Plan for a Microtonal Composition Modulo *p*#

ROBERT P. SCHNEIDER
University of Kentucky
Lexington, KY 40506
robert.schneider@uky.edu

Introduction We present a plan for a musical piece in a microtonal scale, in which the sequence of prime numbers creates melodic patterns arising from the algebraic structure of the scale.

The composition was inspired by a conversation with Neil Sloane of the *On-Line Encyclopedia of Integer Sequences* at the ninth Gathering for Gardner conference in 2010. The author had the pleasure of dining with Dr. Sloane, who had presented musical renderings of a number of integer sequences at the conference, and who enquired whether the author knew of a context in which the sequence of primes would sound musical, as opposed to random. The present plan is an attempt in this direction.

Definition of the microtonal scale In the following, all variables are understood to denote positive integers.

One may define a microtonal sequence of tones in equal temperament by dividing the octave into N distinct pitches, using the well-known formula for the nth tone in the sequence

$$f \cdot 2^{\frac{n-1}{N}},$$

where f denotes the arbitrary frequency of the first tone. The equally-spaced steps of equal-tempered tuning are an analog for the ear of the whole numbers, and we wish to listen for patterns that ripple throughout the integers.

For the present composition, let us divide the octave into $N = p\#$ equal-tempered tones, where p is a prime number chosen by the conductor, and $p\# = 2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p$ denotes the product of all primes less than or equal to p. Then from the above expression we have the formula for the nth tone $T(n)$ in the sequence

$$T(n) = f \cdot 2^{\frac{n-1}{p\#}}.$$

From this sequence $\{T(n)\}$ as n varies over the positive integers, we define a subset of tones to which we will restrict the choice of pitches used in the composition. Let us allow the word "scale" to denote this subset.

We base our scale on the set of integers that are co-prime to $p\#$, i.e. integers n such that $\gcd(n, p\#) = 1$. We will only include the tone $T(m)$ in our scale if m and $p\#$ are co-prime. We begin with tone $T(1)$ as $\gcd(1, p\#) = 1$. Primes less than or equal to p do not define members of the scale, as these primes are not co-prime to $p\#$.

There are $\varphi(p\#)$ such integers less than or equal to $p\#$, where the Euler totient function $\varphi(n)$ counts the number of integers less than or equal to a given n that are co-prime to n. Therefore we have $\varphi(p\#)$ tones comprising one octave of the scale. Choosing $p = 3$ selects a two-tone scale from $3\# = 2 \cdot 3 = 6$ distinct tones in the octave, as $\varphi(6) = 2$. Choosing $p = 5$ selects an eight-tone scale from $5\# = 2 \cdot 3 \cdot 5 = 30$ tones in the octave, as $\varphi(30) = 8$. As we take larger prime values of p in our formula, the number of tones in the octave increases rapidly.

We note that $\gcd(m, p\#) = 1$ also implies $\gcd(p\# - m,\ p\#) = 1$, such that the integers co-prime to $p\#$ are symmetrically distributed around the center of the interval $(1, p\#)$. Then the tones of the scale defined above exhibit the same mirror-symmetry around the central tone $T\left(\frac{p\#+1}{2}\right)$ of the octave. Other symmetries inherent in the integers modulo $p\#$ also translate into patterns in the distribution of tones in the scale.

Musical instructions Clearly every prime greater than p is co-prime to $p\#$, as such a prime is too large to be a factor of $p\#$, and does not itself have any factors other than one. Therefore all of the prime-numbered tones associated with primes greater than p belong to our musical scale.

The composition is carried out by playing the sequence of pitches consisting of $T(1)$ followed by these prime-numbered tones $T(m)$ with $m > p$, sounded one after another in a slow, steady rhythm.

The conductor may choose to set f equal to the pitch he or she prefers in the formula for $T(n)$ above. Let us render the audio tones using a tone generator or synthesizer, calculating the sequence of pitches by the formula. Alternatively, one might mark the positions of the pitches on the fingerboards of a string quartet, or produce the tones by some other method.

The range of pitches used should be limited to one octave to avoid the occurrence of shrill high frequencies, with tones being transposed down an octave whenever m exceeds a multiple of $p\#$. Deep tones should be used to minimize harsh timbres from electronically generated sounds, and to ease the listener faced with strange algorithmic music.

We anticipate from studying the theory of numbers that something of a repeating theme will be established initially. In fact, if q represents the next prime following p, then on the interval (p, q^2) the sequence of primes is identical to the sequence of integers co-prime to $p\#$. Tones numbered on this interval will mimic the predictable opening tones of the scale. This is because any composite integer n has at least one prime factor less than or equal to $\sqrt{n}$. The integers co-prime to $p\#$ lying below q^2 are co-prime to every prime less than q, thus having no prime factors less than or equal to their square roots; therefore these integers are prime.

This apparent theme will decay as the composition progresses, as fewer and fewer integers co-prime to $p\#$ are also primes, yet prime-numbered tones will still fall on the familiar tones of the microtonal scale—degenerating gradually to more random-sounding melodies as only a few primes occur in each octave, then concluding with silence as the density of primes dwindles.

The composition concludes with the first occurrence of one complete octave of the scale, an interval containing $p\#$ tones, which contains no prime-numbered tones. This can occur only when the density of primes has decreased to such a degree that there exist gaps of $p\#$ between consecutive prime numbers. We are guaranteed that such a "silent" octave exists eventually, as there can be shown to exist arbitrarily large gaps between primes among the positive integers.

However, the conclusion of the composition may take a very long time to reach. In the case $p = 3$ noted above, a scale having only two tones in the octave, the composition ends on tone number 23 where the first prime gap of six begins. In the case $p = 5$, with eight tones in the octave, the composition ends on tone 1327, lasting just over 22 minutes at a tempo of 60 beats per minute. In the case $p = 7$ the end of the composition will occur on tone number 20,831,323, lasting over six months at the same tempo. For larger primes, the prime number theorem estimates that an average gap of $p\#$ occurs around tone number $m = p\# \cdot e^{p\#}$, an immense number suggesting performances lasting millions of years.

Concluding remarks Due to the duration of the composition for primes greater than five, an environmental installation might be the best venue for an electronic performance of this piece, in which the sequence of prime numbers may be allowed to continually unfold—perhaps a sturdy solar-powered outdoor structure, self-contained and humming deep tones perpetually.

Number-theoretically, the composition demonstrates that on a certain interval, the sequence of primes corresponds exactly to the sequence of integers co-prime to $p\#$; that to a lessening degree the distribution of primes imitates that of integers co-prime to $p\#$ beyond the interval (p, q^2); and also that large gaps will occur in the sequence of primes. We note that a similar correspondence between primes and integers co-prime to $p\#$ holds on infinitely many intervals, as infinitely many primes p are available from which to define congruence classes modulo $p\#$.

Melodically, we hear a pattern of tones decaying slowly, yet staying within our specially constructed microtonal scale. If we allow our use of the word "music" to encompass tone sequences chosen from a well-defined scale according to some rule or guiding principle, then the planned composition provides one way to listen to what mathematicians poetically call the "music of the primes."

Acknowledgement The author wishes to that Professor Neil Calkin for editorial comments, and for computing the durations of the composition for the first few prime values of p.