INFINITE SERIES FOR $\pi/3$ AND OTHER IDENTITIES

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Abstract. We combine classical identities for π, $\log 2$, and harmonic numbers, to arrive at a nice infinite series formula for $\pi/3$ that does not appear to be well known. In an appendix we give twenty-seven related identities involving π and other irrational numbers. (*Note: This is a LaTeX transcription of the author's first mathematics paper, written in 2006.)*

1. Main identity and proof

Recall the lovely classical identity known in the literature as Leibniz's formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... \tag{1}$$

This identity is immediate from the Maclaurin series expansion of $\arctan x$ at $x = 1$. Here, we prove another simple summation formula that can be used to compute the value of π.

Theorem 1. *We have the identity*

$$\frac{\pi}{3} = \sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-3)}.$$

Proof. We begin with the Maclaurin series for the natural logarithm of 2:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + -... + \frac{1}{2n-1} - \frac{1}{2n}\right) = \log 2 \tag{2}$$

We may rewrite (2) as

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n}\right) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \log 2,$$

thus

$$\sum_{n=1}^{\infty} \frac{1}{4n(4n-2)} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \frac{1}{4} \log 2. \tag{3}$$

Euler [1] found that the difference between the kth harmonic number and $\log k$ approaches a constant $\gamma = 0.5772...$ (the so-called Euler-Mascheroni constant), so we can write

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{2n} - \log(2n)\right) = \gamma. \tag{4}$$

Adding limits from equations (2) and (4), then dividing by 2 (and writing $\log(2n) = \log 2 + \log n$), we find

$$\lim_{n\to\infty} \left(1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n-1} - \frac{1}{2}(\log 2 + \log n)\right) = \frac{1}{2}(\log 2 + \gamma),$$

thus

$$\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}-\frac{1}{2}\log n\right)=\log 2+\frac{1}{2}\gamma. \tag{5}$$

Splitting the sum on the left in half and reorganizing, equation (5) may be rewritten

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+3}+...+\frac{1}{2n-1}\right) \tag{6}$$
$$=\log 2+\frac{1}{2}\gamma-\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{n-1}-\frac{1}{2}\log n\right).$$

Now, substitute $n/2$ for n in (5), then subtract $1/2\cdot\log 2$ from both sides, to arrive at

$$\frac{1}{2}\log 2+\frac{1}{2}\gamma=\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{n-1}-\frac{1}{2}\log n\right). \tag{7}$$

Adding the corresponding sides of (6) and (7), then subtracting $1/2\cdot\log 2+\gamma/2$ from the resulting equation, gives

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+3}+...+\frac{1}{2n-1}\right)=\frac{1}{2}\log 2. \tag{8}$$

Next we will use Leibniz's formula (1), which we can write in the form

$$\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-...+\frac{1}{2n-3}-\frac{1}{2n-1}\right)=\frac{\pi}{4}. \tag{9}$$

If we add the limits from (5) and (9), then divide both sides by 2, we find

$$\lim_{n\to\infty}\left(1+\frac{1}{5}+\frac{1}{9}+...+\frac{1}{2n-3}-\frac{1}{4}\log n\right)=\frac{\pi}{8}+\frac{1}{2}\left(\log 2+\frac{1}{2}\gamma\right).$$

Subtracting $1/2$ times the limit in (5) from this equation gives

$$\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{5}-\frac{1}{6}+-...+\frac{1}{2n-3}-\frac{1}{2n-2}\right)-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+3}+...+\frac{1}{2n-1}\right)$$
$$=\frac{\pi}{8}.$$

Adding $1/2$ times equation (8) to both sides of this expression yields

$$\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{5}-\frac{1}{6}+-...+\frac{1}{2n-3}-\frac{1}{2n-2}\right)=\frac{\pi}{8}+\frac{1}{4}\log 2,$$

which can be rewritten

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-2}\right)=\sum_{n=1}^{\infty}\frac{1}{(4n-2)(4n-3)}=\frac{\pi}{8}+\frac{1}{4}\log 2. \tag{10}$$

Subtracting equation (3) from (10) yields

$$\sum_{n=1}^{\infty}\left(\frac{1}{(4n-2)(4n-3)}-\frac{1}{4n(4n-2)}\right)=\sum_{n=1}^{\infty}\frac{3}{4n(4n-2)(4n-3)}=\frac{\pi}{8}.$$

Finally, multiplication by $8/3$ gives the theorem. □

Remark. *We note that multiplying both sides of Theorem 1 by* 3 *produces a summation formula for* π*, whose* $n = 1$ *term equals* 3*; the remaining terms give the fractional part:*

$$3\sum_{n=2}^{\infty}\frac{1}{n(2n-1)(4n-3)} = 0.14159265...$$

2. Appendix: Further identities

The preceding section is a LaTeX transcription (with a little pruning) of the author's first mathematics paper, written in 2006 at a time when I was almost entirely self taught. During that period I also found a large number of related summation identities by combining Theorem 1 with other equations in the preceding proof, along with well-known zeta values such as Euler's famous solution of the Basel problem (see [1])

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

I give a selection of these formulas below without proof, loosely organized by number of factors in the denominators of the summands. The first identity follows from the telescoping series $(1-1/2)+(1/2-1/3)+(1/3-1/4)+...$, the second and third identities follow by rewriting (2) and (9), respectively, and the rest arise from liberal use of partial fractions.

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \tag{11}$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)} = 2\log 2. \tag{12}$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n-3)} = \frac{\pi}{8} \tag{13}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(4n-3)} = \frac{\pi+2\log 2}{4} \tag{14}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(4n-3)} = \frac{\pi+6\log 2}{6} \tag{15}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(4n-1)} = \frac{6\log 2-\pi}{2} \tag{16}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(4n-1)(4n-3)} = \frac{\log 2}{2} \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(2n+1)} = \frac{2\log 2 - 1}{2} \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-1)(4n-3)} = \frac{\pi - 2}{16} \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)} = \frac{24\log 2 - \pi^2}{6} \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(4n-1)} = \frac{72\log 2 - 12\pi - \pi^2}{6} \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \frac{12 - \pi^2}{6} \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} = \frac{\pi^2 + 24\log 2 - 24}{6} \tag{23}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \frac{\pi^2 - 6}{6} \tag{24}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2 - 9}{3} \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)(4n+1)(4n-1)} = \frac{\pi - 3}{6} \tag{26}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-1)(8n+1)(8n-1)} = \frac{\pi(1+2\sqrt{2}) - 12}{24} \tag{27}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-1)(4n-3)} = \frac{6\log 2 - \pi}{3} \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)(2n+1)} = \frac{\pi^2 - 12\log 2}{6} \tag{29}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)(4n-3)} = \frac{\pi^2 + 8\pi - 24\log 2}{18} \tag{30}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)(4n-1)} = \frac{\pi^2 + 24\pi - 120\log 2}{6} \tag{31}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)(4n-1)(4n-3)} = \frac{168\log 2 - 32\pi - \pi^2}{18} \tag{32}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)(2n-1)(6n+1)(6n-1)} = \frac{52 - 32\log 2 - 27\log 3}{16} \tag{33}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)(2n-1)(3n+1)(3n-1)} = \frac{19 + 16\log 2 - 27\log 3}{10} \tag{34}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} = 10 - \pi^2 \tag{35}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)(4n+1)(4n-1)(8n+1)(8n-1)} = \frac{45 - \pi(3 + 8\sqrt{2})}{6} \tag{36}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)(2n-1)(3n+1)(3n-1)(6n+1)(6n-1)} = \frac{64\log 2 + 27\log 3 - 74}{20} \tag{37}$$

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References

[1] W. Dunham, *Euler: the master of us all*, The Dolciani Mathematical Expositions, 22. Mathematical Association of America, Washington, DC, 1999.