RESULTS FROM A COMPUTATIONAL STUDY OF CYCLOTOMIC PHENOMENA IN THE MOCK THETA FUNCTION $f(q)$

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Abstract. We list some identities found experimentally with a computer during a 2013 study at Emory University, which show Ramanujan's mock theta function $f(q)$ to be involved in cyclotomic-type structures in the limit as q approaches fifth-order roots of unity radially.

Here we collect some surprisingly simple relations the authors observed computationally during a study at Emory University (September–December, 2013) of Ramanujan's mock theta function $f(q)$, a q-series defined [5] as

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1}$$

where $(z;q)_0 := 1, (z;q)_n := \prod_{0\leq i\leq n-1}(1-zq^i)$ is the usual q-Pochhammer symbol.

In our study, the first author programmed SAGE using the following finite formula the second author had found (see [6] for details), to compute the limit of $f(q)$ as q approaches an odd order root of unity $\zeta_m, \zeta_m^2, \zeta_m^3,$ etc. radially from within the unit circle, for many odd values of $m>1$. We let $f(\zeta_m^i)$ denote this limit, which for m odd is given by

$$f(\zeta_m^i) = \frac{4}{3}\sum_{n=0}^{m-1}(-1)^n(-\zeta_m^{-i};\zeta_m^{-i})_n. \tag{2}$$

Note from the denominator of (1) that $f(\zeta_m^i)$ is singular for m even. Folsom-Ono-Rhoades [3], Bringmann-Rolen [1], and others have studied these limits by different methods, in connection to quantum modular forms and other interesting topics.

In our numerics we saw traces of cyclic group theory related to the values $f(\zeta_m^i)$ for odd m. The algebraic structure appears most simply if we use the normalized version

$$\widetilde{f}(\zeta_m^i) := \frac{3}{4}f(\zeta_m^i), \tag{3}$$

which is just the summation on the right-hand side of (2). If we observe that the coefficients of the polynomial

$$\widetilde{F}_m(X) := \prod_{\substack{1\leq i<m \\ \gcd(m,i)=1}} \left(X - \widetilde{f}(\zeta_m^i)\right) \tag{4}$$

are integers, by standard facts about coefficients [2] we have immediately that

$$\sum_i \widetilde{f}(\zeta_m^i),\ \sum_{i\neq j}\widetilde{f}(\zeta_m^i)\widetilde{f}(\zeta_m^j),\ \sum_{i\neq j\neq k}\widetilde{f}(\zeta_m^i)\widetilde{f}(\zeta_m^j)\widetilde{f}(\zeta_m^k),\ \sum_{i\neq j\neq k\neq l}\widetilde{f}(\zeta_m^i)\widetilde{f}(\zeta_m^j)\widetilde{f}(\zeta_m^k)\widetilde{f}(\zeta_m^l), \dots \tag{5}$$

all are integers.

To simplify our calculations, let us take $m = p$ a prime number. Right away we observe computationally that the coefficients indicated in (5) are congruent to 1 modulo p. We also observe that the $\widetilde{f}(\zeta_p^i)$ are cyclic of order p, modulo p:

$$\widetilde{f}(\zeta_p^i)^n \equiv \widetilde{f}(\zeta_p^i)^{n+pk} \pmod{p} \text{ for all } i,k,n \in \mathbb{Z} \tag{6}$$

We found a large number of nice identities for the case $p = 5$, such as this infinite system, which is not hard to prove:

$$\begin{aligned} \widetilde{f}(\zeta_5) + \widetilde{f}(\zeta_5^2) + \widetilde{f}(\zeta_5^3) + \widetilde{f}(\zeta_5^4) &= 4, \\ \widetilde{f}(\zeta_5)^2 + \widetilde{f}(\zeta_5^2)^2 + \widetilde{f}(\zeta_5^3)^2 + \widetilde{f}(\zeta_5^4)^2 &= 4, \\ \widetilde{f}(\zeta_5)^3 + \widetilde{f}(\zeta_5^2)^3 + \widetilde{f}(\zeta_5^3)^3 + \widetilde{f}(\zeta_5^4)^3 &= -11, \\ \widetilde{f}(\zeta_5)^4 + \widetilde{f}(\zeta_5^2)^4 + \widetilde{f}(\zeta_5^3)^4 + \widetilde{f}(\zeta_5^4)^4 &= -76, \ldots \end{aligned} \tag{7}$$

Most strikingly, we see these $\widetilde{f}(\zeta_5^i)$ involved in cyclotomic structures. Firstly, by direct computation we find this simple identity.

Theorem 1. *We have that*

$$\widetilde{f}(\zeta_5)\widetilde{f}(\zeta_5^2)\widetilde{f}(\zeta_5^3)\widetilde{f}(\zeta_5^4) = 1. \tag{8}$$

Then we get the obvious relations

$$\widetilde{f}(\zeta_5^i)\widetilde{f}(\zeta_5^j) = \frac{1}{\widetilde{f}(\zeta_5^k)\widetilde{f}(\zeta_5^l)}, \quad \widetilde{f}(\zeta_5^i) = \frac{1}{\widetilde{f}(\zeta_5^j)\widetilde{f}(\zeta_5^k)\widetilde{f}(\zeta_5^l)}, \tag{9}$$

where $i, j, k, l \in \{1, 2, 3, 4\},\ i \neq j \neq k \neq l$. Direct calculation verifies further nice identities.

Theorem 2. *Certain products $\widetilde{f}(\zeta_5^i)\widetilde{f}(\zeta_5^j), i \neq j$, are equal to roots of unity:*

$$\begin{aligned} \widetilde{f}(\zeta_5)\widetilde{f}(\zeta_5^3) &= \zeta_5 \\ \widetilde{f}(\zeta_5)\widetilde{f}(\zeta_5^2) &= \zeta_5^2 \\ \widetilde{f}(\zeta_5^3)\widetilde{f}(\zeta_5^4) &= \zeta_5^3 \\ \widetilde{f}(\zeta_5^2)\widetilde{f}(\zeta_5^4) &= \zeta_5^4 \end{aligned} \tag{10}$$

At this point it is easy to derive a variety of identities, for example

$$\widetilde{f}(\zeta_5)^3\widetilde{f}(\zeta_5^2)^2\widetilde{f}(\zeta_5^3) = 1, \quad \left(\widetilde{f}(\zeta_5) + \widetilde{f}(\zeta_5^4)\right)\left(\widetilde{f}(\zeta_5^2) + \widetilde{f}(\zeta_5^3)\right) = -1. \tag{11}$$

Then using (3) we arrive at the following lovely relations.

Theorem 3. *For Ramanujan's mock theta function $f(q)$ at fifth-order roots of unity ζ_5^i, we have*

$$\begin{aligned} 9/16\ f(\zeta_5)f(\zeta_5^3) &= \zeta_5 \\ 9/16\ f(\zeta_5)f(\zeta_5^2) &= \zeta_5^2 \\ 9/16\ f(\zeta_5^3)f(\zeta_5^4) &= \zeta_5^3 \\ 9/16\ f(\zeta_5^2)f(\zeta_5^4) &= \zeta_5^4 \end{aligned} \tag{12}$$

and also

$$f(\zeta_5)f(\zeta_5^2)f(\zeta_5^3)f(\zeta_5^4) = 256/81.$$

From the preceding formulas, we arrive at the most elegant finding of this study.

Theorem 4. *At fifth-order roots of unity, we have the symmetric relation*

$$\zeta_5^i f(\zeta_5^i) \ = \ \zeta_5^{-i} f(\zeta_5^{-i}). \tag{13}$$

As noted above, we computed Theorems 1 and 2 directly from the formula (2) letting m be the prime $p = 5$; we have not proved these by algebraic methods, so we don't have a clear intuition as to how the theorems might generalize. We expect there to be analogs to the equations above (but more complicated) for other primes, at least, as presumably Theorem 2 depends in the end on properties of (2) and facts about polynomials at roots of unity, not on the choice of p.

Are there relations like these for other mock theta functions at roots of unity?

Acknowledgement

We thank our Ph.D. advisor, Ken Ono, for his guidance during this study.

References

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