RECURSIVE CONSTRUCTION OF THE PRIMES

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Here we collect and prove some mathematical observations the author used to write a 2010 score for an experimental musical composition [1]. Let us define $\widehat{x}_n$ to be the largest element of the set $P_n \subset \mathbb{Z}^+$ defined by $P_1 := \{2\}$ and, for $n > 1$, by

$$P_n := \{x \in (\widehat{x}_{n-1},\ \widehat{x}_{n-1}^2) \text{ such that } \gcd(x,\ \widehat{x}_{n-1}\#) = 1\}, \tag{1}$$

where, on analogy to *primorial* notation $p\# := 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot ... \cdot p$ for $p \in \mathbb{P}$, we define

$$\widehat{x}_k\# := \prod_{i=1}^{k} \prod_{x_i \in P_i} x_i.$$

We note that the P_n are mutually disjoint.

Proposition 1. *We have that $P_n \subset \mathbb{P}$ for all $n \geq 1$. Moreover, it is the case that*

$$\bigcup_{n=1}^{\infty} P_n = \mathbb{P}.$$

Remark. *Thus, in fact, $\widehat{x}_k\#$ is an instance of the primorial function.*

Proof. We proceed by induction on n to prove $P_n \subset \mathbb{P}$. Clearly we have as a base case that, since $\widehat{x}_1 = 2$ and $2\# = 2$, then $P_2 = \{x \in (2, 4) \text{ such that } \gcd(x, 2) = 1\} = \{3\} \subset \mathbb{P}$. Assume the statement holds true for all P_i with $i < n$. Considering now the set P_n, recall that $x \in \mathbb{Z}^+$ is composite if and only if x has a prime factor $\leq \sqrt{x}$. Since $\widehat{x}_{n-1}$ is prime by the induction hypothesis, and since the elements of P_n are coprime to the elements of $\bigcup_{i=1}^{n-1} P_i$ (which are all of the primes $\leq \widehat{x}_{n-1}$), then each element $x \in P_n$ has no prime factor $\leq \sqrt{x} < \widehat{x}_{n-1}$, thus is prime as well, which concludes our induction argument.

That the primes can be covered by the disjoint sets $P_i,\ i = 1, 2, 3, ...,$ follows from this argument (as well as definition (1) which ensures every prime lies in one of these sets), as $n \to \infty$. ∎

Remark. *We could have more efficiently covered the primes with larger disjoint sets $P_n := \left\{x \in (\widehat{x}_{n-1},\ \overline{x}_{n+1}^2) \text{ such that } \gcd(x,\ \widehat{x}_{n-1}\#) = 1\right\}$ in place of the definition (1), where we let $\overline{x}_i$ denote the smallest element of P_i. However, this approach requires knowledge of P_{n+1} to construct P_n, whereas the present construction is purely recursive. Coverings using smaller sets P_n are possible too, of course; just replace $\widehat{x}_{n-1}$ with any other element of P_{n-1} in (1).*

References

[1] R. Schneider, *Plan for a microtonal composition modulo p#*, Experimental musical score (2010).