, call the summation on the far right $\sum_A$.

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Remark

The RHS converges for q^{-1} outside the unit circle.

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Proof

Substitute $\varphi_n(f;q) = (-1)^n q^{n(n+1)/2} \varphi_n(1/f;q^{-1}) \prod_{k=1}^{n} f(k)$ into $\sum_A$ above.

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$\sum_A$, $\sum_B$ are the summations from above.

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Inspired by continued fraction $1 + \frac{1}{1+\frac{1}{1+\dots}}$ for golden ratio.

Theorem 1 (All Together Now)

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$$\frac{1}{\varphi(f;q)} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{\lambda_i \in \lambda} f(\lambda_i) = 1 + \sum_{n=1}^{\infty} q^n \frac{f(n)}{\varphi_n(f;q)}$$

$$= 1 + \frac{1}{\varphi(f;q)} \sum_{n=1}^{\infty} q^n \, f(n) \varphi_{n-1}(f;q)$$

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\varphi(f;q) &= \sum_{\lambda \in \mathcal{P}*} (-1)^{|\lambda|} q^{|\lambda|} \prod_{\lambda_i \in \lambda} f(\lambda_i) \\
&= 1 - \sum_{n=1}^{\infty} q^n \, f(n) \varphi_{n-1}(f;q) \\
&= 1 - \varphi(f;q) \sum_{n=1}^{\infty} q^n \frac{f(n)}{\varphi_n(f;q)} = 1 - \cfrac{\sum_A}{1 + \cfrac{\sum_B}{1 - \cfrac{\sum_A}{1 + \cfrac{\sum_B}{1 - \dots}}}}
\end{aligned}$$

Famous Consequences

Recall from above

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We have from Theorem 1

$$\prod_{n=1}^{\infty}(1 - f(n)q^n)^{-1} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{\lambda_i \in \lambda} f(\lambda_i).$$

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Corollary 2

Taking $q = 1, X = \mathbb{P}, f(n) := n^{-s}$ gives the Euler product formula for $\zeta(s), \mathrm{Re}(s) > 1$.

Formula for $\zeta(s)$

Corollary 3

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It also follows from Theorem 1 that

$$\zeta(s) = 1 + \sum_{p \in \mathbb{P}} \frac{1}{p^s \prod_{\substack{r \in \mathbb{P} \\ r \leq p}} (1 - r^{-s})}.$$

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where n_λ is the so-called *integer* of λ (the product of its parts), and $\mathcal{P}_{\mathbb{P}}$ is the prime partitions.

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Question

Do nice phenomena arise if we use other subsets of $\mathcal{P}$?

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- $\zeta_{\mathcal{P}}(s)$ doesn't converge... let's see some cases that do.

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Summing over partitions into even parts, we have

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$$\zeta_{\mathcal{P}_{2\mathbb{Z}}}(2) := \sum_{\lambda \in \mathcal{P}_{2\mathbb{Z}}} n_\lambda^{-2} = \frac{\pi}{2}.$$

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$$\zeta_{\mathcal{P}_{2\mathbb{Z}}}(2) := \sum_{\lambda \in \mathcal{P}_{2\mathbb{Z}}} n_\lambda^{-2} = \frac{\pi}{2}.$$

Remark

Note the formal resemblance of the sum to $\zeta(2)$.

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$$\zeta_{\mathcal{P}_{m\mathbb{Z}}}(8) = \frac{\omega_4\pi^4}{m^4\sin(\frac{\pi}{m})\sinh(\frac{\pi}{m})\sin(\frac{\omega_8\pi}{m})\sinh(\frac{\omega_8\pi}{m})},$$

with ω_k a kth root of unity

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with ω_k a kth root of unity, and so on for $s = 2^N$.

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Summing over partitions into distinct parts, we have

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$$\zeta_{\mathcal{P}^*}(2) = \frac{\sinh \pi}{\pi}.$$

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$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right),$$

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as well as product-sum identities that follow from Theorems 1 and 2:

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The proofs of the above corollaries (and many that follow) use another identity of Euler

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right),$$

as well as product-sum identities that follow from Theorems 1 and 2:

$$\prod_{n \in X} (1 - n^{-s})^{-1} = \zeta_{\mathcal{P}_X}(s)$$

and

$$\prod_{n \in X} (1 - n^{-s}) = \zeta_{\mathcal{P}_X^*}(s)$$

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$$\zeta_{\mathcal{P}_{\geq 2}}(3) = \frac{3\pi}{\cosh(\frac{1}{2}\pi\sqrt{3})}.$$

Proof

Follows from a problem by Ramanujan in *Journal of the Indian Math. Soc.*

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Question

Do there exist (non-Riemann) partition zeta functions such that, for the “right" choice of $\mathcal{P}' \subseteq \mathcal{P}$, we have $\zeta_{\mathcal{P}'}(N) = \pi^M \times$ *rational number*?

Partition-theoretic zeta functions

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Remark

Note the similarities in notation and form to MZVs.

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Euler's product identity for sine function.

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and similar formulas exist for $\zeta_{\mathcal{P}}(\{2^t\}^k),\ t \in \mathbb{Z}^+$.

Corollary 10 (Rolen-S.)

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For $m \geq 2, k \in \mathbb{N}$, and $j \geq i$, set

$$\alpha_{i,j} := \frac{\zeta(m(j-i+1))}{\pi^{m(j-i+1)}} \frac{(k-i)!}{(k-j)!}.$$

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Then

$$\zeta_{\mathcal{P}}(\{m\}^k) = \frac{\pi^{mk}}{k!} \det \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{1,k} \\ -1 & \alpha_{2,2} & \alpha_{2,3} & \dots & \alpha_{2,k} \\ 0 & -1 & \alpha_{3,3} & \dots & \alpha_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & \alpha_{k,k} \end{pmatrix}.$$

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For $N \geq 1$ we have $\zeta_{\mathcal{P}}(\{2N\}^k) = \pi^{2Nk} \times$ *rational number*.

Multiple zeta values

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$$= \sum_{\substack{\lambda \in \mathcal{P}^* \\ l(\lambda) = k}} n_\lambda^{-s} = \zeta_{\mathcal{P}^*}(\{s\}^k).$$

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and we have similar formulas of the shape "$\zeta(\{2^t\}^k) = \pi^{2^t k} \times$ *finite sum of fractions*".

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Note

Corollaries 8 through 12 above are related to results of M. Hoffman.

Proofs

Summation formulas in Corollaries 9 and 11

Follow from “doubling" identities for $n \in \mathbb{Z}_{\geq 0}$:

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Determinant formulas

Bell polynomials and formula of Chamberland-Straub.

Simple Structure Laws

Notation

For $\mathcal{P}' \subseteq \mathcal{P}$ we define

$$\eta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} (-1)^{l(\lambda)} n_\lambda^{-s}.$$

Relating partitions to partitions into distinct parts

We have

$$\eta_{\mathcal{P}_X^*}(s) = \frac{1}{\zeta_{\mathcal{P}_X}(s)},\ \zeta_{\mathcal{P}_X^*}(s) = \frac{1}{\eta_{\mathcal{P}_X}(s)}.$$

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Doubling formulas

We have

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The second identity becomes $1/\zeta(2s) = 1/\zeta(s)\sum_{n\in\mathbb{Z}^+} |\mu(n)|n^{-s}$.

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$$\zeta_{\mathcal{P}_X^*}(s)\zeta_{\mathcal{P}_X^*}(2s)\zeta_{\mathcal{P}_X^*}(4s)\zeta_{\mathcal{P}_X^*}(8s)\cdots = \zeta_{\mathcal{P}_X}(s)$$

$$\eta_{\mathcal{P}_X}(s)\eta_{\mathcal{P}_X}(2s)\eta_{\mathcal{P}_X}(4s)\eta_{\mathcal{P}_X}(8s)\cdots = \eta_{\mathcal{P}_X^*}(s)$$

Simple Structure Laws

Question

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What other classical number-theoretic objects might be viewed as specializations of partition-theoretic laws?

Gratitude

Thank you to SASTRA for the invitation to speak...

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Thank you to SASTRA for the invitation to speak...

And thank **you** for listening.